Exercise: Due Friday December 18

6. The key to the degree 3 upper bound for SoS refutations of PHP^n_{n-1} was a degree 3 derivation of the constraint $1-\sum_{i\in[n]} x_{ij} \geq 0$ from the constraints $1-x_{ij}-x_{i'j} \geq 0$ for $i \neq i'$. In this problem you will show that this derivation requires degree n for Sherali-Adams (SA). For simplicity, we drop the hole index, so assume that you are given the set $\mathscr{P}$ of polynomials consisting of $1-x_i-x_j \geq 0$ for all $i \neq j \in [n]$.

   Define a linear function $\widetilde{\mathbb{E}}$ on polynomials by

$$\widetilde{\mathbb{E}}(x_S) = \begin{cases} 1 & \text{if } S=\emptyset \\ 1/(d+1) & \text{if } |S|=1 \\ 0 & \text{otherwise,} \end{cases}$$

   where $x_S = \prod_{i\in S} x_i$.

   Show that $\widetilde{\mathbb{E}}$ is a degree d SA-pseudo-expectation for $\mathscr{P}$ and use this to conclude that if $d < n-1$ then there is no degree d SA-derivation from $\mathscr{P}$ of $1-\sum_{i\in[n]} x_i \geq 0$.