Abstract interpretation

UW CSE P 504

Today

Abstract interpretation

- Lattices
- Abstraction function
- Concretization function
- Transfer function (vs. lub vs. glb)
- Galois connection
- Exercise: concrete examples

Abstract interpretation (intuition)

Abstract domain and abstraction function (intuition)

Program

```
x = 0;
y = read_even();
x = y + 1;
y = 2 * x;
x = y - 2;
y = x / 2;
```

Abstract domain (even, odd, unk)

- `x = 0;` → {x=e}
- `y = read_even();` → {x=e; y=e}

Transfer function (intuition)

Program

```
x = 0;
y = read_even();
x = y + 1;
y = 2 * x;
x = y - 2;
y = x / 2;
```

+	unk	odd	even
unk	unk	unk	unk
odd	unk	even	odd
even	unk	odd	even

Transfer function corresponds to the “abstract execution” of +

Abstract interpretation (a bit more formal)

Set, semilattice, lattice

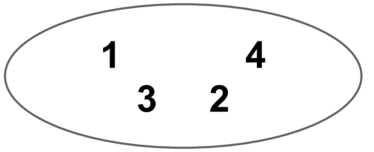
Set, semilattice, lattice

Set

Set, semilattice, lattice

Set

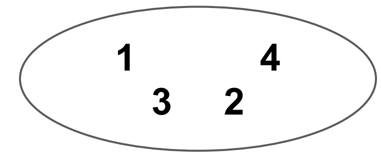
- **unordered** collection of **distinct** objects



Set, semilattice, lattice

Set

- unordered collection of distinct objects

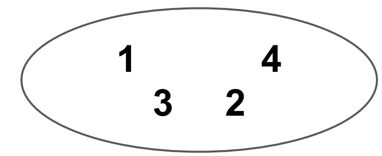


Partially ordered set

Set, semilattice, lattice

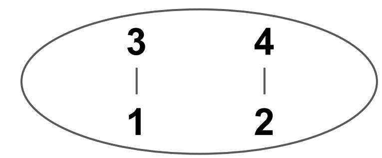
Set

- unordered collection of distinct objects



Partially ordered set

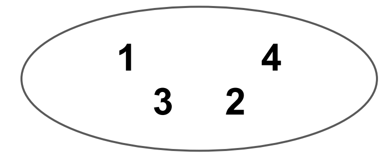
- Binary relationship $\leq$:
  - Reflexive: $x \leq x$
  - Anti-symmetric: $x \leq y \wedge y \leq x \Rightarrow x = y$
  - Transitive: $x \leq y \wedge y \leq z \Rightarrow x \leq z$



Set, semilattice, lattice

Set

- unordered collection of distinct objects



Partially ordered set

- Binary relationship $\leq$:
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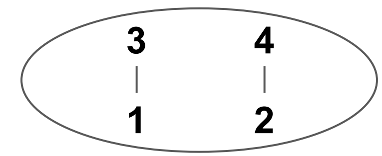


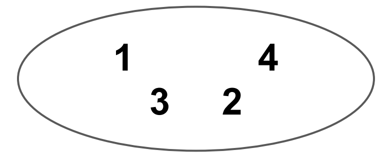
Join semilattice

Meet semilattice

Set, semilattice, lattice

Set

- unordered collection of distinct objects



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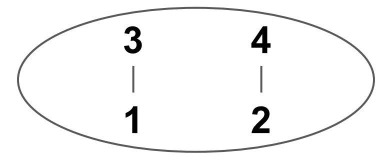
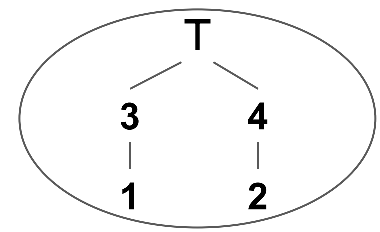


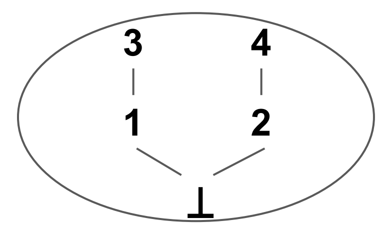
Join semilattice

- Partially ordered set with least upper bound (join)



Meet semilattice

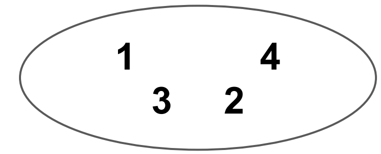
- Partially ordered set with greatest lower bound (meet)



Set, semilattice, lattice

Set

- unordered collection of distinct objects



Partially ordered set

- Binary relationship $\leq$:
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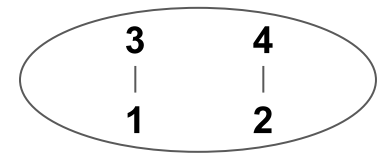

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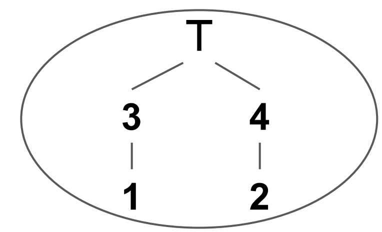
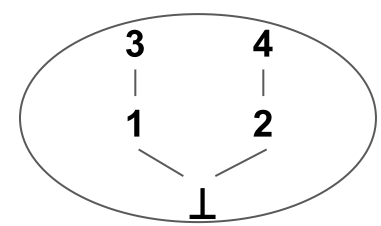


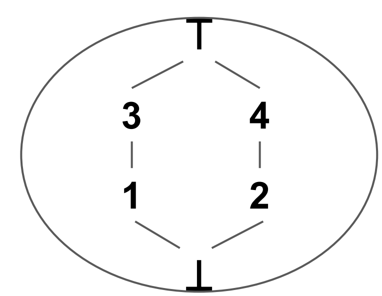
Meet semilattice

- Partially ordered set with greatest lower bound (meet)



Lattice

- Both a join semilattice and a meet semilattice

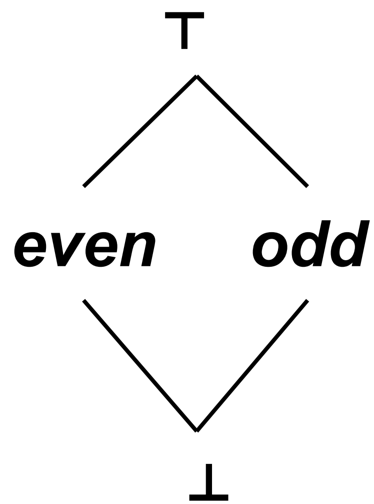


Lattice: example

Abstract domain: ***even, odd, unknown, {}***

Lattice: example

Abstract domain: ***even, odd, unknown*** **(⊤),** ***{}*** **(⊥)**

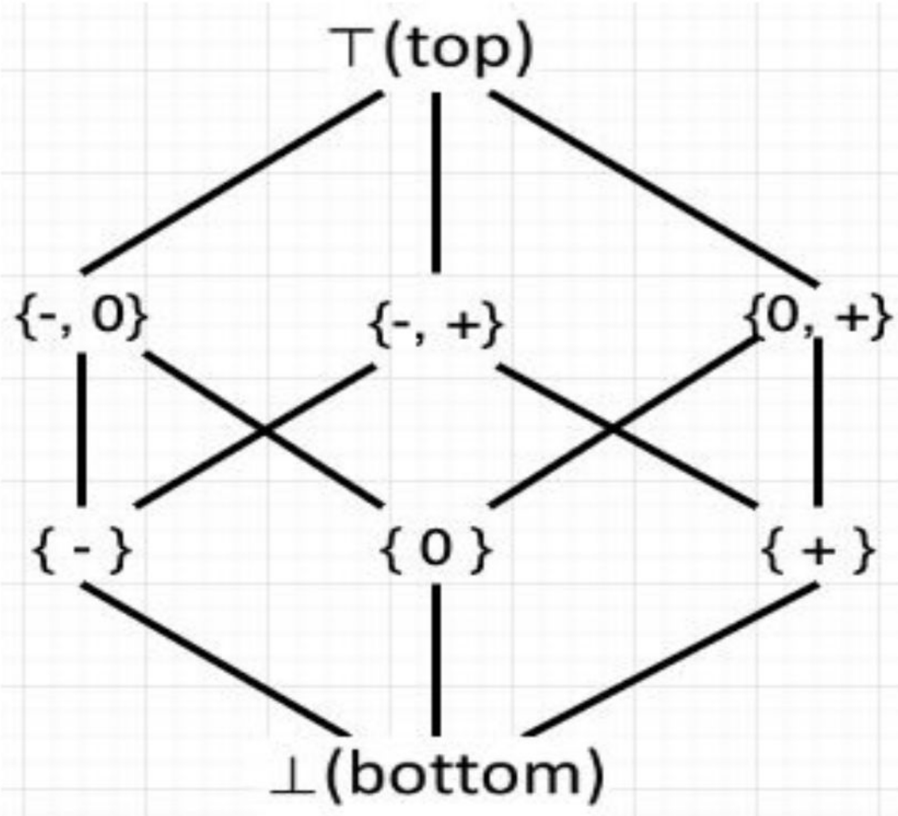


Lattice: example

Abstract domain: ***-, 0, +, unknown, {}***

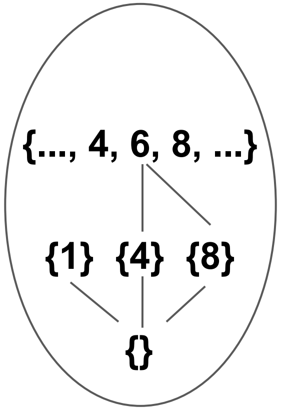
Lattice: example

Abstract domain: *-, 0, +, unknown, {}*

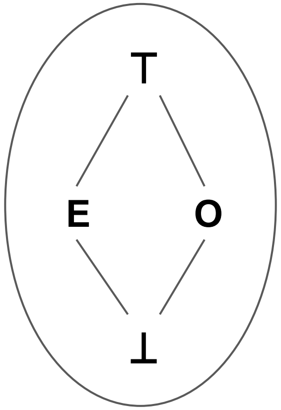


Abstraction function

Concrete ($P(\mathbb{N})$)

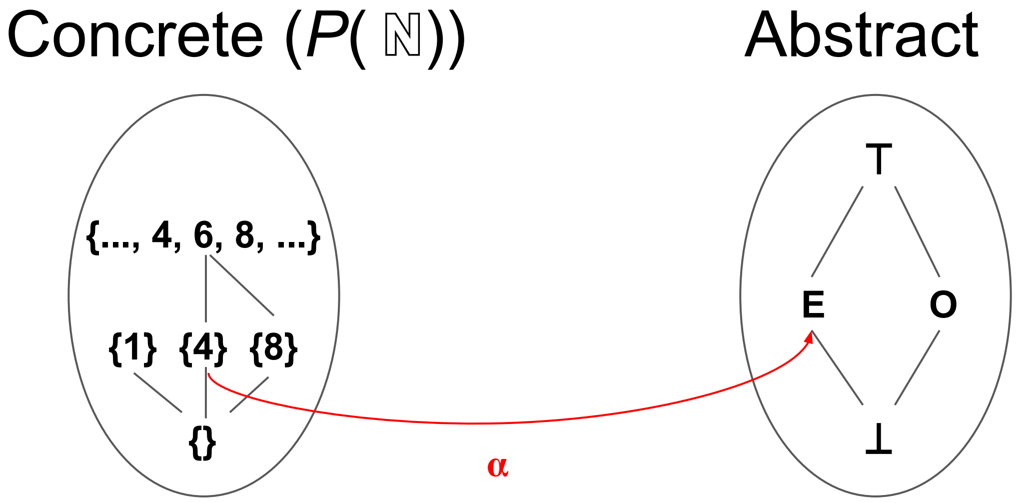


Abstract



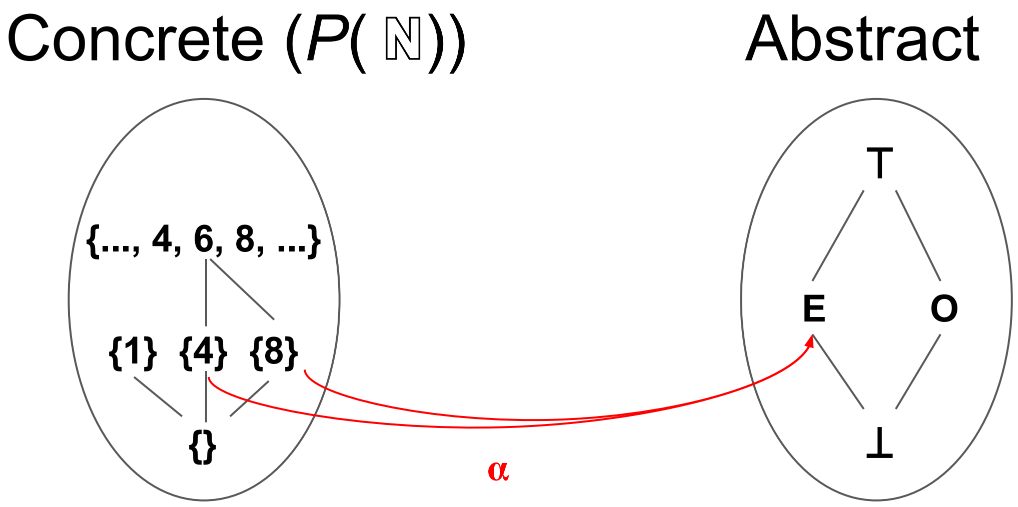
What is the abstraction (α) of {4}?

Abstraction function



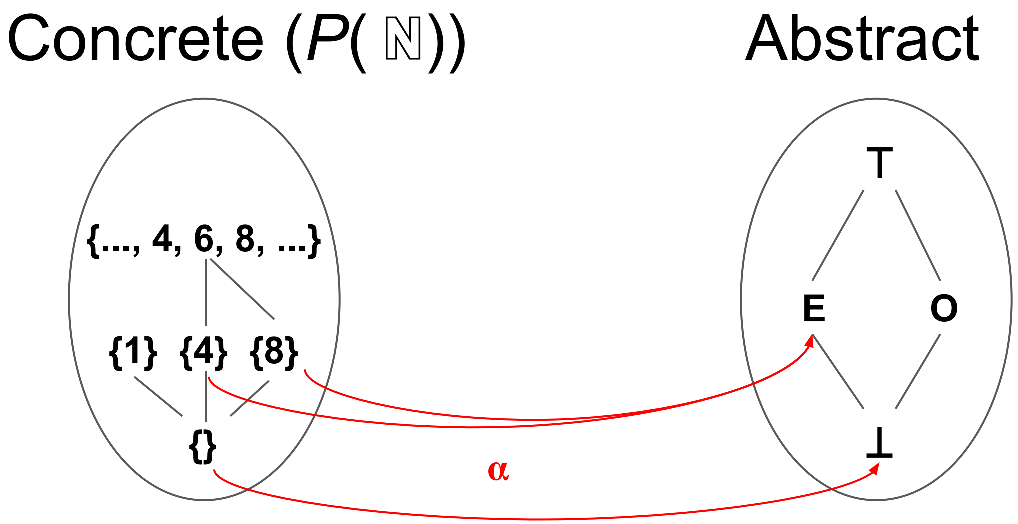
What is the abstraction (α) of {8}?

Abstraction function



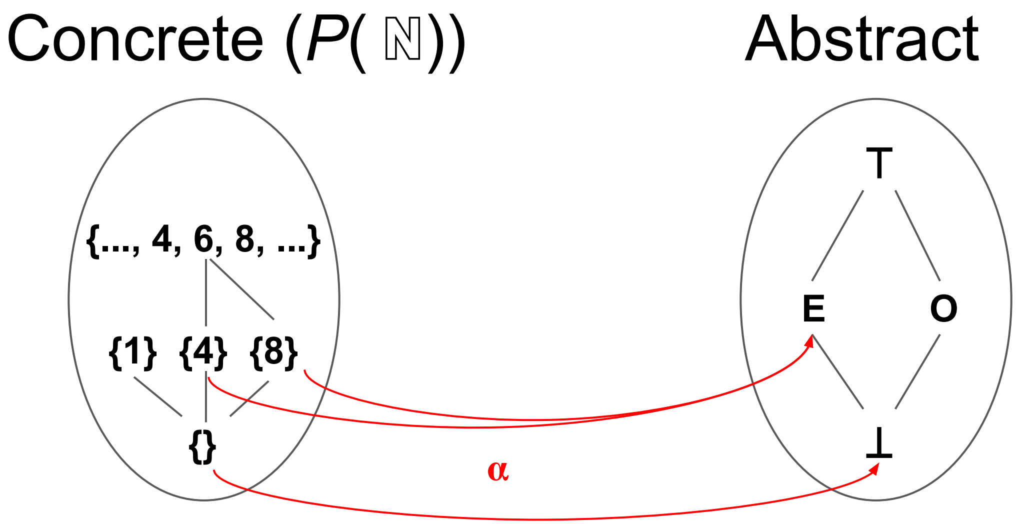
What is the abstraction (α) of {}?

Abstraction function



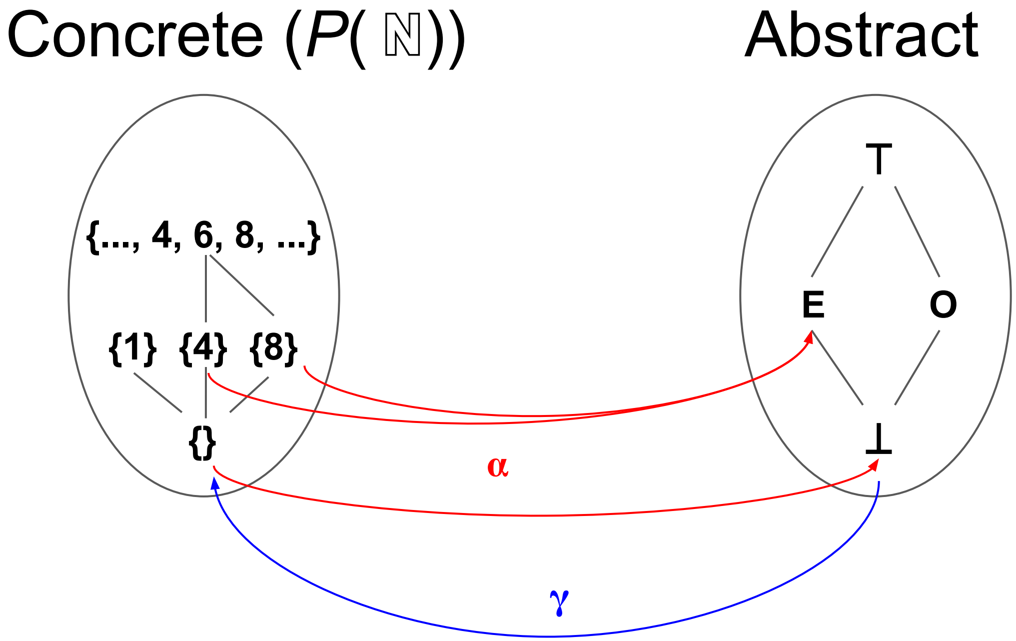
Why do we need an abstraction function?

Concretization function



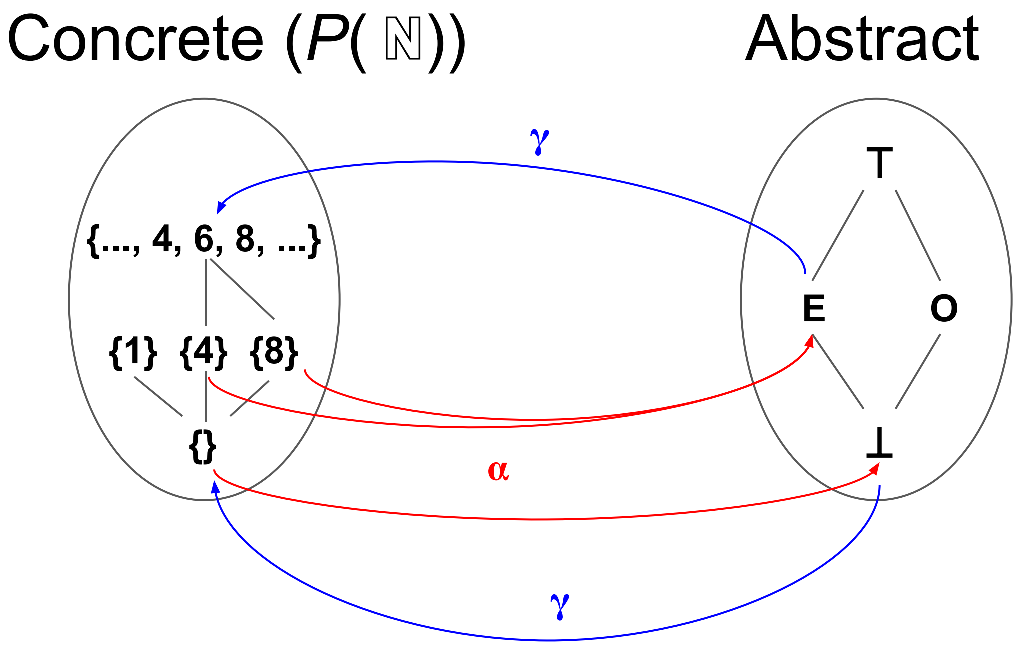
What is the concretization (γ) of ⊥?

Concretization function

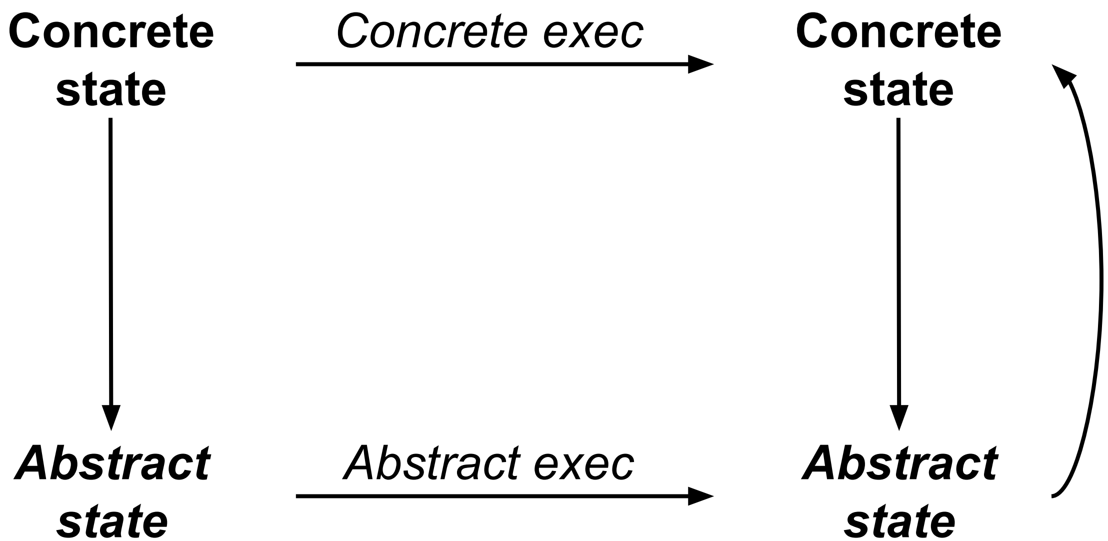


What is the concretization (γ) of **E** ?

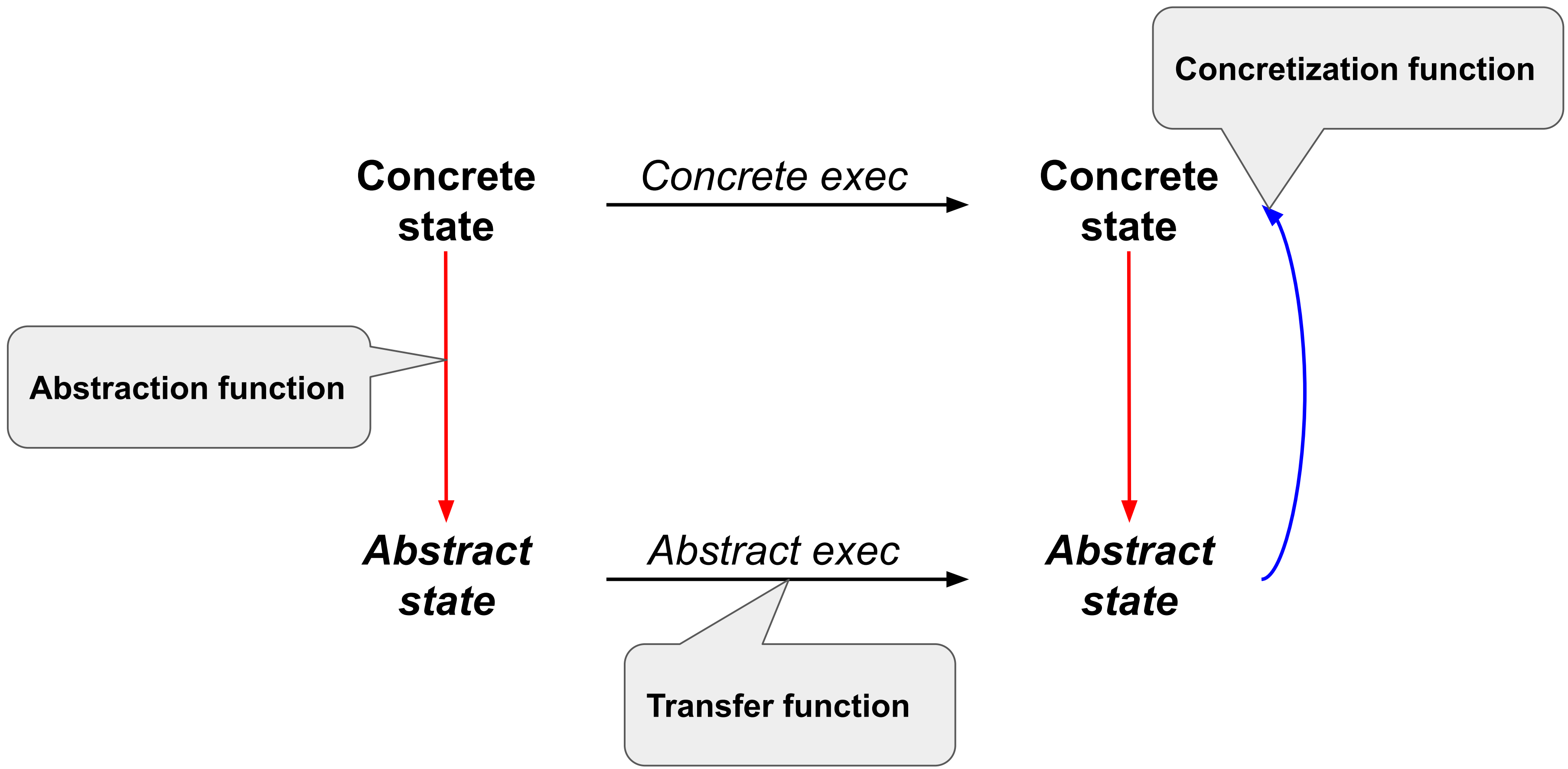
Concretization function



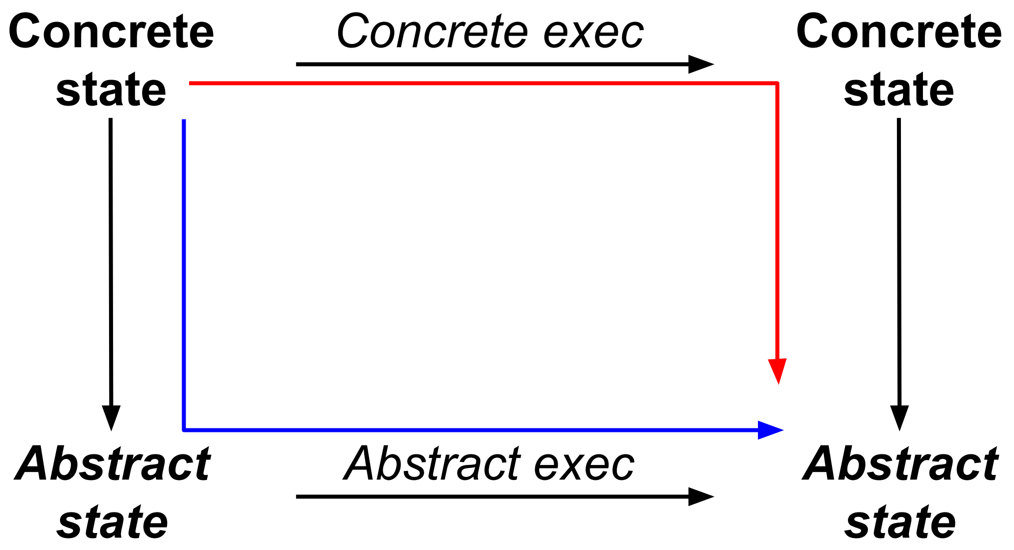
Transfer function



Transfer function

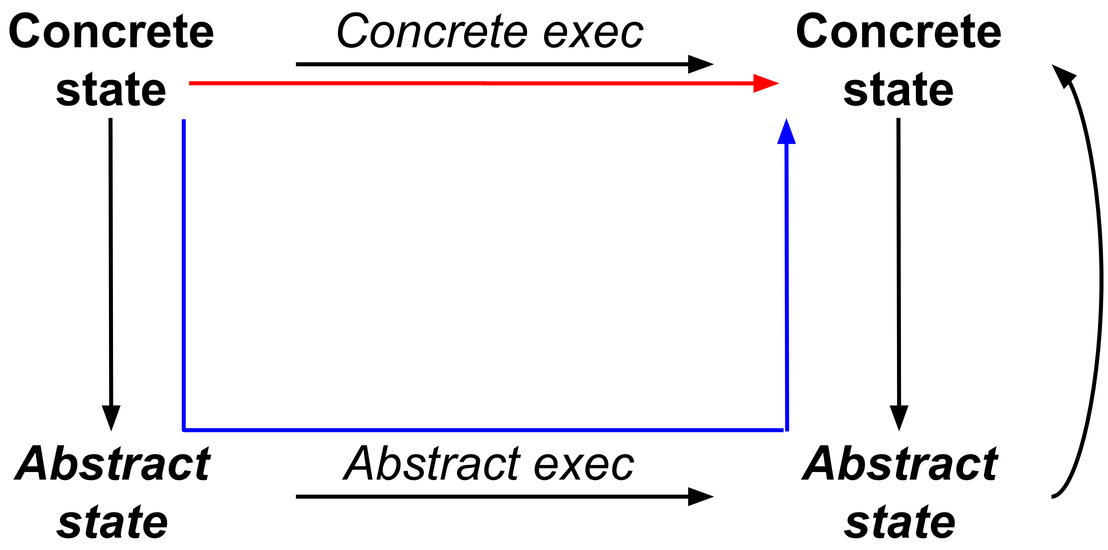


Abstract interpretation: approximation



Do both paths lead to the same abstract state?

Abstract interpretation: approximation



Do both paths lead to the same concrete state?

Abstract interpretation: soundness example

Abstract domain: {*odd, even$_2$, even$_4$, is2, unk*}

x=16 —— 16 / 2 ——→

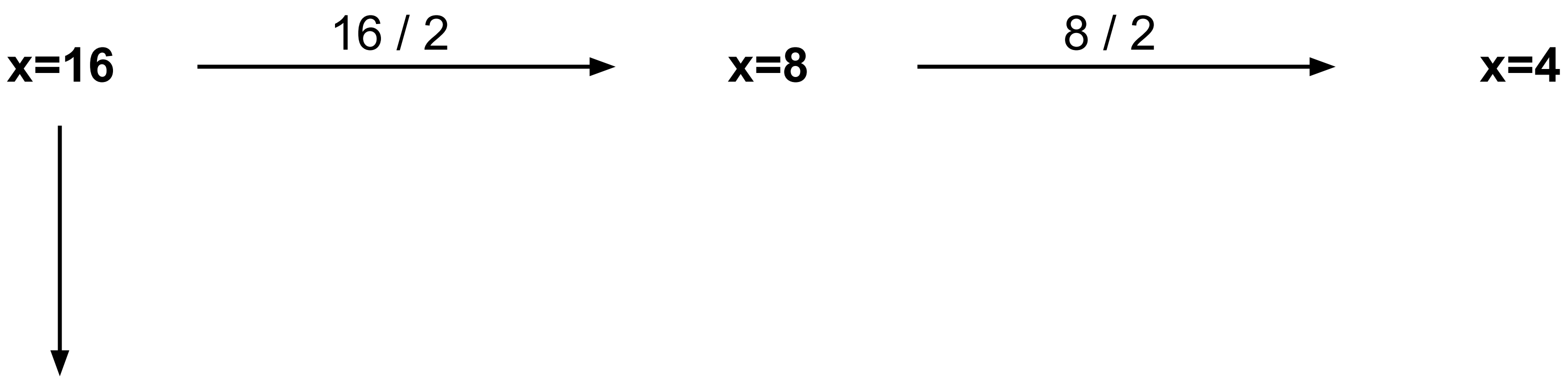
Abstract interpretation: soundness example

Abstract domain: {*odd*, *even*$_2$, *even*$_4$, *is2*, *unk*}

x=16 —16 / 2→ **x=8** —8 / 2→

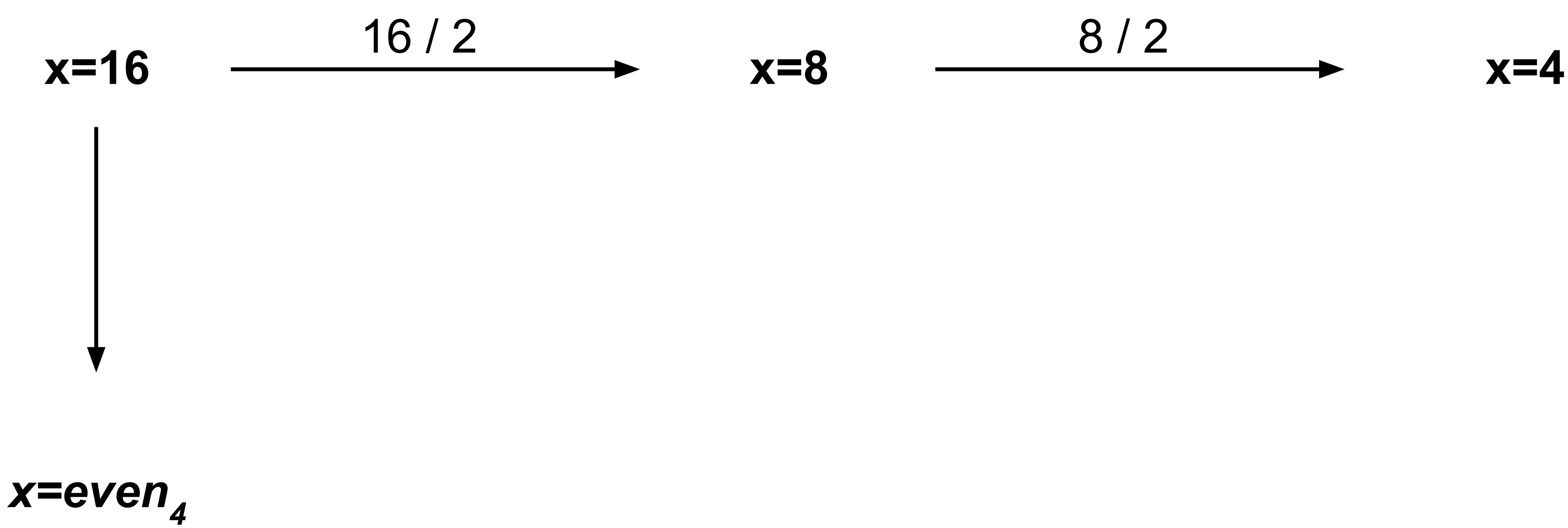
Abstract interpretation: soundness example

Abstract domain: {*odd*, *even*$_2$, *even*$_4$, *is2*, *unk*}



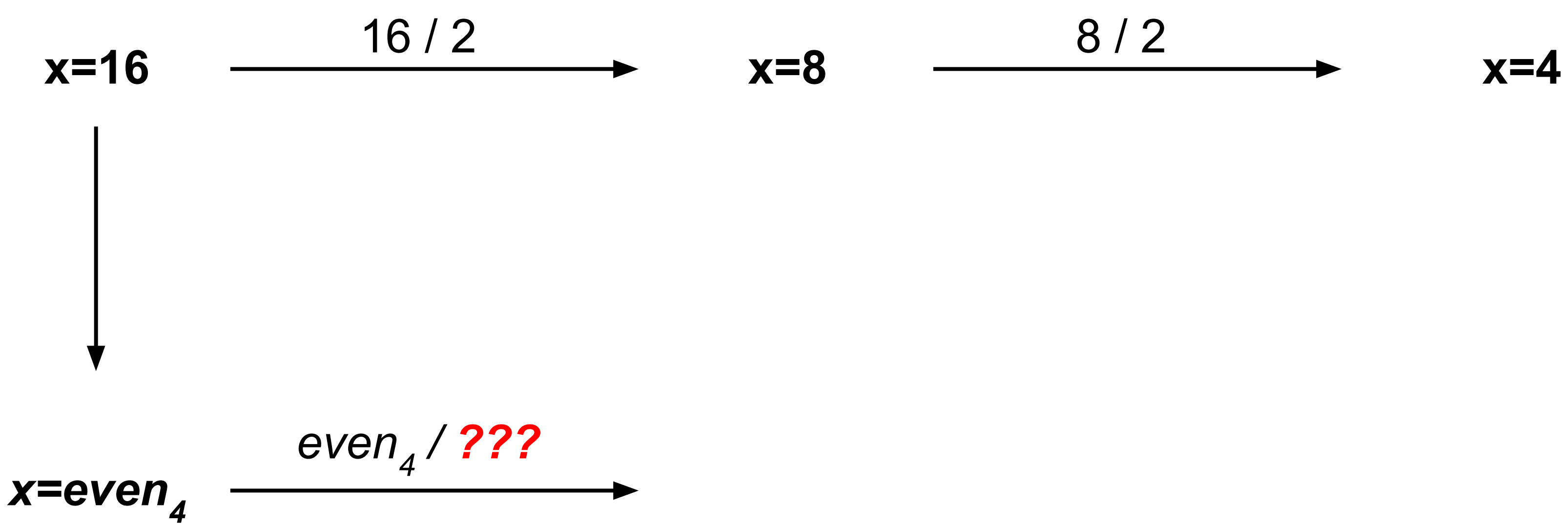
Abstract interpretation: soundness example

Abstract domain: {*odd, even$_2$, even$_4$, is2, unk*}



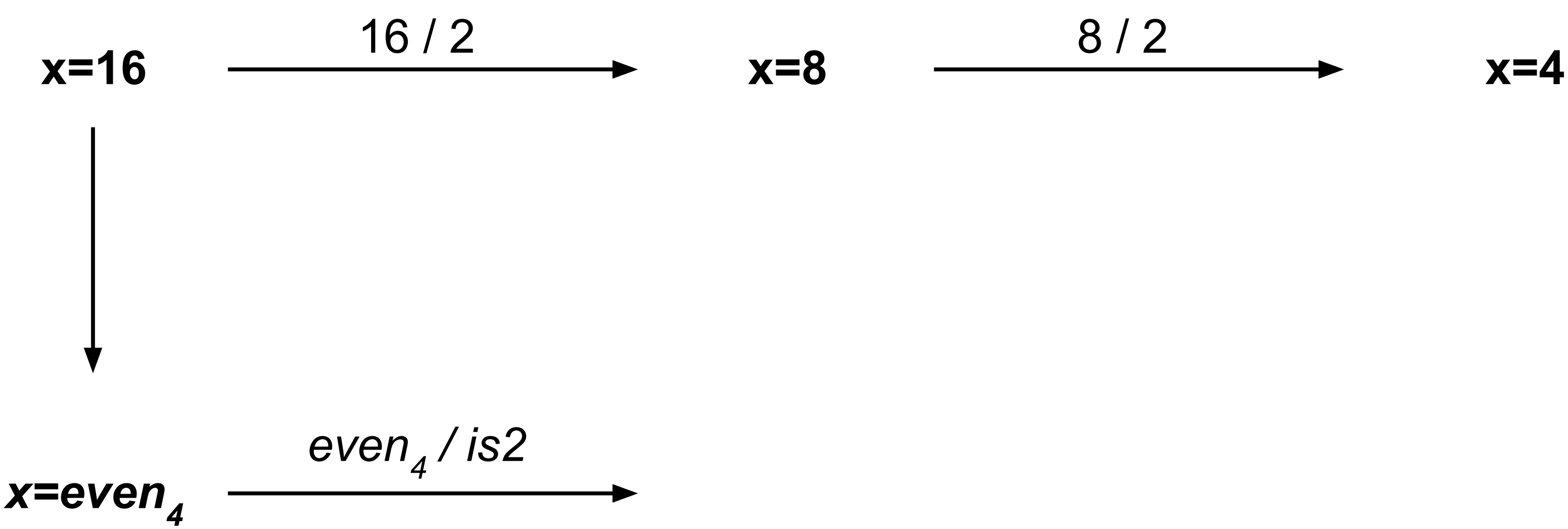
Abstract interpretation: soundness example

Abstract domain: {*odd, even₂, even₄, is2, unk*}



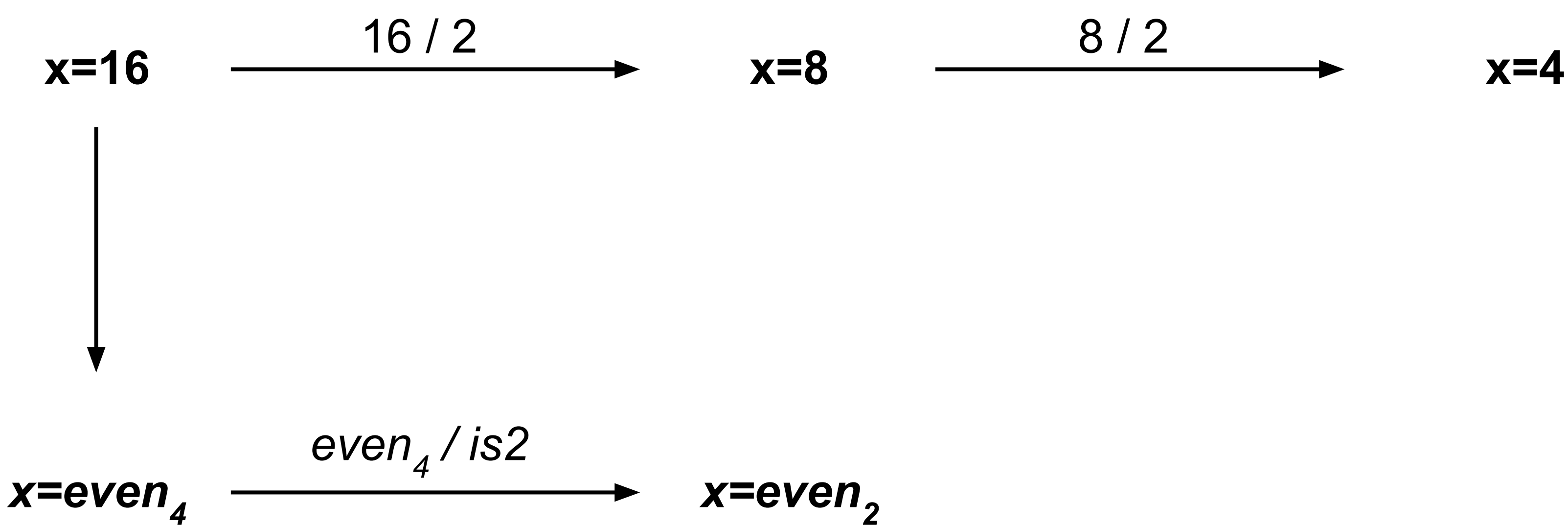
Abstract interpretation: soundness example

Abstract domain: {*odd, even*$_2$*, even*$_4$*, is2, unk*}



Abstract interpretation: soundness example

Abstract domain: {*odd*, *even*$_2$, *even*$_4$, *is2*, *unk*}



Abstract interpretation: soundness example

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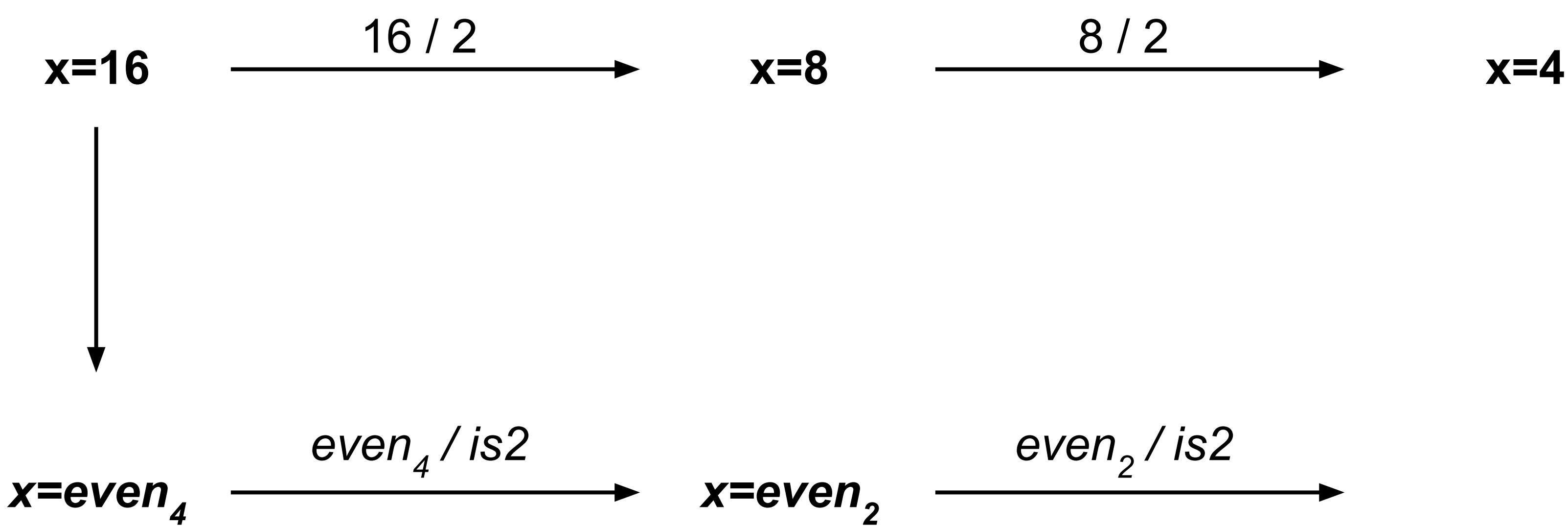

Abstract interpretation: soundness example

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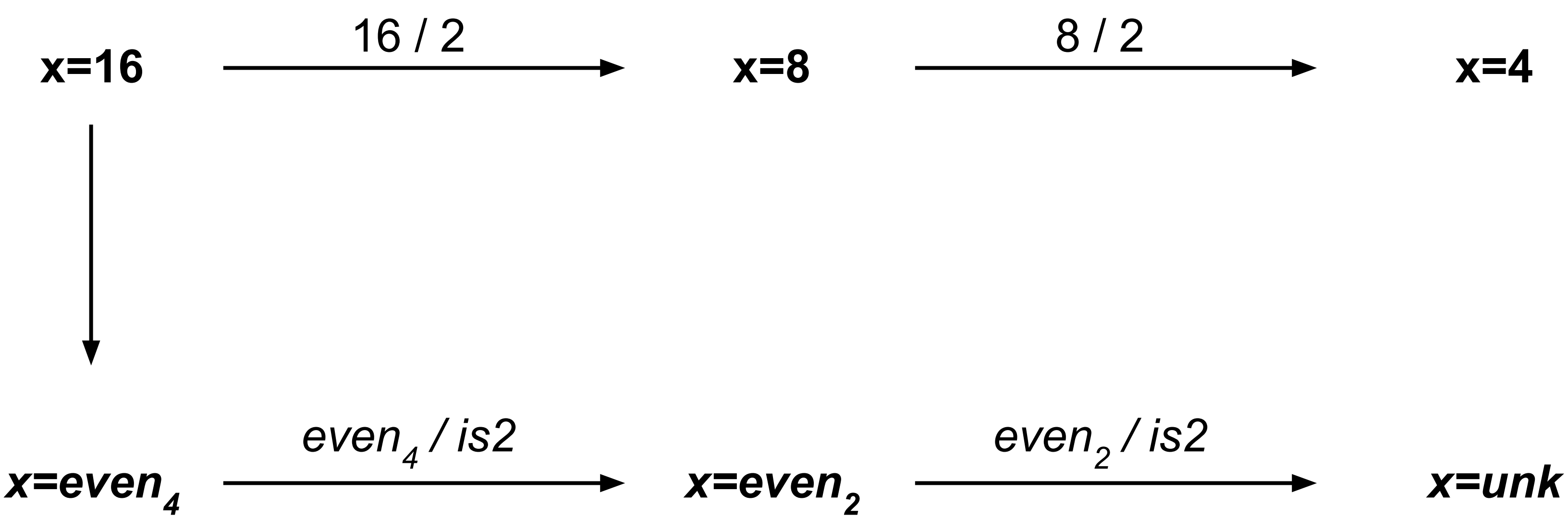

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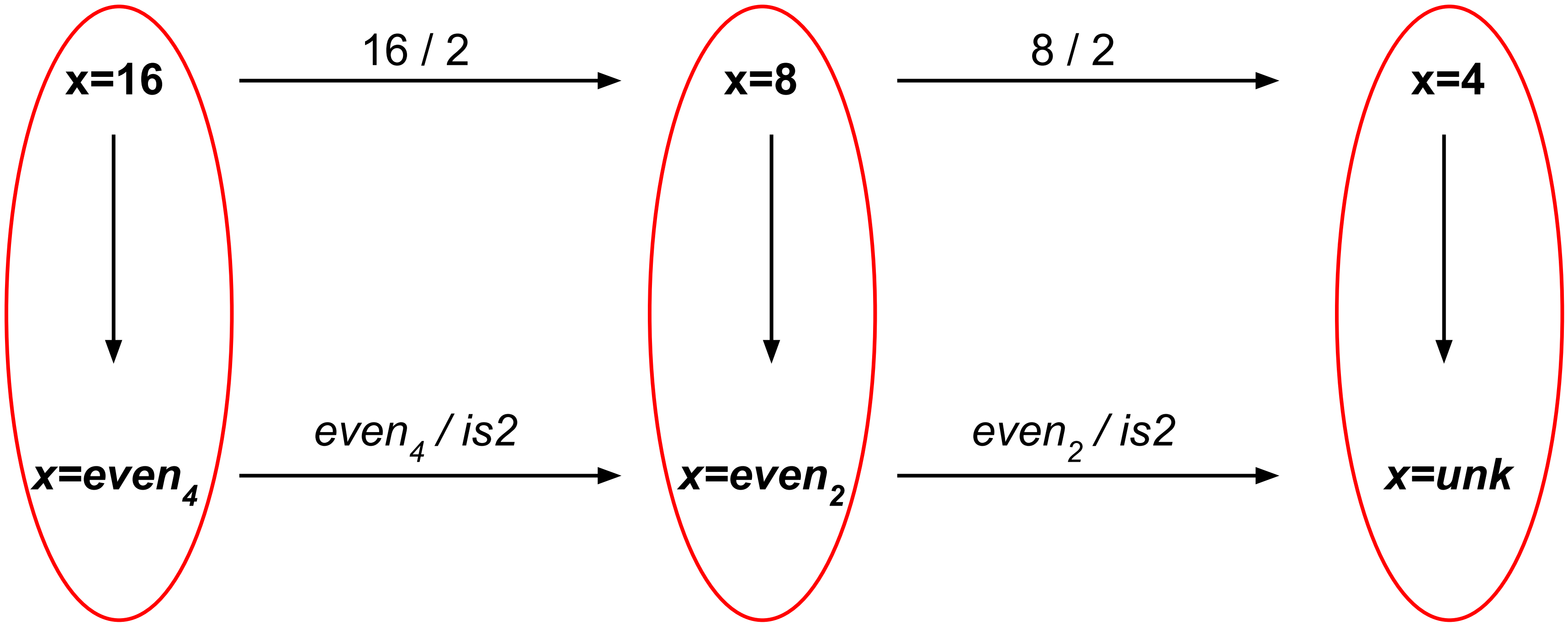
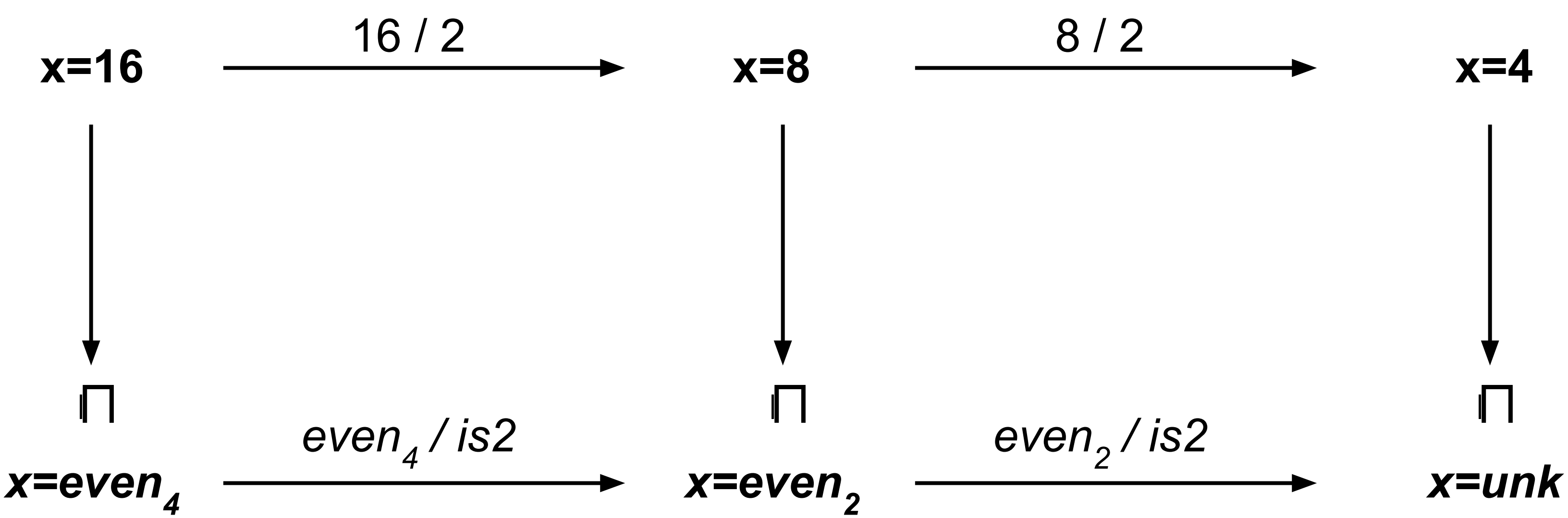
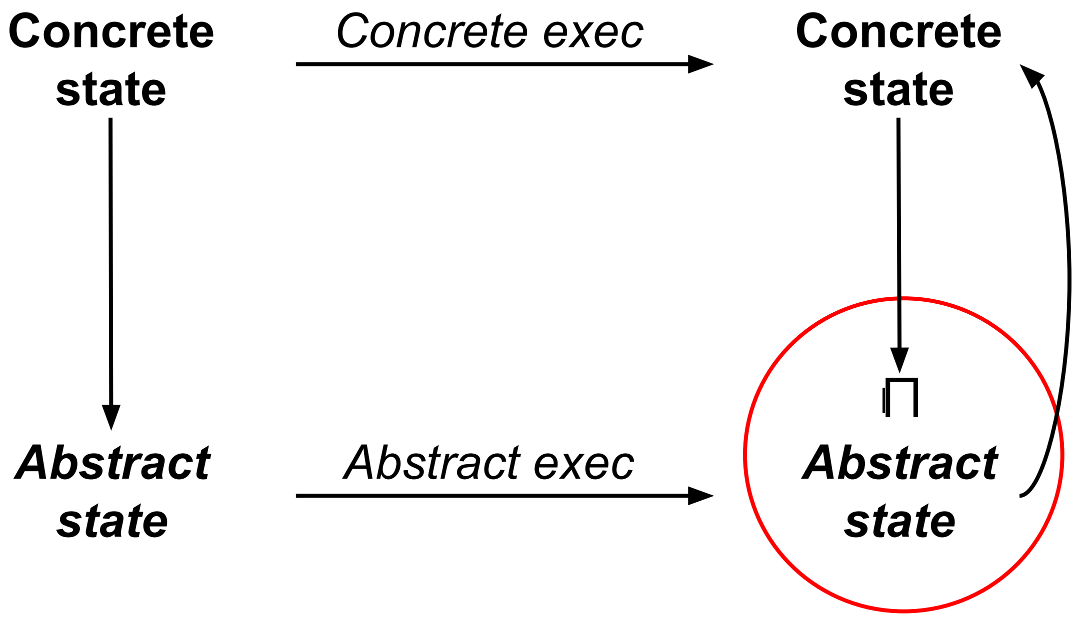


Abstract interpretation: soundness example

Abstract domain: {*odd, even$_2$, even$_4$, is2, unk*}

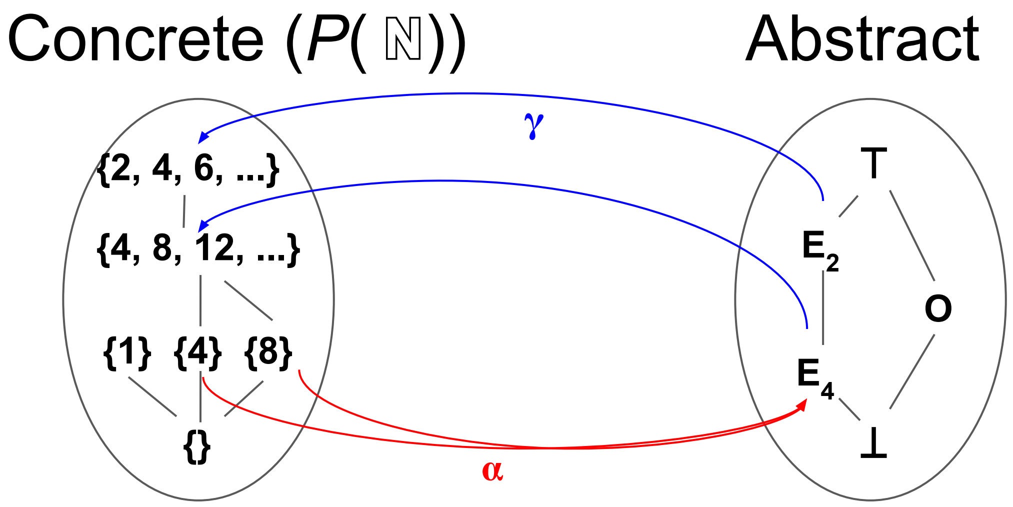


Abstract interpretation: soundness



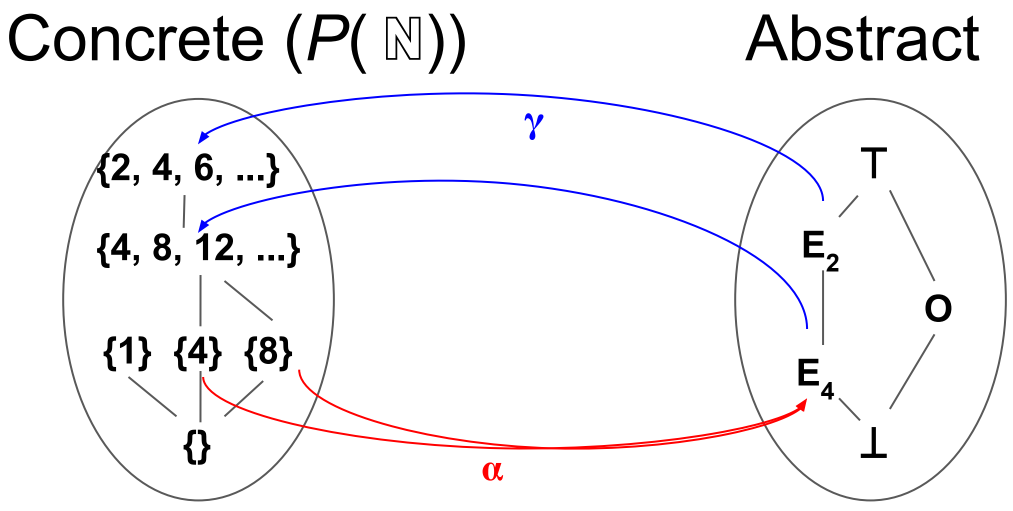
What properties must be satisfied by the abstraction, concretization, and transfer functions?

Sound approximation: properties



What properties must α and γ satisfy?

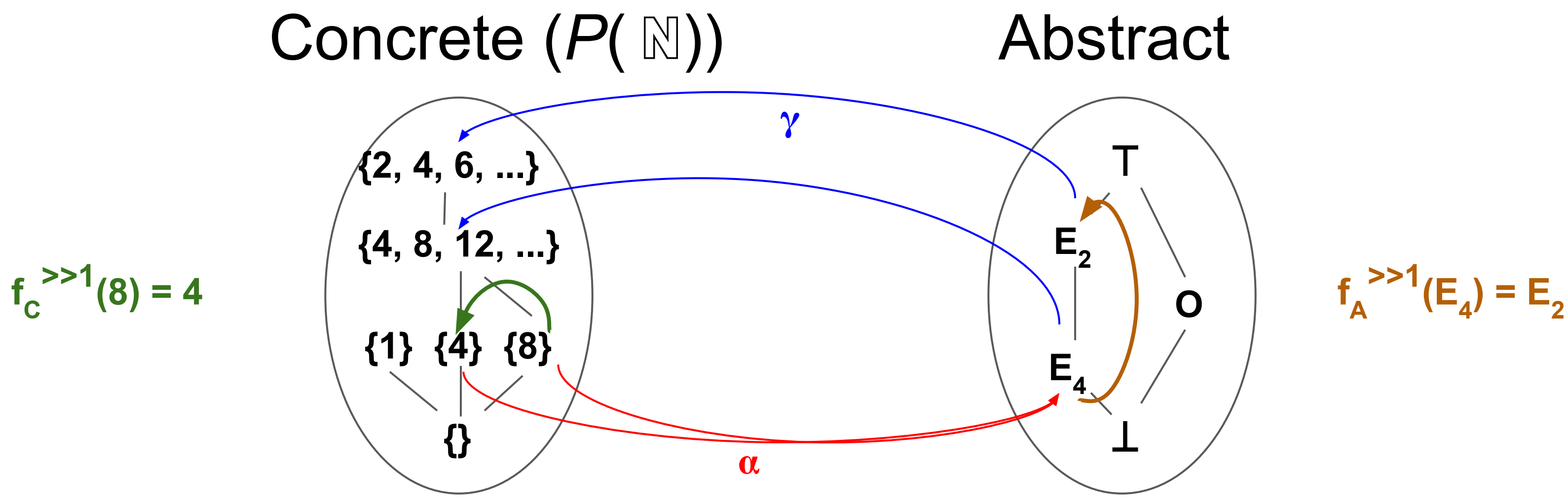
Sound approximation: galois connection



Galois connection

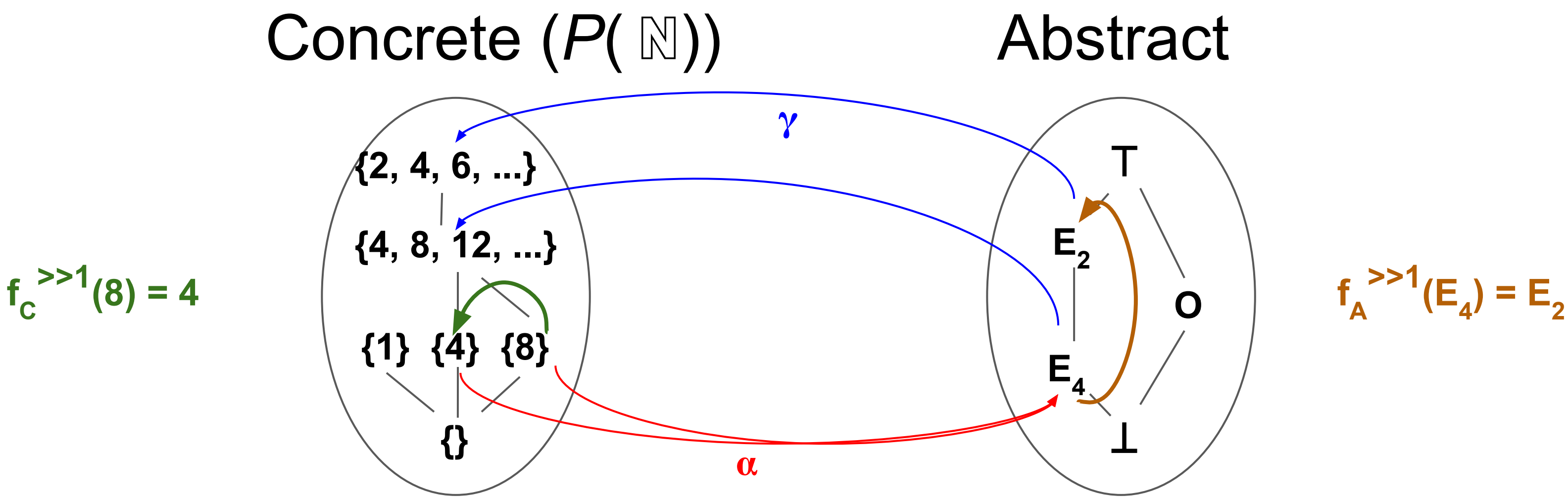
- α: *C* -> *A*
- γ: A -> C
- $\forall c \in C: c \leq \gamma(\alpha(c))$

Sound approximation: properties



What properties must the transfer function(s) satisfy?

Sound approximation: consistency



Transfer function

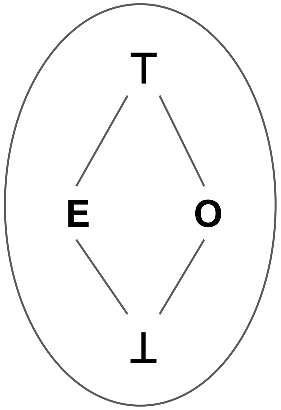
- Consistent with concrete execution
  - c: concrete state; $c' = f_C(c)$
  - a: $\alpha(c)$
  - $a' = f_A(a)$
  - $c'' = \gamma(a')$
  - $c' \leq c''$

Sound approximation: properties

Transfer function

- f_A^+: A x A -> A

+	E	O	T	...
E	E	O	T	
O	O	E	T	
T	T	T	T	
...				



Lub

- lub: A x A -> A

lub(E, O) = T

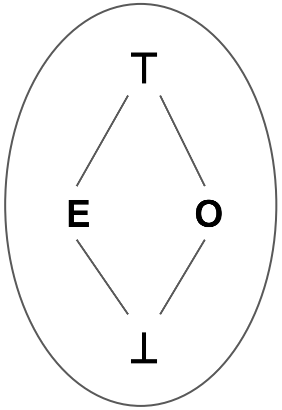
What properties must the lub function satisfy?

Sound approximation: monotonicity

Transfer function

- f_A^+: A x A -> A
- may not be monotone

+	E	O	T	...
E	E	O	T	
O	O	E	T	
T	T	T	T	
...				



Lub

- lub: A x A -> A
- must be monotone

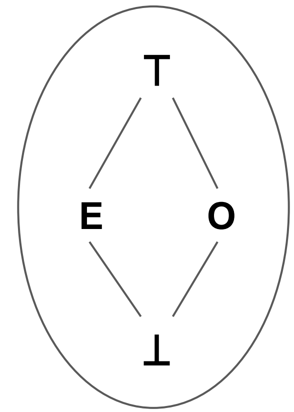
lub(E, O) = T

Sound approximation: join (lub) vs. meet (glb)

Transfer function

- f_A^+: A x A -> A
- may not be monotone

+	E	O	T	...
E	E	O	T	
O	O	E	T	
T	T	T	T	
...				



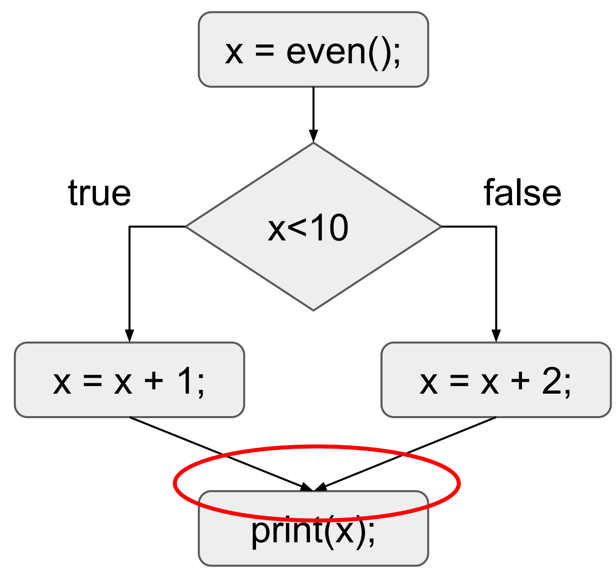
Lub

- lub: A x A -> A
- must be monotone

lub(E, O) = T

```
int x = even();

if (x < 10) {
  x = x + 1;
} else {
  x = x + 2;
}
print(x);
```



Small-group exercise

- Work through two examples:
  - Join vs. meet operation (`f(int a, int b, int c): int`)

```
if (cond) {
  x = a * b;
} else {
  x = a * c;
}
return(x);
```

Which parameters (`a, b, c`)
- will definitely be used?
- may be used?

(cond is independent of the parameters)

  - Termination/fix point iteration

```
int x = 2;
while (x < 10) {
  x = x + 2;
}
```

Is the value of x after the loop an even number? Use an abstract domain with {odd, 2, even$_2$, and even$_4$}

Small-group exercise

- Work through two examples:
  - Join vs. meet operation (`f(int a, int b, int c): int`)

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See Q&A write-up:
https://docs.google.com/document/d/1VEWmFIJVtD2F9ZkXIZ9xeOXGAtkRZATIX13wc1NYmtw

Checker Framework live demo