On the Impact of Combinatorial Structure on Congestion Games

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joint work with Heiner Ackermann and Berthold Vöcking

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Congestion Games – Definition

Congestion game $\Gamma = (\mathcal{N}, \mathcal{R}, (\Sigma_i)_{i \in \mathcal{N}}, (d_r)_{r \in \mathcal{R}})$

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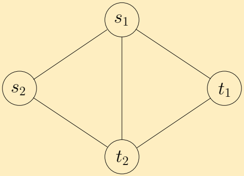

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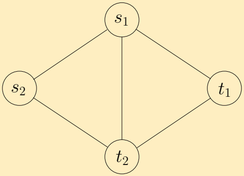

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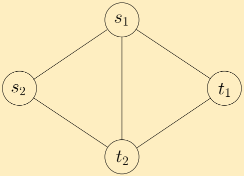

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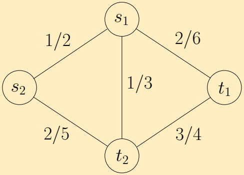

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- Every player wants to **minimize his delay**.

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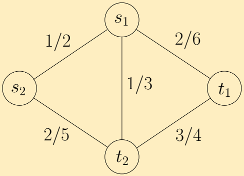

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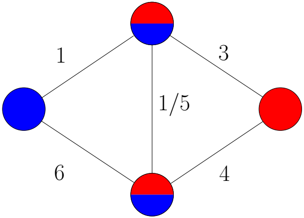
- **delay functions**
  $\forall r \in \mathcal{R} : \; d_r : \mathbb{N} \rightarrow \mathbb{N}$
- Every player wants to **minimize his delay**.
- Every player is faced with **optimization problem with varying delays**.

Congestion Games – Example

$\mathcal{N} = \{1, 2\}$, $\mathcal{R} = E$, $n = 2$, $m = 5$

Σ_1 = set of spanning trees on blue vertices

Σ_2 = set of spanning trees on red vertices



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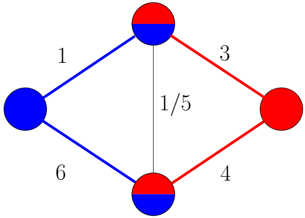

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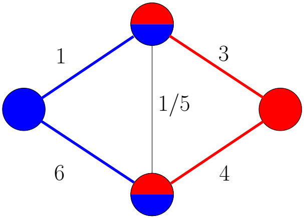
Delay: 7

Opt: 2



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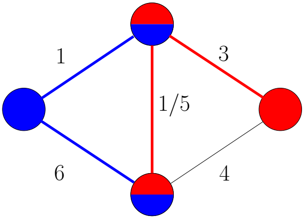

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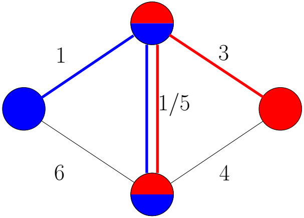

Delay: 8

Opt: 7

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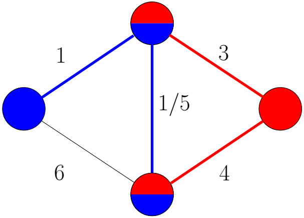

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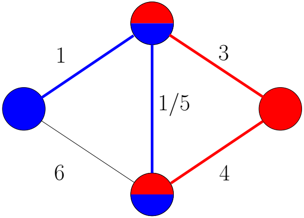

Delay: 7
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A state $S \in \Sigma_1 \times \cdots \times \Sigma_n$ is called pure Nash equilibrium if no player can improve his delay unilaterally.

Questions

Rosenthal (1973)

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- How many best responses are needed to find an equilibrium?
- What is the complexity of computing equilibria?

How many best responses are needed?

1. How many best responses are needed?
2. What is the complexity of finding equilibria?

Known Results

Fabrikant, Papadimitriou, Talwar (STOC 2004)

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In singleton games, all best response sequences have length at most $n^2 m$.

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In singleton games, all best response sequences have length at most $n^2 m$.

Question

What about Spanning Tree congestion games?
Is there a characterization which congestion games converge in polynomial time?

Rosenthal's Potential Function

Properties

- $\Phi\colon \Sigma_1 \times \cdots \times \Sigma_n \to \mathbb{Z}$
- $\forall S : 0 \leq \phi(S) \leq m \cdot n \cdot d_{\max}$.
- If one player decreases his delay by x, then also Φ decreases by x.

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- If one player decreases his delay by x, then also Φ decreases by x.

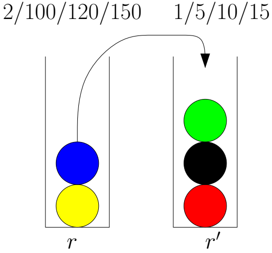
Corollary

- pure Nash equilibria = states in which no player can decrease the potential Φ
- After at most $m \cdot n \cdot d_{\max}$ better responses a pure Nash equilibrium is reached.

Singleton Games

Singleton Games

- Idea: Reduce delays without changing the game!



$$d_r(n_r) > d_{r'}(n_{r'} + 1)$$

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- equivalent delays $\overline{d}_r(x) \leq n \cdot m$

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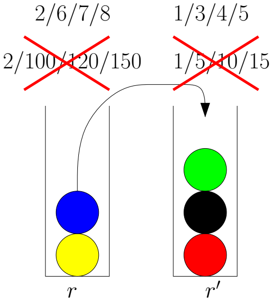

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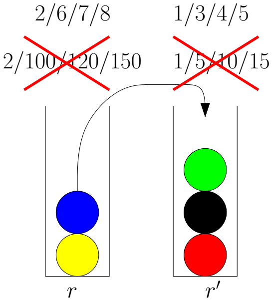

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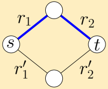
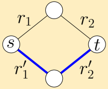
Network Congestion Games



$$d_{r_1}(n_{r_1}) + d_{r_2}(n_{r_2}) > d_{r'_1}(n_{r'_1}+1) + d_{r'_2}(n_{r'_2}+1)$$



What about Spanning Tree Congestion Games?

Theorem

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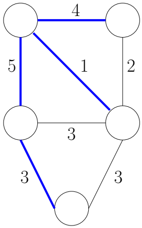
- In matroid congestion games all best response sequences have length at most $n^2 \cdot m \cdot rank$.

Proof

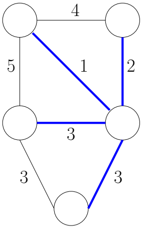
Lemma

- weighted graph: $G = (V, E, w)$
- Let T_0 be a ST, let T^{OPT} be a MST: $w(T_0) \geq w(T^{\mathrm{OPT}})$.

There exists sequence $T_0, \ldots, T_l = T^{\mathrm{OPT}}$ of STs with $w(T_0) \geq w(T_1) \geq \ldots \geq w(T_l)$ with $|T_i \setminus T_{i-1}| = 1$.



$w(T_0) = 13$



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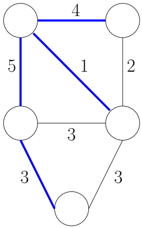

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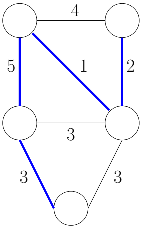

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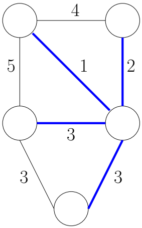

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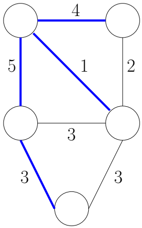

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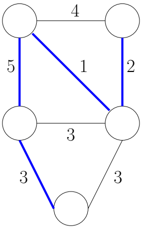

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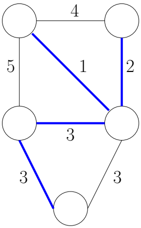

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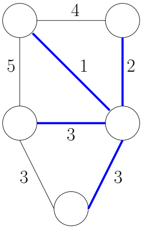

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Fast Convergence beyond Matroids

Theorem:

Let $\mathcal{I}$ be any inclusion-free non-matroid set system. Then, for every n, there exists an n-player congestion game with the following properties.

- each Σ_i is isomorphic to $\mathcal{I}$,
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Conclusion: The matroid property is the maximal property on the individual players' strategy spaces that guarantees polynomial convergence.

Proof Idea for Exponential Convergence

Because of the non-matroid property, one can show:

1-2-exchange property

There exist three resources a, b, c with the property that, if the delays of the other resources are chosen appropriately, an optimal solution of $\mathcal{I}$ contains

- $d(a) < d(b) + d(c) \Rightarrow a \in \mathrm{OPT}$ and $b, c \notin \mathrm{OPT}$,
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Using this property one can interweave the strategy spaces in the form of a counter that yields a best response sequence of length $2^{\Omega(n)}$.

What is the complexity of finding equilibria?

1. How many best responses are needed?
2. What is the complexity of finding equilibria?

PLS

Local Search Problem Π

- set of instances $\mathcal{I}_\Pi$
- for $I \in \mathcal{I}_\Pi$: set of feasible solutions $\mathcal{F}(I)$
- for $I \in \mathcal{I}_\Pi$: objective function $c : \mathcal{F}(I) \to \mathbb{Z}$
- for $I \in \mathcal{I}_\Pi$ and $S \in \mathcal{F}(I)$: neighborhood $\mathcal{N}(S, I) \subseteq \mathcal{F}(I)$

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PLS

Π is in PLS if polynomial time algorithms exists for

- finding initial feasible solution $S \in \mathcal{F}(I)$,
- computing the objective value $c(S)$,
- finding a better solution in the neighborhood $\mathcal{N}(S, I)$ if S is not locally optimal.

PLS-reductions

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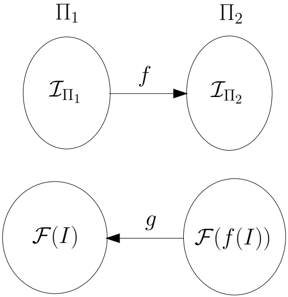
- Polynomial-time computable function $f\colon \mathcal{I}_{\Pi_1} \to \mathcal{I}_{\Pi_2}$.
- Polynomial-time computable function $(S_2 \in \mathcal{F}(f(I)))$ $g\colon S_2 \mapsto S_1 \in \mathcal{F}(I)$



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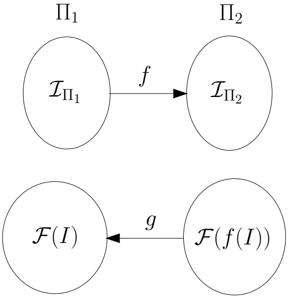
- Polynomial-time computable function $(S_2 \in \mathcal{F}(f(I)))$ $g\colon S_2 \mapsto S_1 \in \mathcal{F}(I)$
- S_2 locally optimal $\Rightarrow g(S_2)$ locally optimal.



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Congestion Games are in PLS

- objective function = Rosenthal's potential function
- $S' \in \mathcal{N}(S)$ if S' is obtained from S by better response of some player

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Fabrikant, Papadimitriou, Talwar (STOC 2004)

Finding a pure Nash equilibrium in network congestion games is PLS-complete.

Threshold Congestion Games

Threshold Congestion Games

$\mathcal{R} = \mathcal{R}_{\mathrm{in}} \dot{\cup} \mathcal{R}_{\mathrm{out}}$. Every player i has two strategies:

- in: an arbitrary subset $\mathcal{S}_i \subseteq \mathcal{R}_{\mathrm{in}}$
- out: $\{r_i\}$ for a unique resource $r_i \in \mathcal{R}_{\mathrm{out}}$ with fixed delay, the so-called threshold t_i

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2-threshold games

Each resource $r \in \mathcal{R}_{\text{in}}$ is contained in the strategies of exactly two players.

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$\mathcal{R} = \mathcal{R}_{\text{in}} \dot\cup \mathcal{R}_{\text{out}}$. Every player i has two strategies:

in: an arbitrary subset $\mathcal{S}_i \subseteq \mathcal{R}_{\text{in}}$

out: $\{r_i\}$ for a unique resource $r_i \in \mathcal{R}_{\text{out}}$ with fixed delay, the so-called threshold t_i

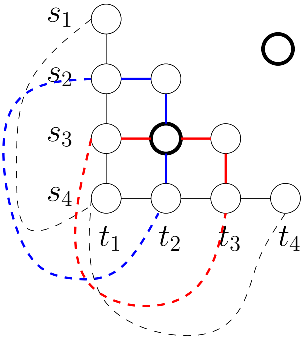
2-threshold games

Each resource $r \in \mathcal{R}_{\text{in}}$ is contained in the strategies of exactly two players.

Theorem

2-threshold congestion games are PLS-complete.

Reduction



$r_{2,3} \in \mathcal{S}_2 \cup \mathcal{S}_3$

Theorem

Network congestion games are PLS-complete for (un)directed networks with linear delay functions.

Conclusions and Open Questions

- 1-2-exchanges ⇒ exponentially long best response sequences
- 1-k-exchanges ⇒ PLS-completeness
- Threshold Congestion Games are a good starting point for PLS-reductions.

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Open Questions

- Are 1-2-exchanges sufficient to construct a state from which every best response sequence is exponentially long?
- How large has k to be in order to prove PLS-completeness?

Thank you!

Questions?