Moden Spectral Graph Theory **Winter 2022**

Lecture 3: Small Set Expansion Problem

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Disclaimer: *These notes have not been subjected to the usual scrutiny reserved for formal publications.*

In this lecture we introduce Cheeger's inequality and use it together with the spectral profile theorem from the previous lecture to design an approximation algorithm for the small set expansion problem on *high* threshold rank. The materials of this lecture is based on the work of Arora, Barak and Steurer [ABS15].

3.1 Cheeger's Inequality

Given a graph $G = (V, E)$ with a random walk matrix P. For a set $S \subseteq V$ define

$$\phi(S) = \frac{\mathcal{E}_P(\mathbf{1}_S, \mathbf{1}_S)}{\pi_0(S)} = \frac{\frac{1}{2}\pi_1(E(S, \overline{S}))}{\pi_0(S)} = \frac{\frac{1}{2}\mathbb{E}_{u\sim\pi_0}\mathbb{E}_{\{u,v\}|u}\mathbb{P}\left[|\{u, v\} \cap S| = 1\right]}{\pi_0(S)} = \mathbb{E}_{u\sim\pi_0(S)}\mathbb{P}_{\{u,v\}|u}\left[v \notin S\right]$$

where $E(S, \overline{S})$ is the set of edges in the cut $(S, \overline{S})$. In other words, $\phi(S)$ is the probability that a walk started at a vertex of S chosen with probability proportional $\pi_0(.)$ leaves S in one step.

Lemma 3.1 (Cheeger's Inequality)**.** *Given a graph $G = (V, E)$, a set $S \subseteq V$ with $\pi_0(S) \leq 1/2$. Then,*

$$\frac{1}{2} \min_{f:S\to\mathbb{R}_{\geq 0}} \frac{\mathcal{E}(f, f)}{\mathrm{Var}(f)} \leq \min_{T\subseteq S} \phi(T) \leq \min_{f:S\to\mathbb{R}_{\geq 0}} \sqrt{\frac{2\mathcal{E}(f, f)}{\mathrm{Var}(f)}}$$

Proof. First, we prove the left side. Fix a set $T \subseteq S$ whit minimum conductance. Let $f = \mathbf{1}_T$. Then, since $\pi_0(T) \leq \pi_0(S) \leq 1/2$,

$$\mathrm{Var}(\mathbf{1}_T) = \pi_0(T) - \pi_0(T)^2 \geq \pi_0(T)/2.$$

This proves the left inequality.

Next, we prove the harder direction. Fix a non-zero function $f : S \to \mathbb{R}_{\geq 0}$, we find a set $T \subseteq \mathrm{supp}(f)$ such that

$$\phi(T) \leq 2\sqrt{\frac{\mathcal{E}(f, f)}{\mathrm{Var}(f)}}$$

Perhaps after renormalization, assume $f \leq 1$. For a threshold $t \geq 0$, define $S_t = \{v : f^2(v) \geq t\}$. Choose a threshold $t \sim [0, 1]$ uniformly at random. Then,

$$\begin{aligned}
\mathbb{E}_t\pi_1(E(S_t, \overline{S_t})) &= \mathbb{E}_t\mathbb{E}_{\{u,v\}\sim\pi_1}\mathbb{P}\left[(f^2(u) < t \wedge f^2(v) > t) \vee (f^2(v) < t \wedge f^2(u) > t)\right] \\
&= \frac{1}{2}\mathbb{E}_{\{u,v\}\sim\pi_1}|f^2(u) - f^2(v)| \\
&\leq \frac{1}{2}\mathbb{E}_{\{u,v\}\sim\pi_1}|f(u) - f(v)| \cdot |f(u) + f(v)| \\
&\underset{Cauchy-Schwarz}{\leq} \sqrt{\frac{1}{2}\mathbb{E}_{\{u,v\}\sim\pi_1}(f(u) - f(v))^2} \cdot \sqrt{\frac{1}{2}\mathbb{E}_{\{u,v\}\sim\pi_1}(f(u) + f(v))^2} \\
&\leq \sqrt{\mathcal{E}(f, f)} \cdot \sqrt{\mathbb{E}_{\{u,v\}\sim\pi_1}f(u)^2 + f(v)^2}
\end{aligned}$$

Furthermore, notice

$$\mathbb{E}_{\{u,v\}\sim\pi_1} f(u)^2 + f(v)^2 = \mathbb{E}_{\{u,v\}\sim\pi_1}\mathbb{E}_{u|\{u,v\}} 2f(u)^2 = \mathbb{E}_{u\sim\pi_0}\mathbb{E}_{\{u,v\}|u} 2f(u)^2 = 2\mathbb{E}f^2.$$

On the other hand,

$$\mathbb{E}_t \pi_0(S_t) = \mathbb{E}_t \mathbb{E}_{u\sim\pi_0}\mathbb{P}\left[t < f(u)^2\right] = \mathbb{E}f(u)^2.$$

Putting these together there must exist a value of t, say t^* such that

$$\phi(S_{t^*}) \leq \frac{\mathbb{E}_t \pi_1(E(S_t, \overline{S_t}))}{\mathbb{E}_t \pi_0(S_t)} \leq \frac{\sqrt{2\mathcal{E}(f,f)}}{\sqrt{\mathbb{E}f^2}} \leq \sqrt{\frac{2\mathcal{E}(f,f)}{\mathrm{Var}(f)}}$$

where the last inequality uses that $\mathrm{Var}(f) = \mathbb{E}f^2 - (\mathbb{E}f)^2 \leq \mathbb{E}f^2$. $\square$

As an immediate consequence of the above lemma we can define a conductance/expansion profile for a graph G, where for any $0 \leq r < 1$,

$$\Phi(r) = \inf_{S:\pi_0(S)\leq r} \phi(S).$$

Then, putting the above lemma next to the improved mixing time bound we can prove an improved mixing time using the expansion profile, namely

$$8\log\epsilon^{-1} \sum_{t=1}^{\log \pi_0(u)^{-1}} \frac{1}{\Phi^2_{\tilde{P}}(2^{-t+2})}$$

3.2 Small Set Expansion Problem

Definition 3.2 (Small-Set Expansion Hypothesis, [RS10])**.** . *For every constant $\eta > 0$, there exists sufficiently small $\delta > 0$ such that given a graph G it is NP-hard to distinguish the following cases,*

Yes: *There exists a vertex set $S \subseteq V$ with $\pi_0(S) \leq \delta$ and conductance $\phi_0(S) \leq \eta$,*

No: *For every set $S \subseteq V$ with $\pi_0(S) \leq \delta$ satisfies $\phi(S) \geq 1 - \eta$.*

Raghavendra and Steurer in [RS10] established a reduction from the Small-Set Expansion problem to Unique Games. More precisely, their work showed that Small-Set Expansion Hypothesis implies the Unique Games Conjecture. This result suggests that the problem of approximating expansion of small sets lies at the combinatorial heart of the Unique Games problem. In words, if one wants to design an algorithm for the Unique Games it is better to start with the small set expansion problem.

In fact, this connection proved useful in the development of subexponential time algorithms for Unique Games by Arora, Barak and Steurer [ABS15]. We remark that it was also conjectured in [RS10] that Unique Games Conjecture is equivalent to the Small-Set Expansion Hypothesis but that is still left open.

3.3 Subexponential Time Algorithms for SSE Problem

For a graph $G = (V, E)$ with random walk matrix P and $0 < \eta < 1$ write

$$\mathrm{rank}_{1-\eta}(P) = |\{\lambda_i(P) : \lambda_i \geq 1 - \eta\}|$$

to denote the number of eigenvalues of P that are at least $1-\eta$ in absolute value. Recall that the first eigenvalue of P is 1, so this rank is at least 1 and if it is equal to 1 (for a constant η), then G is an expander graph. We say a graph G is low-threshold rank of $\text{rank}_{1-\eta}(P)$ is "small".

In the following lemma we prove that if G is a low-threshold rank graph (for a constant η say $\eta = 1/2$) then we can approximate the small set expansion within a constant factor (and in time exponential in $\text{rank}_{1-\eta}(P)$).

Putting this together with Theorem 3.4 gives a sub-exponential time algorithm for the small set expansion problem.

Lemma 3.3 (Subspace Enumeration Algorithm)**.** *Given a graph $G = (V, E)$, there is a $\exp(\text{rank}_{1-\eta}(P))$-time algorithm that given an $\epsilon > 0$, if G has a set S with $\phi(S) \le \epsilon$ outputs a set S' with $\pi_0(S\Delta S') \le 4\epsilon/\eta$. So, $\phi(S') \le O(\epsilon/\eta)$.*

Proof. The algorithm that we will discuss is called the subspace enumeration algorithm and it was first proposed by Kolla [Kol10]. Fix a set $S \subseteq V$ (with $\phi(S) \le \epsilon$). Let $U \subseteq \mathbb{R}^V$ be the span of eigenvectors of P with eigenvalue at least $1-\eta$ (in absolute value). So, $\dim(U) = \text{rank}_{1-\eta}(G)$. Note that $\left\|\frac{\mathbf{1}_S}{\sqrt{\pi_0(S)}}\right\| = 1$; we write $\overline{\mathbf{1}_S} := \frac{\mathbf{1}_S}{\sqrt{\pi_0(S)}}$. We write

$$\overline{\mathbf{1}_S} = \sqrt{1-\gamma}x + \sqrt{\gamma}x^{\perp}$$

for $x \in U$ and $x^{\perp}$ orthogonal to U and both are unit norm functions. It follows that

$$\begin{aligned}
\epsilon \ge \phi(S) = \frac{\mathcal{E}(\mathbf{1}_S, \mathbf{1}_S)}{\pi_0(S)} &= \frac{\langle (I-P)\mathbf{1}_S, \mathbf{1}_S\rangle}{\pi_0(S)} \\
&= 1 - \langle P\overline{\mathbf{1}_S}, \overline{\mathbf{1}_S}\rangle \\
&= 1 - (1-\gamma)\langle Px, x\rangle - \gamma\langle Px^{\perp}, x^{\perp}\rangle \\
&\underset{\langle Px,x\rangle \le 1}{\ge} 1 - (1-\gamma) - \gamma(1-\eta) = \gamma\eta.
\end{aligned}$$

In the third equality we used that $Px \in U$ and thus orthogonal to $x^{\perp}$ and, similarly, $Px^{\perp} \in U^{\perp}$. Therefore,

$$\left\|\overline{\mathbf{1}_S} - x\right\|^2 = \left\|\overline{\mathbf{1}_S}\right\|^2 + \|x\|^2 - 2\langle \overline{\mathbf{1}_S}, x\rangle = 2 - 2\sqrt{1-\gamma} \approx \gamma \le \epsilon/\eta.$$

Now, we search for the x vector by running a $\sqrt{\epsilon/\eta}$ net in the unit sphere of dimension $\text{rank}_{1-\eta}(P) = \dim(U)$ in U. Note that the number of points in this net is no more than $\sqrt{\epsilon/\eta}^{\dim(U)} = \exp(\text{rank}_{1-\eta}(P)\log(\eta\epsilon^{-1}))$. To run such an exhaustive search we can start with all points in the $\sqrt{\epsilon/\eta}^{\dim(U)}$ grid around the origin and project each of its points to the unit sphere. So, we find a vector y such that $\left\|\overline{\mathbf{1}_S} - y\right\|^2 \le 2\epsilon/\eta$.

Now, we construct a set S' based on the vector y.

$$S' := \left\{ v : y_v \ge \frac{1}{2\sqrt{\pi_0(S)}} \right\}$$

Observe that for any $v \in S\Delta S'$ we have $|\overline{\mathbf{1}_S} - y|_v \ge \frac{1}{2\sqrt{\pi_0(S)}}$. This is simply because every coordinate of $\overline{\mathbf{1}_S}$ is either zero or $1/\sqrt{\pi_0(S)}$.

Therefore,

$$2\epsilon/\eta \ge \left\|\overline{\mathbf{1}_S} - y\right\|^2 \ge \frac{\pi_0(S\Delta S')}{4\pi_0(S)}.$$

So, $\frac{\pi_0(S\Delta S')}{\pi_0(S)} \leq 4\epsilon/\eta$. So,

$$\phi(S') = \frac{\mathcal{E}(\mathbf{1}_{S'}, \mathbf{1}_{S'})}{\pi_0(S')} \leq \frac{\mathcal{E}(\mathbf{1}_S, \mathbf{1}_S) + \pi_0(S\Delta S')}{\pi_0(S)(1-4\epsilon/\eta)} \leq \epsilon(1+4\epsilon/\eta) + 5\epsilon/\eta$$

This finishes the proof of the lemma. □

Theorem 3.4. *Let $G=(V,E)$ be a graph with (at most) n vertices such that $\mathrm{rank}_{1-\eta}(P) \geq n^{2\eta/\gamma}$. Then, there is a polynomial time algorithm that finds a set S with $\pi_0(S) \leq 4n^{-\eta/\gamma}$ and $\phi(S) \leq 4\sqrt{\gamma}$.*

Proof. Recall that $\tilde{P} = (I+P)/2$ is the lazy-random walk operator. Let $f_v = \mathbf{1}_v/\pi_0(v)$. Then, for any $t \geq 0$

$$\mathbb{E}_{v\sim\pi_0}\left\|\tilde{P}^t f_v\right\|^2 = \mathbb{E}_{v\sim\pi_0}\mathbb{E}_{u\sim\pi_0}\tilde{P}^{2t}f_v(u)f_v(u) = \mathbb{E}_{v\sim\pi}\pi(v)\frac{\tilde{P}^{2t}(v,v)}{\pi(v)}\cdot\frac{1}{\pi(v)} = \sum_v \tilde{P}^{2t}(v,v) = \mathrm{Tr}(\tilde{P}^{2t})$$

So, for $t = \frac{\log n}{\gamma}$

$$\mathbb{E}_{v\sim\pi_0}\left\|\tilde{P}^t f_v\right\|^2 = \mathrm{Tr}(\tilde{P}^{2t}) \geq \mathrm{rank}_{1-\eta}(P)(1-\eta/2)^{2t} \geq n^{2\eta/\gamma}\cdot n^{-\eta/\gamma} = n^{\eta/\gamma}.$$

On the other hand, notice

$$\mathbb{E}_{v\sim\pi_0}\|f_v\|^2 = \mathbb{E}_{v\sim\pi_0}\frac{1}{\pi_0(v)} = n.$$

So, there must be a vertex u such that $\frac{\|\tilde{P}^t f_u\|^2}{\|f_u\|^2} \geq \frac{\mathbb{E}_{v\sim\pi_0}\|\tilde{P}^t f_v\|^2}{\mathbb{E}_{v\sim\pi_0}\|f_v\|^2} = \frac{n^{\eta/\gamma}}{n}$. Fix such vertex u. So we have,

$$\frac{\mathrm{Var}(\tilde{P}^t f_u)}{\mathrm{Var}(f_u)} = \frac{\left\|\tilde{P}^t f_u\right\|^2 - (\mathbb{E}\tilde{P}^t f_u)^2}{\frac{1}{\pi_0(u)} - 1} \approx \frac{n^{\eta/\gamma}}{n}.$$

We just ignore the -1 in the numerator and denominator. Therefore, by the proof of Theorem 2.7 (in lecture 2) for some $t^* \leq t$ we have $g = \tilde{P}^{t^*} f_u$ satisfies

$$\gamma \geq \frac{\mathrm{Var}(g) - \mathrm{Var}(\tilde{P}g)}{\mathrm{Var}(g)} = \frac{\mathcal{E}_{\tilde{P}^2}(g,g)}{\mathrm{Var}(g)} \underset{Corollary 2.8}{\geq} \frac{\mathcal{E}_{\tilde{P}}(g,g)}{\mathrm{Var}(g)}$$

This is because if for every $i \leq t$, we have $\frac{\mathrm{Var}(\tilde{P}^i f_u) - \mathrm{Var}(\tilde{P}\tilde{P}^i f_u)}{\mathrm{Var}(\tilde{P}^i f_u)} \geq \gamma$ then, by time $t = \frac{\log n}{\gamma}$ we should have $\mathrm{Var}(\tilde{P}^t f_u) \leq \mathrm{Var}(f_u)/n$. Note that since variance is decreasing, $\mathrm{Var}(g) \geq \mathrm{Var}(\tilde{P}^t f_u)$.

Therefore, by Lemma 2.6, there is a function $h : \mathrm{supp}(g) \to \mathbb{R}_{\geq 0}$ that is a level set of g such that $\frac{\mathcal{E}_{\tilde{P}}(h,g)}{\mathrm{Var}(h)} \leq 2\frac{\mathcal{E}_{\tilde{P}}(g,g)}{\mathrm{Var}(g)} \leq 2\gamma$ and

$$\pi(\mathrm{supp}(h)) \leq \frac{4(\mathbb{E}g)^2}{\mathrm{Var}(g)} \leq \frac{4(\mathbb{E}P^{t^*}f_u)^2}{\mathrm{Var}(P^t f_u)} \leq \frac{4}{n^{\eta/\gamma}-1}$$

Therefore, applying Cheeger's inequality 3.1 to h and matrix $\tilde{P}$ we get a set S such that

$$2\sqrt{\gamma} \geq \sqrt{2\frac{\mathcal{E}_{\tilde{P}}(h,h)}{\mathrm{Var}(h)}} \geq \phi_{\tilde{P}}(S) = \frac{\mathcal{E}_{\tilde{P}}(\mathbf{1}_S,\mathbf{1}_S)}{\pi(S)} = \frac{\frac{1}{2}\mathcal{E}_P(\mathbf{1}_S,\mathbf{1}_S)}{\pi(S)} = \frac{1}{2}\phi_P(S)$$

where we used that $I - \tilde{P} = I - (I+P)/2 = \frac{1}{2}(I-P)$. It follows that $\phi_P(S) \leq 4\sqrt{\gamma}$. Furthermore, $\pi(S) \leq \pi(\mathrm{supp}(h)) \leq 4n^{-\eta/\gamma}$. □

Corollary 3.5. *Let G be an n vertex graph, and $\epsilon > 0$. If $\text{rank}_{1-\epsilon^5}(G) \geq n^{2\epsilon}$ then we can find in polynomial time a set $S \subseteq V(G)$ with $\pi_0(S) \leq n^{-\epsilon}$ and $\phi(S) \leq O(\epsilon^2)$*

Proof. Instantiate Theorem 3.4 with $\eta = \epsilon^5, \gamma = \epsilon^4$. □

We remark that after the work of [ABS15], several papers studied higher order variants of Cheeger's inequality and they managed to prove the following theorems:

Theorem 3.6 (Higher Order Cheeger's Inequality, [LOT14; LRTV12])**.** *For any graph G and any $2 \leq k \leq n$ if $\text{rank}_{1-\epsilon}(G) \geq k$ then G all of the following holds true:*

- *G has a set S with $\pi_0(S) \leq 1/k$ and*
$$\phi(S) \leq O(k^2)\sqrt{\epsilon},$$
- *G has a set S with $\phi_0(S) \leq 2/k$ and*
$$\phi(S) \leq O(\log k)\sqrt{\epsilon}$$
- *If G is planar then it has a set with $\phi_0(S) \leq 2/k$ and*
$$\phi(S) \leq O(\sqrt{\epsilon}).$$

Furthermore, such a set can be found in polynomial time.

Although above theorem have found many applications in TCS it didn't lead to a resolution of the SSE hypothesis. This is mainly due to the fact that, unless G is planar or low dimensional, there is a loss of $O(\log k)$ in the conductance of the small set promised by theorem.

Theorem 3.7 (Low trace threshold rank decomposition theorem)**.** *Let G be a regular graph. There is a polynomial time algorithm that on input a graph G and $\epsilon > 0$, outputs a partition $\chi = (P_1, \ldots, P_q)$ of $V(G)$ such that*
$$\phi(\chi) := \mathbb{E}_{\{u,v\}\sim\pi_1}\mathbb{I}\left[\chi^{-1}(u) \neq \chi^{-1}(v)\right] \leq O(\epsilon)$$
and for every $i \in [q]$, $\text{rank}_{1-\epsilon^5}(G[P_i]) \leq n^{2\epsilon}$.

Here we sketch the proof of the theorem: We start by making the graph 1/2-lazy. The idea is to use Corollary 3.3 repeatedly. We start with a single partition V; each time we choose a set $P_j \in \chi$ such that $\text{rank}_{1-\epsilon^5}(G[P_j]) > n^{2\epsilon}$. Then, we use Corollary 3.3 to decompose P_j into two sets $S, P_j - S$ with $\pi_0(S) \leq n^{-\epsilon}\pi_0(P_j)$. Note that, because the graph is 1/2-lazy, the stationary distribution of the walk on $G[P_j]$ is almost the same as π_0. So, we don't need to worry about the updated stationary measure. Then, we substitute P_j with $S, P_j - S$ in χ and we charge any new edge added to the partition the smaller side, i.e., S.

It is not hard to see that such an algorithm always terminates. Furthermore, every edge is part of a "split" at most $O(\epsilon^{-1})$ times. This is because the measure of the smaller side always shrinks by a factor of $n^{-\epsilon}$. So after $O(\epsilon^{-1})$ many steps we get to a set P_j with measure $\pi_0(P_j) \leq n^{-(1-\epsilon)}$. Since G is regular, π_0 is uniform, so $\text{rank}_{1-\eta}(G[P_j]) \leq |P_j| \leq n^{3\epsilon}$.

3.4 Subexponential Time algorithms for Unique Games Problem

A unique game of n variables and alphabet k is an n vertex graph G whose edges are labeled with permutations on the set $[k]$, where every edge $e = (u, v)$ (arbitrarily oriented) is labeled with a permutation $\pi_e : [k] \to [k]$. An assignment to the game is a string $y = (y_1, \ldots, y_n) \in [k]^n$, and the value of y is the fraction of edges $e = (u, v)$ for which $y_v = \pi_e(y_u)$. The value of the game G is the maximum value of y over all $y \in [k]^n$. Khot's Unique Games Conjecture [Kho02] states that for every $\epsilon > 0$, there is an integer $k > 0$, such that given a graph $G = (V, E)$ and permutations π_e for every $e \in E$ it is NP-hard to distinguish between the following cases:

Yes: Game has value at least $1 - \epsilon$, and

No: Game has value at most ϵ.

In this section we now show that this problem can be solved in subexponential time:

Theorem 3.8 (Subexponential Algorithm for Unique Games)**.** *There is an* $\exp(kn^{O(\epsilon)})$ *poly*(n)*-time algorithm that on input a unique game G on n vertices and alphabet size k that has an assignment satisfying* $1 - \epsilon^6$ *of its constraints outputs an assignment satisfying* $1 - O(\epsilon \log \epsilon^{-1})$ *of the constraints.*

We now sketch the proof of this theorem. First, it turns out that, without loss of generality, we can assume that the unique game constraint graph G is d-regular for some (constant) d. If not, one can blow up every vertex v with degree d_v to a cloud of d_v vertices each being connected to one of the neighbors of v. Then, we connect this cloud of vertices with a constant degree expander with a trivial equality bijection on the edges of the expander.

So, from now on, assume G is d-regular. For a unique game $G = (V, E)$, the *label extended graph* of G, denoted $\hat{G}$, is a graph with nk vertices, where for an edge $(u, v) \in E$ and $i, j \in [k]$, we place an edge between (u, i) and (u, j) in $\hat{G}$ if and only if $\pi_e(i) = j$. That is, every vertex $v \in V$ corresponds to the "cloud" $C_v := \{(v, 1), \ldots, (v, k)\}$ in $V(\hat{G})$.

We say that $S \subseteq V(\hat{G})$ is *conflict free* if S intersects each cloud in at most one vertex, i.e., we assign at most one label to each variable. Such a set corresponds to a partial assignment $f = f_S$ for the game G (i.e., a partial function from $V(G)$ to $[k]$). We define the value of a partial assignment f , denoted val(f), to be $2/nd$ times the number of edges $e = (u, v) \in E(G)$ such that both $f(u)$ and $f(v)$ are defined, and $\pi_e(f(i)) = f(j)$. Notice that if an assignment f for the unique game has val$(f) = 1 - \gamma$, then the corresponding conflict-free set $S \subseteq V(\hat{G})$ satisfies $\pi_0(S) = 1/k$ and $\phi(S) \leq 2\gamma$. Thus S is a small set with low expansion.

The main idea behind the algorithm is the fact that the subspace enumeration algorithm Lemma 3.3 discovers (up to some error) *every* subset of $\hat{G}$ of small conducance. Thus, so long as the label-extended graph has low threshold rank, we can find any almost-satisfying assignment (if one exists) in time exponential in this rank by enumerating all nonexpanding subsets and checking if any of them correspond to a (partial) assignment that satisfies almost all the constraints.

Of course, $\hat{G}$ does not necessarily have a low threshold rank, but we can use Theorem 3.7 to partition the constraint graph G into low-threshold rank subsets $\chi = (P_1, \ldots, P_r)$ while only cutting $O(\epsilon)$-fraction of edges. Then, we compute the label-extended graph of each part, $\hat{G}[P_i]$. It is not hard to see that if a graph H has a low threshold rank so does its label extended version (up to an extra loss of factor k). So, we can apply the subspace enumeration algorithm to each $\hat{G}[P_i]$ to find all small set of small conductance. We just pay extra loss of $O(\epsilon)$ due to conductance of the set that we find, and the fraction of missing edges between parts $P_1, \ldots, P_r$.

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