Disclaimer: These notes have not been subjected to the usual scrutiny reserved for formal publications.

In this note we study random walks and mixing time in weighted undirected non-regular graphs $G$. The notation that we define here will be used later to study high dimensional random walks on simplicial complexes.

Given an undirected (weighted) graph $G = (V, E)$ with weight function $w : E \to \mathbb{R}_{\geq 0}$, let $A \in \mathbb{R}^{V \times V}_{\geq 0}$ be the adjacency matrix of $G$.

Consider a natural probability distribution on edges of $G$ where for any edge $\{u, v\}$

$$\pi_1(\{u, v\}) = \frac{w(\{u, v\})}{\sum_{e} w(e)}.$$

This distribution naturally induces a distribution on vertices; given a random edge $e$, we randomly drop one of the endpoints of $e$:

$$\pi_0(v) = \sum_e P_{\pi_1}[e] P[v|e] = \sum_{e: v \in e} \frac{1}{2} \pi_1(e) = \frac{\sum_{u \sim v} \frac{1}{2} w(\{u, v\})}{\sum_{e} w(e)} = \frac{d_w(v)}{\sum_u d_w(u)}.$$

Conversely, suppose we want to sample an edge $e = \{u, v\} \sim \pi_1$. One way is to first sample a vertex $v \sim \pi_0$ and then sample an edge $e|v$, i.e., among all edge incident to $v$ choose one proportional to its weight.

$$P[\{u, v\}|u] := \frac{\pi_1(\{u, v\})}{\sum_{x \sim u} \pi_1(\{u, x\})}$$

We claim this process chooses $e$ with probability $\pi_1(e)$, because,

$$P[e] = P_{\pi_0}[v] P[e|v] + P_{\pi_0}[u] P[e|u] = \pi_0(v) \frac{\pi_1(e) \sum_{f \sim v} \pi_1(f)}{\sum_{f \sim v} \pi_1(f)} + \pi_0(u) \frac{\pi_1(e) \sum_{f \sim u} \pi_1(f)}{\sum_{f \sim u} \pi_1(f)} + \pi_0(v) \frac{w(e) \sum_{f \sim v} w(f)}{\sum_{f \sim v} w(f)} + \pi_0(u) \frac{w(e) \sum_{f \sim u} w(f)}{\sum_{f \sim u} w(f)} = \pi_0(v) \frac{w(e) d_w(v)}{d_w(v)} + \pi_0(u) \frac{w(e) d_w(u)}{d_w(u)} = \frac{\pi(e)}{2} + \frac{\pi(e)}{2} = \pi(e).$$

**Fact 1.1.** Summarizing the above observations,

- To choose $v \sim \pi_0$, we can first choose $e \sim \pi_1$ and then drop one endpoint.
- To choose $e \sim \pi_1$, we can first choose $v \sim \pi_0$ and then choose $e|v$. 

1-1
1.1 Random Walk Operator

We can look at the simple random walk operator on $G$ that says if I am at vertex $u$, I choose an edge incident to $u$ with probability proportional to its weight. Let

$$P[u \to v] = P[\{u, v\} | u].$$

For a function $f : V \to \mathbb{R}$, we have

$$Pf(v) := \mathbb{E}_{(u,v) \sim \pi} [f(u)] = \sum_u P[v \to u] f(u).$$

The following reversibility property will also be used crucially.

$$P(u,v)\pi_0(u) = \pi_0(u)P[\{u, v\} | u] = \pi_1(\{u, v\}) \mathbb{P}[u|\{u, v\}] = \pi_1(\{u, v\}) \mathbb{P}[v|\{u, v\}] = P(v,u)\pi_0(v) \quad (1.1)$$

We equip the linear space $\mathbb{R}^V$ with the following inner product: For two vectors $f,g : V \to \mathbb{R}$,

$$\langle f, g \rangle = \mathbb{E}_{u \sim \pi_0} f(u)g(v) = \sum_v \pi_0(v)f(v)g(v).$$

This naturally defines a norm, where for any such function $f$, $\|f\| = \sqrt{\langle f, f \rangle}$.

Using the above fact, it follows that $P$ is self-adjoint with respect to the above inner product, i.e., for any two $f,g : V \to \mathbb{R}$,

$$\langle f, Pg \rangle = \mathbb{E}_{u \sim \pi_0} [f(v)Pg(v)] = \mathbb{E}_{u \sim \pi_0} [f(v)\mathbb{E}_{u,v} | v | g(u)] = \mathbb{E}_{u,v} | v | \mathbb{E}_{u,v} f(v)g(u) = \mathbb{E}_{u,v} | v | \mathbb{E}_{u,v} f(u)g(v)$$

Similarly, we can show that $\langle Pf, g \rangle$ is also equal to the RHS.

**Fact 1.2.** $\lambda_1 = 1$ and $\lambda_n \geq -1$.

**Proof.** First, observe that the all-ones function $\mathbf{1}$ is an eigenfunction,

$$P\mathbf{1} = \mathbf{1}.$$

Second, we show $\lambda_i \leq 1$ for all $i$. For any eigenfunction $f : V \to \mathbb{R}$, with eigenvalue $\lambda$, i.e., $Pf = \lambda f$, we claim that $\lambda \leq 1$. Say $u = \arg\max_v |f(v)|$. Then,

$$\lambda f(u) = Pf(u) = \mathbb{E}_{u,v} | u | f(v) \leq \mathbb{E}_{u,v} | u | f(v) | \leq \mathbb{E}_{u,v} | u | f(u) = |f(u)|.$$

In the second inequality we used $|f(v)| \leq |f(u)|$ for all $v \in V$. So, we have $\lambda \leq 1$ as desired. Also, observe from the same inequality that $|\lambda| \leq 1$ as desired.

**Fact 1.3.** The matrix $P$ has $n$-real eigenvalues with $n$-orthonormal eigefunctions.

**Proof.** The proof is fairly general and holds for any self-adjoint matrix with respect to an inner-product. First, Suppose $Pf = \lambda f$ for $f \in \mathbb{C}^n$ and $\lambda \in \mathbb{C}$. And, recall for any function $f$, $\langle f, f \rangle = \mathbb{E}_u f(u)\overline{f(u)} \in \mathbb{R}$. Then,

$$\|Pf\|^2 = \langle Pf, Pf \rangle = \langle P^2 f, f \rangle = \lambda^2 \|f\|^2.$$

So, $\lambda^2 \in \mathbb{R}$.

On the other hand, suppose $P$ has two eigenfunctions $f, g \in \mathbb{R}^n$ with eigenvalues $\lambda_f \leq \lambda_g$; then,

$$\langle \lambda_f f, g \rangle = \langle Pf, g \rangle = \langle f, Pg \rangle = \langle f, \lambda_g g \rangle.$$ 

Therefore, since $\lambda_f \neq \lambda_g$ we must have $\langle f, g \rangle = 0$.

It follows by the spectral theorem that $P$ has $n = |V|$ eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$ with corresponding orthonormal eigenfunctions $f_1, \ldots, f_n$ such that for any $1 \leq i < j \leq n$,

$$\langle f_i, f_j \rangle = 0,$$

and for any $i$, $\|f_i\| = 1$.

### 1.2 Mixing

**Theorem 1.4.** For a weighted graph with eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$, if $\max\{|\lambda_2|, |\lambda_n|\} < 1$, then for any vertex $u$,

$$\lim_{t \to \infty} P_t f(u) = Ef.$$

Letting $f(v) = 1_v$, we have $Ef = \pi_0(v)$. So, the above theorem shows that as $t \to \infty$, the distribution of the walk converges to $\pi_0$. In other words, $\pi_0$ is the stationary distribution of the simple random walk on $G$.

The mixing time of a random walk is defined as

$$\tau_{\text{mix}} = \min \left\{ t : \|P^t(u, \cdot) - \pi_0\|_{\text{TV}} = \frac{1}{2} \sum_v |P^t 1_v(u) - \pi_0(v)| \leq \frac{1}{4}, \forall u \in V \right\},$$

(1.2)

where $\tau_{\text{mix}}$ is the first time that the total variation distance of the distribution of the walk started at $u$, from the stationary distribution is at most a constant.

For a function $f : V \to \mathbb{R}$ define

$$\|f\|_p = (E \pi |f|^p)^{1/p}.$$

More generally, we define the $L_p$ mixing time of the walk as

$$\tau_{\text{mix},p} := \min \left\{ t : \left\| \frac{P^t(u, \cdot)}{\pi} - 1 \right\|_p \leq \frac{1}{4}, \forall u \in V \right\}.$$

Note that by Cauchy-Schwarz inequality

$$\sum_v |P^t(u, v) - \pi(v)| = E \pi |P^t(u, \cdot)/\pi - 1| \leq \sqrt{E \pi |P^t(u, \cdot)/\pi - 1|^2} = \left\| \frac{P^t(u, \cdot)}{\pi} - 1 \right\|_2.$$

So, the $L_2$-mixing of a chain implies $L_1$ mixing (but not vice versa). And similarly, for any $p > 1$, $L_p$ mixing implies $L_1$ mixing.

**Theorem 1.5.** Let $G$ be a weighted graph with eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$, such that $\lambda^* := \max\{|\lambda_2|, |\lambda_n|\} < 1$. For any $\epsilon > 0$, any function $f \in V \to \mathbb{R}$ and $t \geq \frac{\log \epsilon}{1 - \lambda^*}$

$$\|P^t f - Ef\| \leq \epsilon \|f\|.$$
Proof. Let $f_1, \ldots, f_n$ be the orthonormal eigenfunctions of $P$ with corresponding eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$. Recall $\lambda_1 = 1$ and $f_1 = 1$. Also note that $P^t$ have exactly the same eigenfunctions with eigenvalues $\lambda_1', \ldots, \lambda_n'$. Therefore, we can write $f = \sum_i (f, f_i) f_i$. First, by (??),

$$P^t f - Ef = P^t \left( \sum_{i=1}^{n} (f, f_i) f_i \right) - \langle f, 1 \rangle 1$$

$$= \sum_{i=1}^{n} \lambda_i' f_i \langle f, f_i \rangle - \langle f, 1 \rangle 1 = \sum_{i=2}^{n} \lambda_i' (f, f_i) f_i$$

Therefore, using the orthonormality of $f_i$’s,

$$\|P^t f - Ef\|^2 = \left\langle \sum_{i=2}^{n} \lambda_i' (f, f_i), \sum_{j} \lambda_j' (f, f_j) \right\rangle$$

$$= \sum_{i=2}^{n} \lambda_i'^2 (f, f_i)^2$$

$$\leq \lambda^{2t} \|f\|^2$$

where in the last inequality we used $|\lambda_i| \leq \lambda^*$ for all $i > 1$. By the above inequality, for $t \geq \frac{\log \frac{1}{1-\lambda^*}}{\log \epsilon}$ we have

$$\|P^t f(u) - Ef\| \leq \lambda^t \|f\| = (1 - (1 - \lambda^*))^t \|f\| \leq e^{-(1-\lambda^*)t} \|f\| = \epsilon \|f\|$$

where in the second inequality we used $1 - x \leq e^{-x}$. \hfill \Box

Corollary 1.6. Suppose for any function $f : V \to \mathbb{R}$, $\epsilon > 0$ and $t \geq \frac{\log \epsilon^{-1}}{1-\lambda^*} \|P^t f - Ef\| \leq \epsilon \|f\|$ then,

$$\tau_{mix,1} \leq \tau_{mix,2} \leq \max_u \frac{\log \epsilon^{-1} \pi(u)^{-1}}{1-\lambda^*}$$

Note that in particular, if we want to bound the mixing time starting at a given vertex $u$, we don’t need the max on the right hand side.

Proof. Fix a vertex $u \in V$ and let $f := 1_u/\pi_0(u)$. Then, Ef = 1.

$$\|P^t f - Ef\|^2 = \|P^t f - (Ef)1\| = \|P^t f\|^2 - 1.$$ 

On the other hand,

$$\|P^t f\|^2 = \langle P^t f, P^t f \rangle = \mathbb{E}_{v \sim \pi_0} P^t f(v) \cdot P^t f(v)$$

$$\begin{aligned}
&= \mathbb{E}_{v \sim \pi_0} \frac{P^t(v, u)}{\pi(u)} \cdot \frac{P^t(v, u)}{\pi(u)} \\
&= \mathbb{E}_{v \sim \pi_0} \left( \frac{P^t(u, v)}{\pi(v)} \right)^2
\end{aligned}$$

Putting these together, $\|P^t f - Ef\|^2 = \left\| \frac{P^t(u, \cdot)}{\pi(\cdot)} - 1 \right\|^2$. Finally, since $\|f\|^2 = \frac{1}{\pi(u)}$, letting $\epsilon$ of Theorem 1.5 $\epsilon \pi(u)$ proves the claim. \hfill \Box
Theorem 1.5 allows us to estimate the probability of any event. Say $A \subseteq V$ and we want to estimate $\pi_0(A)$. For $f = 1_A$ we have $Ef = \pi_0(A)$ and $(f, f) \leq \pi_0(A) < 1$. So, we can use $P^t f(u)$, for a vertex $u \in V$, as an estimate of $Ef$. By Cauchy-Schwartz inequality,

$$\mathbb{E}_v |P^t f(v) - Ef| \leq \sqrt{\mathbb{E}_u |P^t f(u) - Ef|^2} = \|P^t f - Ef\|.$$ 

So, for $t = \frac{\log \epsilon^{-1} \log \pi_0(u)^{-1}}{1-\lambda^*}$

$$|P^t f(u) - F| \leq \frac{1}{\pi_0(u)} \mathbb{E}_v |P^t f(v) - F| \leq \epsilon \|f\|_{Theorem 1.5}$$

Having this, we can simply estimate $\pi_0(A)$ within an additive error of $\epsilon$ (with probability $1 - \delta$) by averaging $k$ independent $f(X_t)$, where $X_t$ is the $t$-th state that the simple random walk started at $u$ lands on. Here, $k = O\left(\frac{1}{\epsilon^2 \log \delta}\right)$ and $t$ is as defined above.