CSE 599s: Modern Spectral Graph Theory **Winter 2022**

Problem Set 1

Deadline: Feb 9 (at 11:59 PM) in gradescope

P1) a) Show that the cycle C_n with vertices $1, \dots, n$ does not satisfy local-to-global properties. In particular, construct a function $f : [n] \to \mathbb{R}$ such that f is locally correlated,

$$\mathbb{E}_{\{i,j\}\sim\pi_1}(f(i) - f(j))^2 \le 1$$

but

$$\mathbb{E}_{i,j\sim\pi_0}(f(i) - f(j))^2 \ge \Omega(n^2).$$

b) Given a graph $G = (V, E)$ suppose that there are k disjoint sets $S_1, \dots, S_k \subseteq V$ such that $\phi(S_i) \le \epsilon$ for all $1 \le i \le k$. Prove that $\lambda_k(P) \ge 1 - \Omega(\epsilon)$, where $\lambda_k(.)$ is the k-th largest eigenvalue of P.

c) Prove that $\lambda_k(C_n) \ge 1 - \Omega(k/n)$.

P2) Given a graph $G = (V, E)$ with a random walk operator P, in class we argued that $\phi(S) = \mathbb{P}\left[X_1 \notin S | X_0 \sim_{\pi_0} S\right]$ is the probability that a random walk started at a vertex v chosen according to π_0 from S, leaves S in 1-step.

a) Let $A \in \mathbb{R}^{n\times n}$ be a positive semi-definite matrix. Show that (for any inner-product $\langle ., .rangle)$ any unit vector $x \in \mathbb{R}^n$ and any $k \ge 1$,

$$\langle Ax, x\rangle^k \le \langle A^k x, x\rangle.$$

b) Suppose that P is the random walk operator on a graph G with respective distributions π_0, π_1. Furthermore, assume $P \succeq_{\pi_0} 0$. Show that

$$\mathbb{P}\left[X_k \in S | X_0 \sim_{\pi_0} S\right] \ge (1 - \phi(S))^k.$$

c) Recall that the L2 mixing time of the (lazy) walk P on G is the smallest time t such that for any starting vertex u,

$$\left\| \frac{P^t(u, .)}{\pi_0(.)} - 1 \right\|_{\pi_0} = \sum_v \pi_0(v) \left(\frac{P^t(u, v)}{\pi_0(v)} - 1 \right)^2 \le 1/4.$$

Suppose that G has a set S with $\pi_0(S) = 1/\sqrt{n}$ and $\phi(S) \ll 1$. Show that the L2 mixing time of the (lazy) random walk on G is at least $\Omega(\frac{\log n}{\phi(S)})$. Note that such a set is not necessarily a a barrier to L1 mixing and L1 mixing of G could be $O(\log n)$.

P3) Given two symmetric matrices $A, B \in \mathbb{R}^{n\times n}$; recall that $A \bullet B = \text{trace}(AB)$. Prove that

$$A \bullet B \le \sum_{i=1}^{n} \lambda_i(A) \cdot \lambda_i(B),$$

where $\lambda_i(A)/\lambda_i(B)$ are the i-th smallest eigenvalues of A/B respectively.