CSE 599s: Modern Spectral Graph Theory **Winter 2022**

Problem Set 2

Deadline: March 1st(at 11:59 PM) in gradescope

P1) For $w \in \mathbb{F}_2^{n_1} \otimes \mathbb{F}_2^{n_2}$ suppose that every column of w is a codeword in C_1 (a binary linear code of length n_1) and every row of w is a codeword of C_2 (a binary linear code of length n_2). Prove that w is a codeword of $C_1 \otimes C_2$.

P2) Let c, d be integers, and let $\gamma\delta \in (0, 1)$. Define a (c, d)-regular (γ, δ)-expander to be a bipartite graph (L, R, E) with vertex sets L, R such that all vertices in L have degree c, and all vertices in R have degree d; and the additional property that every set of vertices $L' \subseteq L$, such that $|L'| \leq \delta|L|$, has at least $(1 - \gamma)c|L'|$ neighbors. Given such a expander, one can naturally assign an expander code to be a binary linear code whose parity check matrix is the same as the adjacency matrix of this bipartite graph. Recall that L will be the set of bits of the code and R will be the constraints.

a) Give a random construction of such code. To do that you basically need to construct a random (c, d) regular bipartite graph. Note that we must have $c|L| = d|R|$. Here is a natural distribution: put c copies of every left vertex and d copies of every right vertex then choose a uniformly random perfect matching from the left to the right; finally merge all copies. Prove that given $\gamma > 0$, for m, n sufficienlty large enough, the code is (γ, δ) regular for some constant $\delta > 0$. What is the best δ you can get from this random construction?

b) If C is a (c, d, γ, δ)-expander code and $\gamma < 1/2$, then $\delta_C \geq \delta$.

c) In this part we show (c, d, γ, δ)-expander code C as defined above is $(d, \alpha, \beta, \delta)$-smooth, provided $\gamma < 1/6$, $\alpha < (1/3 - 2\gamma)\delta d$ and $\beta = \frac{\alpha}{(1/3-2\gamma)d}$. Namely, we want to show for any given $R_0 \subseteq R$ with $|R_0| \leq \alpha|R|$, there exists $L' \subseteq L$ with $|L'| \leq \beta|L|$ and $R_0 \subseteq R' \subseteq R$ such that the code defined by the induced subgraph $G(L - L', R - R')$ has distance δ.

Hint: Construct the sets L' and R' iteratively. Initially set $L' = \emptyset$ and $R' = R_0$. Then iterate as follows: While there exists a vertex $u \in L - L'$ such that u has more than $c/3$ neighbors in R', add u' to L' and add all the neighbors of u' to R'. Show that this process stops in $t \leq \beta n$ steps, and that the induced graph on $(L - L') \cup (R - R')$ is a (good) expander (in the sense of part (b)).

P3) Given a probability distribution π_i over $X(i)$, faces of dimension i of a complex X, let π_{i-1} be defined as follows: Choose $\sigma \sim \pi_i$ and drop one of the elements of σ uniformly at random, i.e., $\pi_{i-1}(\tau) = \sum_{\sigma \in X(i): \tau \subset \sigma} \frac{\pi_i(\sigma)}{i+1}$. Let $P^{\downarrow}_{i \to i-1}$ and $P^{\uparrow}_{i-1 \to i}$ be as defined in class.

(a) Show that for any $f \in \mathbb{R}^{X(i)}, g \in \mathbb{R}^{X(i-1)}$,

$$\langle P^{\downarrow}_{i \to i-1} f, g \rangle_{\pi_{i-1}} = \langle f, P^{\uparrow}_{i-1 \to i} g \rangle_{\pi_i}.$$

(b) Show that for any $f \in \mathbb{R}^{X(i)}$,

$$\left\| P^{\downarrow}_{i \to i-1} f \right\|^2_{\pi_{i-1}} \leq \|f\|^2_{\pi_i}$$