1. **The doubling trick** You are given an online algorithm $\mathcal{A}$ that guarantees $\text{Regret} \leq T^p$, for some $p \in (0, 1)$, but it has parameters that must be chosen as a function of $T$. Using this algorithm as a black-box, we will construct an algorithm with a regret bound $O(T^p)$ that holds simultaneously for all $T$. In particular, we will analyze the following transformation:

   ```
   for epoch $m = 0, 1, 2, \ldots$ do
     Reset $\mathcal{A}$ with parameters chosen for $T = 2^m$
     for rounds $t = 2^m, \ldots, 2^{m+1} - 1$ do
       Run $\mathcal{A}$
   ```

   Essentially, the algorithm initially guesses $T = 1$, and when it observes this guess was too low, it doubles it’s initial guess and re-starts $\mathcal{A}$. Hence, this is called the “doubling trick.”

   To show the desired regret bound, consider any $T$, and

   (a) Show that the regret on rounds 1 through $T$ is less than or equal to the regret on epochs $m = 0$ through the end of epoch $m_T = \lceil \log_2(T) \rceil$. Then, use the regret bound for $\mathcal{A}$ to bound the cumulative regret for these epochs.

   (b) Simplify the bound from (a) to show that it is upper bounded by a constant times $T^p$. Hint: Use the fact that for $x \neq 1$,

   $$
   \sum_{k=0}^{n} x^k = \frac{x^{n+1} - 1}{x - 1}.
   $$

2. **Constructing a transformation to get stronger bounds** In class we considered a one dimensional problem with linear loss functions $f_t(w) = g_tw$, where the adversary chooses $g_t \in [-1, 1]$. The goal was low regret with respect to $W = [-1, 1]$. The Follow-The-Leader (FTL) algorithm did very badly when the adversary played $g_t$ according to the sequence $(0.5, 1, -1, 1, -1, \ldots)$. We then showed that with an appropriate regularization term, the Follow-The-Regularized-Leader (FTRL) for linear functions achieves $\text{Regret} \leq \sqrt{2T}$ against the best fixed $w^* \in [-1, 1]$ (since $G = 1$ and $R = 1$).
However, in hindsight, one might not feel that competing with a fixed point is so great; after all a simple alternating strategy (playing 0, −1, 1, −1, 1, . . .) would have achieved loss \( O(−T) \), while any fixed strategy has loss \( O(1) \). Show a transformation (using the FTRL algorithm as a subroutine) that gives a no-regret algorithm against a competitor set \( W' \) that includes this alternating strategy. Give the regret bound for this algorithm, and compare it to the regret bound achieved by applying FTRL directly to the problem.

Hint: Use a transformation that takes the original one-dimensional problem, and maps it into a two-dimensional online linear optimization problem. You will need to transform both the loss functions and the points played.

3. **Convex functions and global lower bounds**

Recall that a function \( f \) is convex if

\[
f(αw + (1 − α)w') \leq αf(w) + (1 − α)f(w')
\]

for any \( α ∈ [0, 1] \) and for all \( w \) and \( w' \) in \( f \)'s domain. One of the key properties of convex functions is that a (sub)gradient of the function at a particular \( w \) gives information about the global structure of the function. In particular:

(a) Prove that for a differentiable convex function \( f : \mathbb{R}^n \to \mathbb{R} \), for all \( w \) and \( w_0 \) in the domain of \( f \),

\[
f(w) \geq f(w_0) + \nabla f(w_0)(w − w_0),
\]

where \( \nabla f(w_0) \) is the gradient of \( f \) evaluated at \( w_0 \). That is, a first-order Taylor expansion of a convex function gives a lower bound on the function. Hint: Use the fact that

\[
\nabla f(w) \cdot w' = \lim_{\delta \to 0} \frac{f(w + \delta w') - f(w)}{\delta}.
\]

(b) Show that the previous condition is sufficient, that is, any function \( f : \mathbb{R}^n \to \mathbb{R} \) such that Eq. (1) holds for all \( w, w_0 \) in the domain of \( f \) is convex. Hint: Apply Eq. (1) twice at a carefully chosen point.

(c) Consider a convex \( f \) and assume a \( w^* \in \arg \min_w f(w) \) exists. (Aside: often we write \( w^* = \arg \min_w f(w) \), but this is sloppy, because the argmin need not be unique. This sloppiness is usually fine, because we don’t care which argmin we get. Technically, we define \( \arg \min_{w \in W} f(w) = \{ w^* \in W \mid f(w^*) \leq f(w), \forall w \in W \} \). Show that by evaluating \( f \) and computing its gradient at any point \( w \), we can find a half-space that contains \( w^* \) (and hence a half-space that does not contain \( w^* \)). Recall that a half-space is a set of points \( \{ w \mid a \cdot w \geq b \} \) for some \( a \in \mathbb{R}^n \) and \( b \in \mathbb{R} \).

(d) Consider a convex \( f \) in one dimension, defined on \([0, D] \), such that there exists a \( w^* \in \arg \min_{w \in [0, D]} f(w) \). Show that we can find a \( w' \) such that \( |w^* − w'| \leq \varepsilon \) by making only \( \lfloor \log_2 \frac{D}{\varepsilon} \rfloor \) queries to an oracle that computes \( \nabla f(w) \).
(e) Suppose $\mathbf{0} \in \mathbb{R}^n$ is a subgradient of a convex function $f : \mathcal{W} \rightarrow \mathbb{R}^n$ at $w^*$ with $f(w^*)$ finite. Show that $w^* \in \arg \min_{w \in \mathcal{W}} f(w)$.

4. Convex sets and randomization

A set $C$ is convex if for any $w_1, w_2 \in C$, and any $\alpha \in [0, 1]$, we have $\alpha w_1 + (1 - \alpha)w_2 \in C$.

(a) Let $\mathcal{W} \subseteq \mathbb{R}^n$ be a convex set, with $w_1, \ldots, w_k \in \mathcal{W}$, and let $\theta_1, \ldots, \theta_k \in \mathbb{R}$ that satisfy $\theta_i \geq 0$ and $\sum_{i=1}^{k} \theta_i = 1$. Show that $\bar{w} = \sum_{i=1}^{k} \theta_i x_i$ is also in $\mathcal{W}$. We say that $\bar{w}$ is a convex combination of the $w_i$.

(b) Now, let $w_1, \ldots, w_k \in \mathbb{R}^n$ be arbitrary points, and let

$$\Delta^k = \left\{ \theta \in \mathbb{R}^k \mid \theta_i \geq 0, \sum_{i=1}^{k} \theta_i = 1 \right\}$$

be the $k$-dimensional probability simplex (the set of probability distributions on $k$ items). Show that the convex hull of the $w_i$,

$$\text{conv}(w_1, \ldots, w_k) = \{ \theta \cdot w \mid \theta \in \Delta^k \}$$

is in fact a convex set.

(c) Let $w_1, \ldots, w_k \in \mathbb{R}^n$ be arbitrary points, let $\mathcal{W} = \text{conv}(w_1, \ldots, w_k)$, and let $f(w) = g \cdot w$ be a linear loss function on $\mathcal{W}$. Show that for any $w \in \mathcal{W}$, there exists a probability distribution such that choosing a $w_i$ according to the distribution and then playing the chosen $w_i$ against $f$ produces the same expected loss as just playing $w$. Conversely, show that for any probability distribution on $w_1, \ldots, w_k$, there exists a $w \in \mathcal{W}$ that gets the same expected regret. When might it be preferable to represent such a strategy as a distribution $\theta \in \Delta^k$, and when might it be preferable to represent such a strategy as a point $w \in \mathcal{W}$? (Hint: consider $n$ and $k$).