Separation of the Monotone NC Hierarchy *

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Abstract

We prove tight lower bounds, of up to n^ϵ, for the monotone depth of functions in monotone-P. As a result we achieve the separation of the following classes.

1. *monotone-NC $\neq$ monotone-P.*
2. *$\forall i \geq 1$, monotone-$NC^i \neq$ monotone-NC^{i+1}.*
3. *More generally: For any integer function $D(n)$, up to n^ϵ (for some $\epsilon > 0$), we give an explicit example of a monotone Boolean function, that can be computed by polynomial size monotone Boolean circuits of depth $D(n)$, but that cannot be computed by* **any** *(fan-in 2) monotone Boolean circuits of depth less than Const $\cdot D(n)$ (for some constant Const).*

Only a separation of monotone-NC^1 from monotone-NC^2 was previously known.

Our argument is more general: we define a new class of communication complexity search problems, referred to below as DART games, and we prove a tight lower bound for the communication complexity of every member of this class. As a result we get lower bounds for the monotone depth of many functions. In particular, we get the following bounds:

1. *For st-connectivity, we get a tight lower bound of $\Omega(\log^2 n)$. That is, we get a new proof for Karchmer-Wigderson's theorem, as an immediate corollary of our general result.*
2. *For the k-clique function, with $k \leq n^\epsilon$, we get a tight lower bound of $\Omega(k \log n)$. Only a bound of $\Omega(k)$ was previously known.*

1 Introduction

A Boolean function $f : \{0,1\}^n \rightarrow \{0,1\}$ is *monotone* if flipping a bit from 0 to 1 in any argument to f cannot cause the value of f to change from 1 to 0. A *monotone Boolean circuit* is an indegree-two single-output circuit over the monotone base $\{\wedge, \vee\}$. The *size* of a circuit is the number of gates in the circuit, and the *depth* of a circuit is the length of the longest path between a circuit input and the circuit output. The *monotone size* of a function is defined to be the smallest size of a monotone circuit for that function, and the *monotone depth* of the function is defined to be the smallest depth of a monotone circuit for that function.

In his breakthrough paper in 1985, Razborov [Ra85a] proved a super-polynomial lower bound for the monotone-size of the Clique function, and as a conclusion obtained the separation of monotone-P from monotone-NP. Using Razborov's technique, exponential lower bounds for the monotone size of other functions were proved by Andreev [An85], and an exponential lower bound for the monotone size of the Clique function was finally proved by Alon and Boppana [AlBo87]. A simpler proof for that lower bound was recently presented by Haken [Ha95].

Those lower bounds, and other lower bounds for the monotone size of functions, immediately translate into corresponding lower bounds (of up to n^ϵ) for the monotone depth of the same functions. Lower bounds for the size, however, cannot give the separation of classes of monotone depth (e.g., the monotone-NC hierarchy), as those classes are sub-classes of monotone-P. Thus, in order to achieve a separation of those classes, one needs to prove lower bounds for the monotone depth of functions in monotone-P. Hence, in order to achieve a separation of classes of monotone depth, one needs to prove depth lower bounds directly, and not as a consequence of size lower bounds.

In 1988, Karchmer and Wigderson [KaWi88] obtained the important result that the monotone depth of the *st*-connectivity function is $\Omega(\log^2 n)$. Since *st*-connectivity is in monotone-NC^2, the separation of monotone-NC^1 from monotone-NC^2 was obtained. Since then, however, no better lower bounds for the monotone depth of functions in monotone-P were proved, and no larger gaps between the monotone depth of a function and the logarithm of its monotone size were obtained. Other proofs for monotone-$NC^1 \neq$ monotone-NC^2 were later presented in [GrSi92] (where a separation of monotone-L from monotone-NC^1 was also proved), and in [KaRaWi91].

Some other direct lower bounds for the monotone

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depth of functions are known. In particular, a tight lower bound of $\Omega(n)$ was proved for the monotone depth of the Matching function [RaWi90]. The Matching function, however, is not in monotone-P, as a super-polynomial lower bound for its monotone size was proved in [Ra85b]. If sub-exponential upper bounds for the monotone size of the Matching function were shown then the result of [RaWi90] would have given a separation of classes of monotone depth (by a padding argument). It is still open, however, whether such an upper bound exists.

For more information about the early results in monotone complexity see the excellent survey of [BoSi90].

In this paper, we prove tight lower bounds of up to n^ϵ, for the monotone depth of functions in monotone-P. In particular, for $D(n) = n^\epsilon$ (for some constant ϵ), we give an explicit example of a function in monotone-P that can be (uniformly) computed by a family of monotone Boolean circuits of polynomial size and of depth $D(n)$, but that cannot be computed by any family of monotone Boolean circuits of depth less than $Const \cdot D(n)$ (for some constant $Const$). By a padding argument, the same result follows for any function $D(n) \leq n^\epsilon$ as well. Hence, the following corollaries follow immediately:

1. monotone-NC $\neq$ monotone-P.
2. $\forall i \geq 1$, monotone-$NC^i \neq$ monotone-NC^{i+1}.

1.1 Relevance of Monotone Complexity

Monotone complexity has always attracted many researchers. Since the result of [Ra85a], many papers on monotone complexity have appeared, and this includes many interesting papers in the last 3 years (e.g., [Ya94, GoHå95, Ha95, AmMa96, BeUl97, SiTs97, Ju97]). Is monotone complexity research useful ?

Although the separation results of [Ra85a], and [KaWi88], are among the most famous and most impressive results in complexity theory, it is still under debate whether monotone complexity is worth pursuing.

Indeed, by the results of [Ra85b, Ta88], the monotone size of a function may be exponentially larger than its non-monotone size. By the result of [RaWi90], the monotone depth of a function may be exponentially larger than its non-monotone depth. By the results of [Ra89, RaRu93, Ra94], some of the techniques used so far to obtain lower bounds for monotone circuits are not strong enough to obtain non-monotone separations such as P $\neq$ NP. It is therefore widely accepted that the known lower bounds for monotone complexity are only a very small step towards a separation of non-monotone classes.

On the other hand, monotone complexity is relevant for non-monotone complexity, at the very least because a separation theorem for non-monotone complexity classes (e.g., NC $\neq$ P) automatically gives the separation of the corresponding monotone classes as well. Therefore, if one is not able to separate monotone-NC from monotone-P then one is not able to separate NC from P either. Furthermore, although the known techniques for proving lower bounds for monotone complexity are not very likely to give significant lower bounds for non-monotone complexity, it is not unlikely that these techniques will be combined with some new techniques to obtain non-monotone lower bounds, or that monotone complexity will affect non-monotone complexity in some other way.

In addition, monotone complexity is also interesting in its own right. Indeed, determining the monotone size (or depth) of a function is a very natural combinatorial problem, and monotone complexity may be relevant for several other complexity issues. One important example is propositional proof theory, where following [Ra94] and Bonet et. al [BoPiRa95], reductions to monotone complexity were extensively used. In particular, using techniques developed in the sequence of papers [ImPiUr94, BoPiRa95, Kr95], Pudlak [Pu95] used monotone complexity to obtain an impressive exponential lower bound for the length of cutting planes proofs (see also, [CoHa95, Fu96]). Other applications of monotone complexity are also known.

1.2 Methods and Other Results

We use Karchmer and Wigderson's communication complexity approach [KaWi88] (see also [Ka88]). In this approach, a lower bound for the monotone depth of a function f is obtained by proving a lower bound for the complexity of the following communication game: Player I is given an input x, such that $f(x) = 1$. Player II is given an input y, such that $f(y) = 0$. The goal of the two players is to find a coordinate i such that $x_i = 1$, and $y_i = 0$.

Our proof begins by defining a new class of communication games, which we call *dart* games. Briefly stated, a dart game is a game of the following type: Player I is given $x_1, \ldots, x_n$, where for every i, $x_i \in \{1, \ldots, m\}$. Player II is given $y_1, \ldots, y_n$, where for every i, y_i is a coloring of $\{1, \ldots, m\}$. The goal of the two players is to solve a DNF search problem R, depending only on $e_1, \ldots, e_n$, where e_i is the color of x_i in the coloring y_i.

A *structured* communication protocol for a dart game is (briefly stated) one where the players reveal the variables e_i one by one, that is, in each round Player I sends x_i (for some i) and Player II answers with y_i. Our main theorem shows that if m is much larger than n (say $m \geq n^{20}$) then any communication protocol for a dart game can be simulated by a structured protocol of the same complexity (up to a multiplicative constant). Since structured protocols are usually very easy to analyze, this gives a general tight lower bound for the communication complexity of every dart game. It turns out

that this lower bound implies lower bounds for the communication complexity of many monotone Karchmer-Wigderson's games, and hence gives lower bounds for the monotone depth of many functions.

The separation of the monotone NC hierarchy is then obtained by proving a lower bound for a variant, called *GEN* (see [JoLa77]), of the monotone P-complete problem *Path Systems* (see [Co74]). As mentioned above, our argument is general enough to prove lower bounds for many other functions. In particular, we get a new proof for Karchmer-Wigderson's $\Omega(\log^2 n)$ lower bound for *st*-connectivity, on a graph with n vertices, and a new (tight) lower bound of $\Omega(k \cdot \log n)$ for the monotone depth of the k-Clique function for small cliques ($k \leq n^\epsilon$).

In this version of the paper, many of the proofs of claims and lemmas, as well as many other details, are omitted.

2 Communication Games

We consider the standard 2-party Communication Complexity model of Yao [Ya79]. For an excellent survey of communication complexity see [KuNi96].

Let X, Y, Z be finite sets, and let $R \subseteq X \times Y \times Z$. For two subsets $A \subseteq X, B \subseteq Y$, a *communication protocol P for R over the domain $A \times B$* specifies, for each $(x, y) \in A \times B$, the exchange of information bits by two players, Player I and Player II, that initially receive as inputs x and y respectively, and finally agree on a value $P(x, y) \in Z$ such that $(x, y, P(x, y)) \in R$.

The *communication complexity of such a protocol P* is the maximum, over all $(x, y) \in A \times B$, of the number of bits exchanged by the two players on the input pair (x, y) when using P. The *communication complexity $C_R(A, B)$ of R over the domain $A \times B$* is the minimum, over all protocols P for R over $A \times B$, of the complexity of P. Finally, the *communication complexity of the relation R* is $C_R(X, Y)$, which will also be denoted $CC(R)$.

We think of R also as a function from the domain $X \times Y \times Z$ to the range $\{TRUE, FALSE\}$, where $R(x, y, z) = TRUE$ iff $(x, y, z) \in R$.

2.1 DART Games

Denote by $[m]$ the set $\{1, \ldots, m\}$. For every $n, m \in \mathbb{N}$, we define a class of communication games $DART(m, n)$. A communication game, given by the relation $R \subseteq X \times Y \times Z$, is in $DART(m, n)$ if the following holds:

1. $X = [m]^n$. I.e., the input for Player I is a sequence $x = (x_1, x_2, \ldots, x_n)$, with $x_j \in [m]$ for every j.
2. $Y = (\{0,1\}^m)^n$. I.e., the input for Player II is a sequence $y = (y_1, y_2, \ldots, y_n)$ of binary colorings of $[m]$, that is, each y_j is an m-bit string. We think of y_j also as a function $y_j : [m] \to \{0, 1\}$.
3. The relation $R(x, y, z)$ depends only on the sequence $(y_1(x_1), y_2(x_2), \ldots, y_n(x_n))$, and on z, (where $y_j(x_j)$ denotes the x_j-th bit in the string y_j). I.e., if $x, x' \in X$ and $y, y' \in Y$ satisfy that for every j, $y_j(x_j) = y'_j(x'_j)$ then for every z, $R(x, y, z) = R(x', y', z)$.

   Hence, $R(x, y, z)$ can be described as $R((e_1, \ldots, e_n), z)$, where $e_j \stackrel{\text{def}}{=} y_j(x_j)$.
4. $R((e_1, \ldots, e_n), z)$ is a DNF-Search-Problem, i.e., there exists a DNF tautology $F_R(e_1, \ldots, e_n)$, with set of clauses Z, such that $R((e_1, \ldots, e_n), z) = TRUE$ iff z is a satisfied clause of $F_R(e_1, \ldots, e_n)$.

For a relation R, we will say that R is a dart relation, and use the notation $R \in DART(m, n)$ if the corresponding game is in $DART(m, n)$.

2.2 Structured and General Protocols

A *structured* communication protocol for a dart game is a protocol of the following type: In each round of the protocol, Player I sends the value of x_j for some index j, and Player II answers with $y_j(x_j)$. Thus in one round the players exchange $\lceil \log_2 m \rceil + 1$ bits of information, and find out the value of one $y_j(x_j)$. [1] A structured protocol can also be described as a decision tree for the corresponding DNF-search-problem, over the variables $(e_1, \ldots, e_n)$ (for the exact definition see [LoNeNaWi95]).

For a dart relation R, denote by $SC(R)$ (that is, the *Structured Complexity* of R) the number of rounds in the shortest structured protocol that solves R. Note, that if the number of rounds in a structured protocol is k then the communication complexity is $k \cdot (\lceil \log_2 m \rceil + 1)$. Recall that the communication complexity of the best general protocol for the relation R is denoted by $CC(R)$.

Thus, structured communication protocols for dart games are very limited. In each round, each player is allowed to give information on only one variable. It is, therefore, not very surprising that for many interesting relations, it is very easy to determine $SC(R)$ exactly. It turns out, however, that in many interesting cases general communication protocols for dart games can be simulated by structured ones ! In these cases, a lower bound for the structured complexity of a relation (i.e., $SC(R)$) gives a lower bound for the general communication complexity (i.e., $CC(R)$) as well.

Our main theorem shows that if m is larger than some polynomial in n ($m \geq n^{20}$) then structured protocols for $DART(m, n)$ games are as powerful (up to a multiplicative constant) as general protocols. The constant

[1] W.l.o.g. it can be assumed that both players know the index j, and therefore j does not have to be transmitted. Also, w.l.o.g. it can be assumed that the protocol depends only on the values of $y_j(x_j)$-s, transmitted in previous rounds, and not on the entire communication.

20 here is not optimal. Obtaining the best possible constant is not the focus of this paper.

Theorem 2.1 *Assume that $m \geq n^{20}$, and let $R \subseteq X \times Y \times Z$ be a relation in $DART(m,n)$. Then*

$$CC(R) = SC(R) \cdot \Omega(\log m).$$

(Recall that $SC(R)$ denotes the number of rounds in the shortest structured protocol for R, and not the communication complexity of that protocol).

2.3 Multi-Color DART Games

So far, we have defined dart games using colorings with two colors only. This is done for simplicity, and because for most applications two colors are enough. The main theorem, however, is true when one allows up to m^δ colors (for some small constant δ). Let us therefore generalize the definition of dart games to the case of r colors.

For every $n, m, r \in \mathbb{N}$, let us define the class $DART_r(m,n)$. A communication game, given by the relation $R \subseteq X \times Y \times Z$, is in $DART_r(m,n)$ if the following holds:

1. $X = [m]^n$.
2. $Y = (\{1,\ldots,r\}^m)^n$. I.e., the input for Player II is a sequence $(y_1, y_2, \ldots, y_n)$ of r-colorings of $[m]$. We think of y_j also as a function $y_j : [m] \to [r]$.
3. The relation $R(x,y,z)$ depends only on the sequence $(y_1(x_1), y_2(x_2), \ldots, y_n(x_n))$, and on z. Hence, $R(x,y,z)$ can be described as $R((e_1,\ldots,e_n), z)$, where $e_j \stackrel{\text{def}}{=} y_j(x_j)$.
4. $R((e_1,\ldots,e_n), z)$ is a DNF-Search-Problem in $\{(e_i = j)\}_{i \in [n], j \in [r]}$.

   That is, there exists a tautology F_R, such that F_R is a disjunction of conjunctions of expressions of the form $(e_i = j)$, and such that Z is the set of clauses of F_R, and such that $R((e_1,\ldots,e_n), z) = TRUE$ iff z is a satisfied clause of F_R.

As before, a structured communication protocol for R is a protocol of the following type: In each round of the protocol, Player I sends the value of x_j for some index j, and Player II answers with $y_j(x_j)$. Thus in one round, the players exchange $\lceil \log_2 m \rceil + \lceil \log_2 r \rceil$ bits of information, and find out the value of one $y_j(x_j)$. As before, denote by $SC(R)$ the number of rounds in the shortest structured protocol that solves R.

The following theorem is a generalization of Theorem 2.1 to the case of r colors, where $r \leq m^\delta$ (for some small constant $\delta > 0$). For simplicity, we take $\delta = 1/1000$, which is not optimal.

Theorem 2.2 *Assume that $m \geq n^{20}$, and $m \geq r^{1000}$, and let $R \subseteq X \times Y \times Z$ be a relation in $DART_r(m,n)$. Then*

$$CC(R) = SC(R) \cdot \Omega(\log m).$$

3 Separation of Depth-Classes

In this section, we use Theorem 2.1 to prove the separation of the monotone-NC hierarchy. In this version of the paper, only a sketch of the proof is given.

First, let us recall the connection between communication complexity and monotone depth : For a monotone Boolean function $f : \{0,1\}^l \to \{0,1\}$, define the relation R_f by

$$R_f = \{ (x,y,i) \in f^{-1}(1) \times f^{-1}(0) \times [l] \mid x_i = 1, y_i = 0 \}.$$

The communication game played on R_f is therefore the following : Player I gets an l-bit string, on which f evaluates to 1. Player II gets an l-bit string, on which f evaluates to 0. Their goal is to agree on a bit position i, in which player I's string has a 1, and player II's string has a 0. Below, we will refer to that game as the monotone KW-game for the function f.

The following observation was discovered by Yannakakis (unpublished), and by Karchmer and Wigderson who realized its full potential. (Here we only need the monotone form) :

Lemma 3.1 *[KaWi88] $CC(R_f)$ is equal to the monotone depth of f.*

3.1 The *GEN* Function

The first insights leading to our separation results came from choosing a convenient function capturing the difficulty of the class monotone-P. Let us describe a variant of the very first P-complete function known [Co74], which Cook called: *Path Systems*. In this paper we call this function *GEN* (for "GENeration"), in analogy with Jones and Laaser's non-monotone version of the function [JoLa77] (see also [BaMc91]):

The *GEN* function: The input for *GEN* is a string of l^3 bits $(t_{ijk})_{1 \leq i,j,k \leq l}$. For $1 \leq k \leq l$, we say that 1 *generates* k if $k = 1$, or for some i and j such that $t_{ijk} = 1$, 1 generates i and 1 generates j (where "1 generates i" and "1 generates j" are defined recursively in the same way). The function *GEN* determines whether 1, called the source, generates l, called the target. That is, $GEN(t_{111},\ldots,t_{lll}) = 1$ iff 1 generates l. In this context, we will refer to the set $\{1,2,\ldots,l\}$ as the set of *GEN*-elements.

It is not hard to verify that *GEN* is a monotone Boolean function, computable by a monotone polynomial size circuit family. Our main goal here is to prove lower bounds for the monotone depth of *GEN*, as well as for some variations of it.

3.2 The *PYRGEN* Game

Let $n = \binom{d+1}{2}$. We will now define the communication game $PYRGEN(m,d)$, that will later be related to *GEN*, and to some variations of it. The game $PYRGEN(m,d)$ will be in the class $DART(m,n)$.

Recall that in a $DART(m,n)$ game, Player I receives a sequence of n integers in $[m]$, and Player II receives n binary colorings of $[m]$. For the *PYRGEN* game, it is convenient to index each player's sequence by (i,j), where $1 \leq j \leq i \leq d$, and to imagine that the sequence is laid out in a d-level pyramidal fashion, where the index i denotes the level in the pyramid, and the index j denotes the exact position in that level. Denote by $(x_{i,j})_{1\leq j\leq i\leq d}$ the sequence for Player I, and by $(y_{i,j})_{1\leq j\leq i\leq d}$ the sequence for Player II, and as before for every i,j, denote $e_{i,j} = y_{i,j}(x_{i,j})$. The goal of the two players is to find (i,j), such that one of the following is satisfied:

1. $i = 1, j = 1$, and $e_{i,j} = 0$, or
2. $i = d$, and $e_{i,j} = 1$, or
3. $i \leq d-1$, and $(e_{i,j} = 1) \wedge (e_{i+1,j} = 0) \wedge (e_{i+1,j+1} = 0)$.

In other words, the goal is to find either a position in the bottom of the pyramid, with $e_{i,j} = 1$, or a position at the top (note that there is only one such position) with $e_{i,j} = 0$, or a "pyramid-triangle" $(i,j),(i+1,j),(i+1,j+1)$, such that $(e_{i,j} = 1) \wedge (e_{i+1,j} = 0) \wedge (e_{i+1,j+1} = 0)$. It is not hard to verify that it is always possible to achieve one of these goals.

3.3 The Complexity of *PYRGEN*

A simple protocol shows that $SC(PYRGEN(m,d)) \leq 2d-1$, and therefore that $CC(PYRGEN(m,d)) \leq (2d-1)(\log_2 m + 2)$. On the other hand, we have:

Lemma 3.2 *$SC(PYRGEN(m,d)) \geq d$.*

Using Lemma 3.2, and Theorem 2.1, we obtain that the *general* communication complexity of the $PYRGEN(m,d)$ game satisfies:

Corollary 3.3 *Assume $m \geq d^{40}$. Then $CC(PYRGEN(m,d)) = \Theta(d \cdot \log m)$.*

3.4 The Monotone Depth of *GEN*

We prove a lower bound for the communication complexity of the monotone KW-game for *GEN*. This is proved by a reduction to the communication complexity of *PYRGEN*. Let $l \stackrel{\text{def}}{=} l(m,d) \stackrel{\text{def}}{=} m \cdot \binom{d+1}{2} + 2$. We will consider a set of l *GEN*-elements. The first element, 1, will be the source, and the last element, l, will be the target. The other $m \cdot \binom{d+1}{2}$ elements are indexed by $((i,j),k)$, where $1 \leq j \leq i \leq d$, and $1 \leq k \leq m$, that is, (i,j) is a vertex of the pyramid, and $k \in [m]$. We therefore have m *GEN*-elements corresponding to each vertex of the pyramid. We think of the source as placed below the bottom of the pyramid, and we think of the target as placed above the top of the pyramid.

We say that a triple (v_1,v_2,v_3) of *GEN*-elements is **consistent with the structure of the pyramid** in one of the following cases:

1. v_1, v_2 are both the source, and v_3 corresponds to a vertex at the bottom of the pyramid (i.e., $v_3 = ((d,j),k)$, for some j,k).
2. v_3 is the target, and v_1, v_2 both correspond to vertices at the top of the pyramid (i.e., $v_1 = ((1,1),k_1)$, and $v_2 = ((1,1),k_2)$, for some k_1,k_2).
3. (v_1,v_2,v_3) corresponds to a triangle of the pyramid. That is, for some $i \leq d-1$, and some j, and some k_1,k_2,k_3, we have $v_1 = ((i+1,j),k_1), v_2 = ((i+1,j+1),k_2)$, and $v_3 = ((i,j),k_3)$.

Lemma 3.4 *$CC(PYRGEN(m,d))$ is at most the communication complexity of the monotone KW-game for the GEN function with $l(m,d)$ elements.*

We can now set $m = d^{40}$ and obtain:

Corollary 3.5 *For some $\epsilon > 0$, the monotone depth of GEN (with l elements) is $\Omega(l^\epsilon)$.*

Corollary 3.6 *Monotone-NC $\neq$ monotone-P.*

3.5 A Tight Monotone Depth Hierarchy

As before, consider a set of $l = l(m,d)$ *GEN*-elements. As before, the first element, 1, will be the source, and the last element, l, will be the target, and the other $m \cdot \binom{d+1}{2}$ elements are indexed by $((i,j),k)$, where (i,j) is a vertex of the pyramid, and $k \in [m]$. As before, a triple (v_1,v_2,v_3) of *GEN*-elements can be consistent or inconsistent with the structure of the pyramid.

To achieve tight lower bounds for monotone-P, let us introduce the following variation of *GEN*. Note that the *PYRAMID-GEN function* is not to be confused with the *PYRGEN game* defined in Section 3.2. Of course the definition of *PYRAMID-GEN* is specifically targeted for the game *PYRGEN*.

The *PYRAMID-GEN* function: The input is a string of l^3 bits $(t_{ijk})_{1\leq i,j,k\leq l}$. First, for every triple (i,j,k) that is not consistent with the structure of the pyramid, change t_{ijk} to 0. Now apply the *GEN* function on the new sequence (t_{ijk}). The output of *PYRAMID-GEN* on this input will be the output of *GEN*.

In other words: the function *PYRAMID-GEN* determines whether 1 (the source) generates l (the target), using only triples that are consistent with the structure of the pyramid. Note that triples that are not consistent with the structure of the pyramid can be removed from the input, and therefore the relevant input is of length lower than l^3.

Proposition 3.7 *PYRAMID-GEN(m,d) can be solved by a monotone polynomial size circuit family of depth $O(d \cdot \log m)$.*

Since the proof of Lemma 3.4 will apply to the function *PYRAMID-GEN* as well, we have:

Lemma 3.8 *$CC(PYRGEN(m,d))$ is at most the communication complexity of the monotone KW-game for the function PYRAMID-GEN(m,d).*

Corollary 3.9 *Assume $m \geq d^{40}$, then the monotone depth of PYRAMID-GEN(m,d) is $\Theta(d \cdot \log m)$.*

Now fix $m = d^{40}$, and use a standard padding argument (when needed), to get the following corollary:

Corollary 3.10 *There exist constants $\epsilon, c > 0$, such that for any integer function $D(n) \leq n^{\epsilon}$, there exists an explicit monotone function $F : \{0,1\}^n \rightarrow \{0,1\}$, that can be (uniformly) computed by a family of monotone Boolean circuits of polynomial size and of depth $D(n)$, and cannot be computed by any family of monotone Boolean circuits of depth less than $c \cdot D(n)$.*

As a result we obtain the following corollary:

Corollary 3.11 *For every $i \geq 0$, monotone-$NC^i \neq$ monotone-NC^{i+1}.*

4 Other Applications

In this section, we prove other applications of Theorem 2.1 and Theorem 2.2. Although both applications are very easy, in this version of the paper only a short sketch is given :

4.1 Lower Bound for *st*-Connectivity

As mentioned above, a tight lower bound of $\Omega(\log^2 n)$ was proved for *st*-connectivity in [KaWi88]. Here we show how to obtain that lower bound as an immediate consequence of Theorem 2.1. The proof, moreover, is different from the one given in [KaWi88].

For m, n, define the game $CONN(m,n)$ to be the following $DART(m,n)$ game: The input for Player I is $(x_i)_{1 \leq i \leq n}$, and for Player II $(y_i)_{1 \leq i \leq n}$. As before, for every i, define $e_i = y_i(x_i)$. The goal of the players is to find i such that one of the following is satisfied:
1. $i = 1$, and $e_i = 1$, or
2. $i = n$, and $e_i = 0$, or
3. $i \leq n-1$, and $(e_i = 0) \wedge (e_{i+1} = 1)$, or
4. $i \leq n-1$, and $(e_i = 1) \wedge (e_{i+1} = 0)$.

We claim that $SC(CONN(m,n)) = \lceil \log_2(n+1) \rceil$, (the upper bound and the lower bound are both trivial). Thus, by Theorem 2.1 we have for $m \geq n^{20}$, $CC(CONN(m,n)) = \Omega(\log n \cdot \log m)$.

To see the connection to *st*-connectivity, consider $n \cdot m + 2$ vertices: the two special vertices, s and t, and $n \cdot m$ other vertices, indexed by (i,j), where $1 \leq i \leq n$, and $1 \leq j \leq m$. It is not hard to see that given any protocol for the monotone KW-game for *st*-connectivity on these vertices, the two players can use that protocol to solve $CONN(m,n)$.

Now fix $m = n^{20}$, and use Lemma 3.1 to get that the monotone depth of *st*-connectivity is $\Omega(\log^2 n)$.

4.2 Lower Bound for k-Clique

For the monotone depth of the k-clique function, on a graph with n vertices, a lower bound of $\Omega(k)$ was proved in [RaWi90]. Obviously, for $k = \Omega(n)$, that lower bound is tight. For smaller values of k, however, the lower bound is not tight. Here we prove that for $k \leq n^{\epsilon}$ (for some small constant $\epsilon > 0$), the monotone depth of k-clique is $\Omega(k \cdot \log n)$. Obviously, this lower bound is tight.

For m, k, define the game $CLQ(m,k)$ to be the following $DART_{(k-1)}(m,k)$ game: The input for Player I is $(x_i)_{1 \leq i \leq k}$, and for Player II $(y_i)_{1 \leq i \leq k}$. As before, for every i, define $e_i = y_i(x_i)$. Note that $e_i \in [k-1]$. The goal of the players is to find i, j, such that $e_i = e_j$.

We claim that $SC(CLQ(m,k)) = k$, (the upper bound and the lower bound are both trivial). Thus, by Theorem 2.2 we have for $m \geq k^{1000}$, $CC(CLQ(m,k)) = \Omega(k \cdot \log m)$.

To see the connection to k-clique, consider $k \cdot m$ vertices, indexed by (i,j), where $1 \leq i \leq k$, and $1 \leq j \leq m$. It is not hard to see that given any protocol for the monotone KW-game for k-clique on these vertices, the two players can use that protocol to solve $CLQ(m,k)$.

Now fix $m = k^{1000}$, and use Lemma 3.1 to get that for $k \leq n^{1/1001}$, the monotone depth of k-clique is $\Omega(k \cdot \log n)$.

5 Thickness and Predictability

In this section, we present some of our main tools, and notations, used for the proof of Theorem 2.1. The "average-degree", $AVDEG_j(A)$, defined below, is analogous to the *predictability* notion, introduced by [EdImRuSg91]. Our proof uses tools and intuitions from [EdImRuSg91].

Let $X = [m]^n$. As before, let A be a subset of X. The bipartite graph $GRAPH_1(A)$ is defined in the following way: Consider a bipartite graph with disjoint vertex sets $V_L = [m]$ (the "left nodes"), and $V_R = [m]^{n-1}$ (the "right nodes"). The set of edges E contains all the pairs $(x_1, (x_2, \ldots, x_n))$ s.t., $(x_1, x_2, \ldots, x_n) \in A$. In other words, the set of edges is the set A, where each $(x_1, x_2, \ldots, x_n) \in A$ is viewed as an edge between the "left" node x_1 and the "right" node $(x_2, \ldots, x_n)$.

For every $1 \leq j \leq n$, the bipartite graph $GRAPH_j(A)$ is now defined in the same way, where as before $V_L = [m]$, $V_R = [m]^{n-1}$, and each $(x_1, x_2, \ldots, x_n) \in A$ is viewed as an edge between the "left" node x_j and the "right" node $(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n)$.

For the graph $GRAPH_j(A)$, define the set $\hat{V}_j$ to be the set of all nodes in V_R with non-zero degree (that is, the set of right nodes with non-zero degree). The average-degree $AVDEG_j(A)$ is defined to be the average degree of a right node in $\hat{V}_j$ in the graph $GRAPH_j(A)$, that is, $AVDEG_j(A) = |A|/|\hat{V}_j|$. Using different notations, we define $AVDEG_j(A)$ by

$$AVDEG_j(A) = \frac{|A|}{|A_{[n]\setminus\{j\}}|},$$

where $A_{[n]\setminus\{j\}}$ denotes the projection of A on $(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n)$, that is, $|A_{[n]\setminus\{j\}}|$ is the number of assignments to $(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n)$ such that there exists at least one assignment to x_j satisfying $(x_1, \ldots, x_{j-1}, x_j, x_{j+1}, \ldots, x_n) \in A$.

Observe that $AVDEG_j(A)$ ranges from m to 1. When $AVDEG_j(A) = 1$, x_j is fixed as a function of $(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n)$. In that case, "the j^{th} slot is totally predictable" in the sense that knowing $(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n)$ determines the value of x_j. When $AVDEG_j(A) = m$, the degree of every right node is precisely m, (since the average degree of the right nodes is m and clearly m is also the maximum degree of any right node). In that case, $(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n)$ gives no information on x_j.

While $AVDEG_j(A)$ is the *average* degree of right nodes in $\hat{V}_j$, we will also be interested in the minimal degree of such a right node. Define $MINDEG_j(A)$ to be the minimal degree of a right node in $\hat{V}_j$, in the graph $GRAPH_j(A)$.

The **thickness of** A is now defined by

$$Thickness(A) = \min_{1 \leq j \leq n} MINDEG_j(A).$$

5.1 Some Useful Observations

The following observation is analogous to [EdImRuSg91, Lemma 4]:

Claim 5.1 *Let $A' \subseteq A$. Then for every j,*

$$AVDEG_j(A') \geq \frac{|A'|}{|A|} \cdot AVDEG_j(A).$$

The projection set $A_{[n]\setminus\{j\}}$ can be viewed as a subset of $[m]^{n-1}$. For $n \geq 2$ and for every $i \in \{1, \ldots, j-1, j+1, \ldots, n\}$, one can define, as before, the bipartite graph $GRAPH_i(A_{[n]\setminus\{j\}})$, and the minimal and average degrees, $MINDEG_i(A_{[n]\setminus\{j\}})$ and $AVDEG_i(A_{[n]\setminus\{j\}})$. The following claim shows that the thickness of $A_{[n]\setminus\{j\}}$ is never smaller than the thickness of A, that is, projections never decrease the thickness.

Claim 5.2 *For any j,*

$$Thickness(A_{[n]\setminus\{j\}}) \geq Thickness(A).$$

5.2 The Thickness Lemma

The following lemma is our most important technical tool. It shows that if for a set A, $AVDEG_j(A)$ is large for every j, then there exists a large subset A' of A, such that $Thickness(A')$ is large. We will first state the lemma in a general form, and then restate it in a simpler form that will be used herein.

Lemma 5.3 *If for some $1 > \delta > 0$, and for every j, $AVDEG_j(A) \geq \delta \cdot m$ then for any $\alpha \geq 0$, there exists $A' \subset A$, such that: $|A'| \geq (1-\alpha)|A|$, and $Thickness(A') \geq \Delta$, where*

$$\Delta \stackrel{\text{def}}{=} \frac{(1-\alpha)\delta m}{n \cdot [1 + \alpha^{-1} \cdot \ln(\delta^{-1})]}.$$

Corollary 5.4 *Assume that $m \geq n^{20}$. If for every j, $AVDEG_j(A) \geq 4 \cdot m^{19/20}$ then there exists $A' \subset A$, such that: $|A'| \geq |A|/2$, and*

$$Thickness(A') \geq m^{17/20}.$$

6 Proof of the Main Theorem

In this section we sketch the proof of Theorem 2.1. The extension of Theorem 2.1 to the case of multi-color dart games (Theorem 2.2) is straight forward, and is not discussed in this version of the paper.

Assume that $m \geq n^{20}$, and assume (for simplicity) that $m^{1/20}$ is larger than some big constant (say $m^{1/20} \geq 1000$). As before, we denote by X the set $[m]^n$, we denote by Y the set $(\{0,1\}^m)^n$, and we denote by $R \subseteq X \times Y \times Z$ a relation in $DART(m,n)$. As before, we denote by A a subset of X, and we denote by B a subset of Y. As before, for a relation R, and for two subsets $A \subseteq X$, and $B \subseteq Y$, we denote by $C_R(A,B)$ the deterministic communication complexity of the relation R, over the domain $A \times B$.

We measure the size of A, B by $\alpha = \log_2(|X|/|A|)$, $\beta = \log_2(|Y|/|B|)$, that is, α, β are the number of bits of information known about A, B respectively. We will be interested in sets A with $Thickness(A) \geq m^{17/20}$. Such a set A is said to be *thick*.

For any $\alpha, \beta, k \geq 0$, and $m \geq 1000^{20}$, denote by $GAMES_m[\alpha, \beta, k]$ the set of all triples (R, A, B) such that for some $n \leq m^{1/20}$:

1. R is a relation in $DART(m,n)$, s.t. $SC(R) \geq k$.

2. A is a thick subset of X, s.t. $\log_2(|X|/|A|) \leq \alpha$, i.e.,

$$Thickness(A) \geq m^{17/20},$$

$$|A| \geq 2^{-\alpha} \cdot |X|.$$

3. B is a subset of Y, s.t. $\log_2(|Y|/|B|) \leq \beta$, i.e.,

$$|B| \geq 2^{-\beta} \cdot |Y|.$$

$COMP_m[\alpha, \beta, k]$ is now defined to be the minimum of $C_R(A, B)$, over all triples $(R, A, B) \in GAMES_m[\alpha, \beta, k]$. We will prove here a general lower bound for $COMP_m[\alpha, \beta, k]$.

Given α, β, k, m, let $(R, A, B) \in GAMES_m[\alpha, \beta, k]$ be a triple with minimal $C_R(A, B)$, that is,

$$C_R(A, B) = COMP_m[\alpha, \beta, k].$$

To bound $C_R(A, B)$ we will consider two cases:

1. CASE 1: For every j, $AVDEG_j(A) \geq 8 \cdot m^{19/20}$.
2. CASE 2: For some j, $AVDEG_j(A) < 8 \cdot m^{19/20}$.

6.1 A Recursive Bound in CASE 1

To bound $C_R(A, B)$ in the first case, we use the following lemma:

Lemma 6.1 *For any $\alpha, \beta, k, m \geq 0$, with $\beta \leq m^{2/20}$, and $m \geq 1000^{20}$, and for any $(R, A, B) \in GAMES_m[\alpha, \beta, k]$, if for every $1 \leq j \leq n$, $AVDEG_j(A) \geq 8 \cdot m^{19/20}$ then*

$$C_R(A, B) \geq MIN(\, COMP_m[\alpha + 2, \beta, k]\,,$$

$$COMP_m[\alpha, \beta + 1, k]\,) + 1.$$

Proof. First, we prove that $C_R(A, B)$ is not 0 (proof omitted in this version). Now, let P be the best protocol for solving R over $A \times B$, that is, a protocol with communication complexity $C_R(A, B)$. Consider the first bit transmitted by P. That bit is transmitted either by Player I or by Player II.

If Player II transmits the first bit then partition the set B into $B = B_0 \cup B_1$, according to the bit transmitted, that is, B_0 is the set of inputs (for Player II) where 0 is transmitted, and B_1 is the set of inputs where 1 is transmitted. Obviously, $|B_0| + |B_1| = |B|$. W.l.o.g., assume that $|B_0| \geq |B|/2$, and consider the triple (R, A, B_0). The protocol P solves R on $A \times B_0$, using only $C_R(A, B) - 1$ communication bits (since one bit was already transmitted). Since (R, A, B_0) is obviously in $GAMES_m[\alpha, \beta + 1, k]$, we have in this case $COMP_m[\alpha, \beta + 1, k] \leq C_R(A, B) - 1$.

If Player I transmits the first bit then partition the set A into $A = A_0 \cup A_1$, according to the bit transmitted, and assume w.l.o.g. that $|A_0| \geq |A|/2$. A_0 is not necessarily thick, and therefore (R, A_0, B) is not necessarily in $GAMES_m[\alpha + 1, \beta, k]$. However, since for every j, $AVDEG_j(A) \geq 8 \cdot m^{19/20}$, we know by Claim 5.1 that for every j, $AVDEG_j(A_0) \geq 4 \cdot m^{19/20}$. Therefore, by Corollary 5.4, there exists $A' \subset A_0$ such that $|A'| \geq |A_0|/2$, and $Thickness(A') \geq m^{17/20}$. Therefore, $(R, A', B) \in GAMES_m[\alpha + 2, \beta, k]$. Since P solves R on $A' \times B$, using only $C_R(A, B) - 1$ communication bits, we have $COMP_m[\alpha + 2, \beta, k] \leq C_R(A, B) - 1$. ∎

6.2 A Recursive Bound in CASE 2

In the second case, we use the following lemma to bound $C_R(A, B)$:

Lemma 6.2 *For any $\alpha, \beta, k, m \geq 0$, with $\beta \leq m^{2/20}$, $k \geq 1$, and $m \geq 1000^{20}$, and for any $(R, A, B) \in GAMES_m[\alpha, \beta, k]$, if for some $1 \leq j \leq n$, $AVDEG_j(A) < 8 \cdot m^{19/20}$ then*

$$C_R(A, B) \geq COMP_m[\alpha + 3 - (\log_2 m)/20, \beta + 1, k - 1].$$

Proof. W.l.o.g., assume that $j = n$, that is,

$$AVDEG_n(A) < 8 \cdot m^{19/20}.$$

Since $Thickness(A) \geq m^{17/20}$,

$$MINDEG_n(A) \geq m^{17/20}.$$

Denote by R_0 the restriction of the relation R to the first $n - 1$ coordinates, derived by fixing $e_n \stackrel{\text{def}}{=} y_n(x_n)$ to be 0, and denote by R_1 the restriction derived by fixing $e_n \stackrel{\text{def}}{=} y_n(x_n)$ to be 1. Obviously, both R_0, R_1 are relations in $DART(m, n - 1)$. Since $SC(R) \geq k$, at least one of $SC(R_0), SC(R_1)$ is $\geq k - 1$. W.l.o.g., assume that

$$SC(R_0) \geq k - 1.$$

We will prove the lemma by showing the existence of $A' \subset [m]^{n-1}$, and $B' \subset (\{0,1\}^m)^{n-1}$, such that

$$(R_0, A', B') \in GAMES_m[\alpha + 3 - (\log_2 m)/20, \beta + 1, k - 1],$$

and $C_{R_0}(A', B') \leq C_R(A, B)$. Therefore, we will have

$$C_R(A, B) \geq C_{R_0}(A', B') \geq$$

$$COMP_m[\alpha + 3 - (\log_2 m)/20, \beta + 1, k - 1],$$

which proves the lemma.

For every subset $U \subset [m]$, let us define sets $A_U \subset [m]^{n-1}$, $B_U \subset (\{0,1\}^m)^{n-1}$. The sets A', B' above will be the sets A_U, B_U for some particular choice of U.

- The set A_U is defined in the following way: $(x_1, \ldots, x_{n-1}) \in A_U$ iff there exists an element $v \in U$, such that $(x_1, \ldots, x_{n-1}, v) \in A$. In other words, A_U is the set of all right nodes in the graph $GRAPH_n(A)$ that are connected by an edge to (at least one) element of the set U (viewed as a subset of the set of left nodes).

- The set B_U is defined in the following way: $(y_1, \ldots, y_{n-1}) \in B_U$ iff there exists a coloring $w \in \{0,1\}^{[m]}$, such that all elements of U are colored 0 by w, and such that $(y_1, \ldots, y_{n-1}, w) \in B$.

Claim 6.3 $\forall U \subset [m]$, $C_{R_0}(A_U, B_U) \leq C_R(A, B)$.

To complete the proof of the lemma, we still have to prove that for some $U \subset [m]$, $(R_0, A_U, B_U) \in GAMES_m[\alpha + 3 - (\log_2 m)/20, \beta + 1, k - 1]$. To prove this we still have to show that for some U:

1. $|A_U| \geq 2^{-[\alpha+3-(\log_2 m)/20]} \cdot m^{n-1}$,
2. $|B_U| \geq 2^{-[\beta+1]} \cdot 2^{m\cdot(n-1)}$, and
3. $Thickness(A_U) \geq m^{17/20}$.

We will use a probabilistic argument:

Let U be a random subset of $[m]$, of size $m^{5/20}$ (we assume for simplicity that $m^{5/20}$ is an integer). The following claim shows that with high probability $A_U = A_{[n]\setminus\{n\}}$, (that is, A_U contains every single element of $A_{[n]\setminus\{n\}}$).

Claim 6.4 *For a random set U of size $m^{5/20}$,*

$$Prob_U \left[A_U = A_{[n]\setminus\{n\}}\right] \geq 3/4.$$

The following claim shows that with high probability B_U is large.

Claim 6.5 *For a random set U of size $m^{5/20}$,*

$$Prob_U \left[|B_U| \geq |B|/2^{m+1}\right] \geq 3/4.$$

By Claim 6.4, and Claim 6.5 it follows that with probability of at least $1/2$ we have both:
1. $A_U = A_{[n]\setminus\{n\}}$, and
2. $|B_U| \geq |B|/2^{m+1}$.
Take a set U that satisfies both. Since $A_U = A_{[n]\setminus\{n\}}$, we have by Claim 5.2,

$$Thickness(A_U) \geq Thickness(A) \geq m^{17/20}.$$

Also, since $|A|/|A_{[n]\setminus\{n\}}| = AVDEG_n(A) \leq 8 \cdot m^{19/20}$, we have

$$|A_U| \geq \left(8 \cdot m^{19/20}\right)^{-1} \cdot |A| \geq \left(8 \cdot m^{19/20}\right)^{-1} \cdot 2^{-\alpha} \cdot m^n =$$

$$2^{-[\alpha+3-(\log_2 m)/20]} \cdot m^{n-1}.$$

Since $|B_U| \geq |B|/2^{m+1}$, we have

$$|B_U| \geq 2^{-\beta} \cdot 2^{m\cdot n}/2^{m+1} = 2^{-(\beta+1)} \cdot 2^{m\cdot(n-1)}.$$

Thus, A_U, B_U satisfy the required properties, and Lemma 6.2 follows. ■

6.3 Explicit Bound for $COMP_m[\alpha, \beta, k]$

Lemma 6.1, and Lemma 6.2 immediately give the following recursive bound for $COMP_m[\alpha, \beta, k]$.

Corollary 6.6 *For any $\alpha, \beta, k, m \geq 0$, with $\beta \leq m^{2/20}$, $k \geq 1$, and $m \geq 1000^{20}$, $COMP_m[\alpha, \beta, k] \geq MIN(C_1, C_2, C_3)$, where*

$$C_1 = COMP_m[\alpha + 2, \beta, k] + 1,$$
$$C_2 = COMP_m[\alpha, \beta + 1, k] + 1,$$
$$C_3 = COMP_m[\alpha + 3 - (\log_2 m)/20, \beta + 1, k - 1].$$

Using the recursive bound, it is now easy to prove (by induction) explicit bounds for $COMP_m[\alpha, \beta, k]$: Denote by $BOUND_m[\alpha, \beta, k]$ the function:

$$BOUND_m[\alpha, \beta, k] \stackrel{\text{def}}{=} k \cdot [(\log_2 m)/20 - 5]/2 - \alpha/2 - \beta.$$

Theorem 6.7 *For any $\alpha, \beta, k \geq 0$, and $m \geq 1000^{20}$,*

$$COMP_m[\alpha, \beta, k] \geq BOUND_m[\alpha, \beta, k].$$

One can now take in Theorem 6.7; $\alpha = 0, \beta = 0$, to get for $m \geq 1000^{20}$,

$$COMP_m[0, 0, k] \geq k \cdot [(\log_2 m)/20 - 5]/2 = k \cdot \Omega(\log m),$$

which proves Theorem 2.1.

7 Conclusions

We have shown that for m larger than some polynomial in n, the communication complexity of the best protocol for a $DART(m, n)$ game is bounded from below by the communication complexity of the best structured protocol for that game. As a result, we obtained lower bounds for the monotone-depth of several functions.

We claim that our method gives lower bounds for the monotone depth of many other functions. Informally, we argue the following:
1. The monotone Karchmer-Wigderson's games corresponding to many functions can be reduced to dart games.
2. Proving lower bounds for the best structured protocol is usually not hard.

More formally, it is not hard to see that **every** dart game is in fact (a sub-case of) a monotone Karchmer-Wigderson's game for some function !

As for the lower bounds for the best structured protocol, we have already seen several examples were the argument was very easy (or trivial). In general, as mentioned above, given a relation R (with a DNF tautology F_R), the structured complexity of R is the same as the depth of the best decision tree for the corresponding DNF-search problem, over the variables $e_1, \ldots, e_n$. As observed by V. Chvatal and E. Szemeredi, this is also

the same as the depth of the best regular Resolution proof for F_R (for details see [LoNeNaWi95]).

We can therefore conclude that any lower bound for regular Resolution implies a lower bound for the corresponding dart game. As mentioned in the introduction, lower bounds for monotone complexity were used before to derive lower bounds for propositional proof systems (e.g., for Cutting-Planes and for Resolution). Here, we conclude that the other direction is also possible.

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