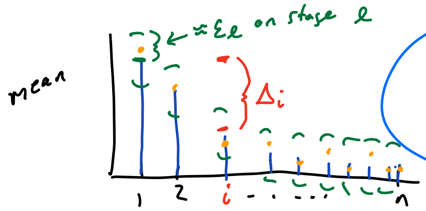
## Elimination-style algorithm

**Input**: $n$ arms $\mathcal{X} = \{1, \ldots, n\}$, confidence level $\delta \in (0,1)$.
Let $\mathcal{X}_1 \leftarrow \mathcal{X}, \ell \leftarrow 1$
for $\ell = 1, 2, \ldots$
  $\epsilon_\ell = 2^{-\ell}$
  Pull each arm in $\mathcal{X}_\ell$ exactly $\tau_\ell = \lceil 2\epsilon_\ell^{-2} \log(\frac{4\ell^2|\mathcal{X}|}{\delta}) \rceil$ times
  Compute the empirical mean of these rewards $\widehat{\theta}_{i,\ell}$ for all $i \in \mathcal{X}_\ell$
 $\mathcal{X}_{\ell+1} \leftarrow \mathcal{X}_\ell \setminus \{i \in \mathcal{X}_\ell : \max_{j \in \mathcal{X}} \widehat{\theta}_{j,\ell} - \widehat{\theta}_{i,\ell} > 2\epsilon_\ell\}$
 $\ell \leftarrow \ell + 1$
**Output**: $\mathcal{X}_{\ell+1}$ (or play the last arm forever in the regret setting)

WLOG assume $\theta_1 > \theta_2 \geq \ldots \geq \theta_n$. $\Delta_i = \theta_1 - \theta_i$

Time horizon $T$ can be set in advance.

Define $\tilde{R}(T) = \max_i \mathbb{E}\left[\sum_{t=1}^{T} X_{t,i} - X_{t,I_t}\right] = \max_i \sum_{i=2}^{n} \Delta_i \mathbb{E}[T_i]$.




Intuition: $i$th arm is eliminated in stage $\ell$ if $\Delta_i \geq 2\epsilon_\ell$

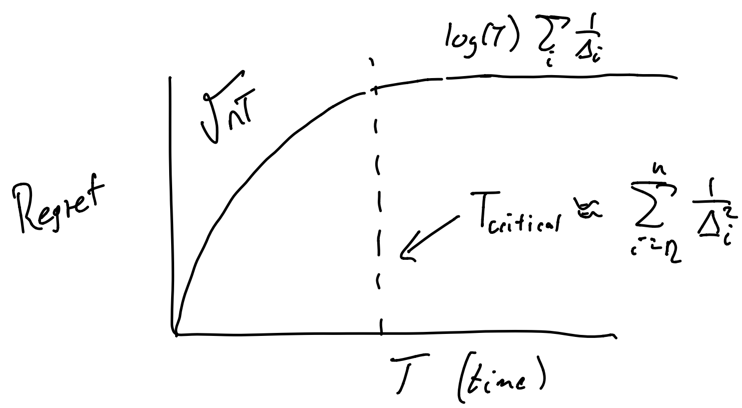
$$= \sum_{t=1}^{T} \theta_i - \mathbb{E}[\theta_{I_t}] = T\theta_i - \sum_{i=1}^{n} \sum_{t=1}^{T} \mathbb{1}\{I_t = i\}\theta_i$$

$$= T\theta_i - \sum_{i=1}^{n} \theta_i T_i$$

$$T_i = \sum_{t=1}^{T} \mathbb{1}\{I_t = i\}$$

Under conditions*, the above algorithm satisfies

$$\tilde{R}(T) \leq \min\left\{ c\sqrt{nT\log(T)},\ c\sum_{i=2}^{n} \frac{\log(T)}{\Delta_i} \right\}$$



Good events: $\mathcal{E}_{i,\ell} = \{ |\hat{\theta}_{i,\ell} - \theta_i| \le \varepsilon_\ell \}$

1st Step: $\mathbb{P}\left( \bigcup_{\ell=1}^{\infty} \bigcup_{i=1}^{n} \mathcal{E}_{i,\ell}^c \right) \le \delta.$

2nd Step: best arm is never eliminated: $1 \in \mathcal{X}_\ell \quad \forall \ell \in \mathbb{N}$

3rd Step: If $i \in \mathcal{X}_\ell$ and $\Delta_i = \theta_1 - \theta_i > 4\varepsilon_\ell$ then $i \notin \mathcal{X}_{\ell+1}$

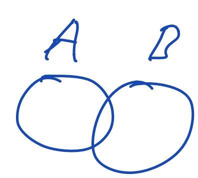
$\implies \max_{i \in \mathcal{X}_\ell} \Delta_i \le 8\varepsilon_\ell$ (Recall $\varepsilon_\ell = 2^{-\ell}$)

4th Step: Show w.p. $\ge 1-\delta$, for any $\nu \ge 0$ we have

$$\sum_i T_i \Delta_i \le \nu T + \sum_{i=2}^{n} c(\nu \vee \Delta_i)^{-1} \log\left( \frac{\log((\nu \vee \Delta_i)^{-1}) n}{\delta} \right).$$

5th Step: Pick $\delta = 1/T$ and convenient values of $\nu \ge 0$.

$a \vee b = \max\{a, b\}$




Step 1: Recall $\mathbb{P}(A \cup B) \le \mathbb{P}(A) + \mathbb{P}(B)$

$$\mathbb{P}\left( \bigcup_{\ell=1}^{\infty} \bigcup_{i=1}^{n} \mathcal{E}_{i,\ell}^c \right) \le \sum_{\ell=1}^{\infty} \sum_{i=1}^{n} \mathbb{P}(\mathcal{E}_{i,\ell}^c) \le \sum_{\ell=1}^{\infty} \sum_{i=1}^{n} \frac{\delta}{2\ell^2 n} \le \delta$$

Condition Assume that if $X_{t,i} \sim \nu_i$ w/ $\mathbb{E}[X_{t,i}] = \theta_i$

then for any $\lambda \ge 0$ $\exists\, \sigma > 0$ such that

$$\mathbb{E}[\exp(\lambda(X_{t,i} - \theta_i))] \le \exp(\lambda^2 \sigma^2 / 2).$$

We say $X_{t,i} - \theta_i$ is $\sigma^2$-sub-Gaussian distributed.

Ex. If $X_{t,i} \sim \text{Bernoulli}(\theta_i)$ $(X_{t,i} \in \{0,1\})$

then $X_{t,i} - \theta_i$ is sub-Gaussian w/ $\sigma^2 = 1/4$

Ex. $X_{t,i} \sim N(\theta_i, \kappa^2)$ " " $\sigma^2 = \kappa^2$

<u>Lemma</u>) Let $Z_1, \dots, Z_t$ be $\sigma^2$-sub-Gaussian R.V.s, then $\mathbb{P}\left(\frac{1}{t}\sum_{s=1}^{t} Z_s > \varepsilon\right) \leq \exp\left(-\frac{t\varepsilon^2}{2\sigma^2}\right)$

$\mathcal{E}_{i,\ell} = \{ |\hat{\theta}_{i,\ell} - \theta_i| \leq \varepsilon_\ell \}$ then

$$\mathbb{P}(|\hat{\theta}_{i,\ell} - \theta_i| > \varepsilon_\ell) = 2\mathbb{P}(\hat{\theta}_{i,\ell} - \theta_i > \varepsilon_\ell)$$

$$= 2\,\mathbb{P}\left(\frac{1}{T_\ell}\sum_{s=1}^{T_\ell} Z_s > \varepsilon_\ell\right) \leq 2\exp\left(-\frac{T_\ell \varepsilon_\ell^2}{2\sigma^2}\right)$$

Assume $\sigma^2 = 1$

$$\leq 2\frac{\delta}{4\ell^2 n} = \frac{\delta}{2\ell^2 n}$$

$$T_\ell = \left\lceil 2\varepsilon_\ell^{-2} \log\left(\frac{4\ell^2 n}{\delta}\right) \right\rceil$$

Step 2: best arm never removed.

Arm 1 is removed at stage $\ell$ if

$$\max_{j \in \hat{\mathcal{X}}_\ell} \hat{\theta}_{\ell,j} - \hat{\theta}_{\ell,1} > 2\varepsilon_\ell .$$

Well, for <u>any</u> $j \neq 1$ we have $\quad \Delta_j = \theta_1 - \theta_j$

$$\hat{\theta}_{\ell,j} - \hat{\theta}_{\ell,1} = \underbrace{\hat{\theta}_{\ell,j} - \theta_j} + \underbrace{\theta_1 - \hat{\theta}_1} - \Delta_j$$

Apply good event $\rightarrow$ $\leq \varepsilon_\ell + \varepsilon_\ell - \Delta_j$

$$\leq 2\varepsilon_\ell$$

$$\Longrightarrow \max_{j \in \hat{\mathcal{X}}_\ell} \hat{\theta}_{\ell,j} - \hat{\theta}_{\ell,1} \leq 2\varepsilon_\ell, \quad 1 \text{ is not thrown out.}$$

Step: If $i \in \mathcal{X}_\ell$ and $\Delta_i = \theta_1 - \theta_i > 4\varepsilon_\ell$ then $i \notin \mathcal{X}_{\ell+1}$

Removed if : $\max_{j \in \mathcal{X}_\ell} \hat{\theta}_{\ell,j} - \hat{\theta}_{\ell,i} > 2\varepsilon_\ell$

$$\max_{j \in \mathcal{X}_\ell} \hat{\theta}_{\ell,j} - \hat{\theta}_{\ell,i} \geq \hat{\theta}_{\ell,1} - \hat{\theta}_{\ell,i}$$

$$= \underbrace{\hat{\theta}_{\ell,1} - \theta_1} + \underbrace{\theta_i - \hat{\theta}_{\ell,i}} + \Delta_i$$

$$\geq -\varepsilon_\ell + -\varepsilon + 4\varepsilon_\ell$$

$$> 2\varepsilon_\ell .$$

$\Longrightarrow$ arm $i$ is removed.

4th step

$$\sum_i T_i \Delta_i \le \nu T + \sum_{i=2}^{n} c(\nu \vee \Delta_i)^{-1} \log\left(\frac{\log((\nu \vee \Delta_i)^{-1})\, n}{\delta}\right).$$

$$T = \sum_{i=1}^{n} T_i$$

$$\sum_{i=2}^{n} T_i \Delta_i = \sum_{i:\Delta_i \le \nu} T_i \Delta_i + \sum_{i:\Delta_i > \nu} T_i \Delta_i$$

$$\le \nu T + \sum_{i:\Delta_i > \nu} T_i \Delta_i$$

$$= \nu T + \sum_{i:\Delta_i > \nu} \sum_{\ell=1}^{\infty} \Delta_i \tau_\ell \mathbb{1}\{i \in \mathcal{X}_\ell\}$$

$$\le \nu T + \sum_{i:\Delta_i > \nu} \sum_{\ell=1}^{\infty} \Delta_i \tau_\ell \mathbb{1}\{\Delta_i \le 8\varepsilon_\ell\}$$

$$= \nu T + \sum_i \sum_{\ell}^{\log_2(8(\Delta_i \vee \nu)^{-1})} \varepsilon_\ell \tau_\ell$$

$$\tau_\ell \simeq \varepsilon_\ell^{-2}$$

$$\le \nu T + \sum_i \Delta_i^{-1}$$