Exponential-Time Hypothesis (ETH)

The worst-case complexity of 3-SAT on n variable formulas with $O(n)$ clauses is $\geq 2^{\delta n}$ for some $\delta > 0$.

Could have as many $\underline{m} = \Theta(n^3)$ clauses

IS graph N nodes $N = O(n+m) = O(n^3)$

$$2^{\Omega(n)} = 2^{\Omega(N^{1/3})}$$

Sparsification Lemma: Let $\varepsilon > 0$, $k \geq 3$ be a constant.
There is a $2^{\varepsilon n}$ poly(n) time that takes a k-CNF formula F and produces k-CNFs $F_1, \ldots, F_{2^{\varepsilon n}}$ s.t. $F \equiv \bigvee_i F_i$ and each F_i has $(\underline{k/\varepsilon})^{O(k)} n$ clauses

Cor: ETH $\Rightarrow$ ETH for sparse formulas ($O(n)$ clauses)

Proof: Suppose ETH for sparse formulas is false.
For any $\varepsilon' > 0$ can solve sparse formulas in $< 2^{\varepsilon' n}$ time.

negation of ETH

$\boxed{\text{Choose some } \varepsilon'' > 0 \text{ and we'll solve general formulas in } < 2^{\varepsilon'' n} \text{ time.}}$

Set $\varepsilon' = \varepsilon''/3$, $\varepsilon = \varepsilon''/3$.

$2^{\varepsilon''/3 n}$ poly time to create $F_1 \ldots F_{2^{\varepsilon n}}$

run our sparse alg in $2^{\varepsilon' n} = 2^{\varepsilon'' n/3}$ time per formula

$\mathrm{poly}(n)\cdot 2^{\frac{2\varepsilon'}{3}n}$ time to decide sat for all of $F_1, \ldots, F_{2^{\varepsilon n}}$. ∎

contradicts ETH

Cor (1) ETH $\Rightarrow$ $\exists$ constant $\delta_k > 0$ s.t. k-CNF formulas in n vars requires time $\geq 2^{\delta_k n - o(n)}$ time

(2) ETH is equivalent to each of the following

(a) IS, VERTEX-COVER, DOMINATING SET on graphs with n edges require time 2^{cn} for $c>0$

(b) CLIQUE on n-vertex graphs

(c) SUBSET-SUM on n, n-bit integers

(d) HAMILTONIAN-PATH, CIRCUIT

(e) 3-COLOR, k-COLOR on n-edge graphs

ETH is very robust

A. K. Dewdney

Proof of Sparsification Lemma:

Key idea (h, p)-flower

heart, petal

set of literals

- a collection of (h+p)-clauses whose intersection has size h.

heart $H \neq \emptyset$

Threshold θ_p $\quad \theta_0 = 2$ increasing

$C_1 \ldots C_s$ $\quad s \geq \theta_p$

$P = \{C_i \setminus H : i \in [s]\}$ set of petals $|P| \geq \theta_p$

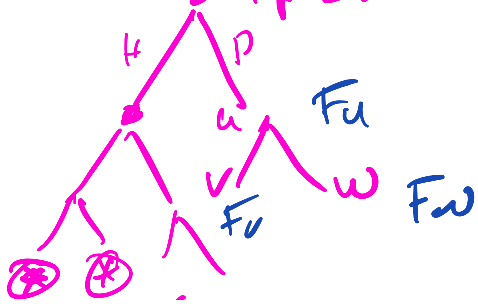
If assignment satisfies all clauses C_S, "Either an assignment satisfies H or satisfies all clauses in P"

Reduce: if formula contains clause C, D, $C \subseteq D$, remove D.

Function Sparsify(F')

- $F' \leftarrow$ reduce(F')
- if there is some (h,p)-flower F^* in F' (for any h,p)
  - Choose the flower s.t. $h+p$ is minimized and h is maximized (p is minimized)
  - H be heart of F^*
  - P be set of petals of F^*
  - Sparsify($F' \cup \{H\}$)
  - Sparsify($F' \cup P$)
- else
  - Append formula F' to the list of output formulas

r F_r = reduce(F)



$F_u \equiv F_v \vee F_w$

$\beta_1 = 2 \quad \beta_j = \sum_{i=1}^{j-1} 4\alpha\beta_i\beta_{j-i}$ $\quad \theta_0 = 2$ $\quad \theta_j = \alpha\beta_j$ $\quad \alpha \geq 2$ $\quad j < k$

We say that F' is <u>k'-sparsified</u> iff
for every $j \leq k'$
no (h,p) flowers with <u>$h+p=j$</u>

<u>Lemma</u> Suppose that F_u is <u>k'-sparsified</u>

(a) If $|C'| = h < j \leq k'$ there are fewer than θ_{j-h} $\#p$ j-clauses $C \in F_u$ containing C'

$\geq \theta_{j-h}$ clauses containing C'

common intersection H

$C' \subseteq H$

$|H| \geq h$

<u>$\theta_{j-|H|} \leq \theta_{j-h}$</u>

(b) For $j \leq k'$ F_u has fewer than $2\theta_{j-1} n/j$ <u>j-clauses</u>.

$2n$ literals among the j-clauses

If not then are $2n\theta_{j-1}$ literal occurrences

$\therefore \exists$ one literal in at least Θ_{j-1} j-clauses that would be a $(1, j-1)$-flower contradiction sparsity

Cor every leaf has at most $C_{n,\varepsilon} n$ clauses

for $C_{n,\varepsilon} = \sum_{j=1}^{n} \frac{2\Theta_{j-1}}{j}$

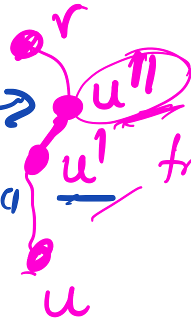
Pf Every leaf is k-sparsified

Lemma If a new clause C' with $|C'| = i < j$ eliminates any new j-clause from F_a then eliminates at most $\boxed{2\Theta_{j-i} - 2}$ total j clauses both old and new

C'

Proof Suppose that i eliminates $\geq 2\Theta_{j-i}-1$ total j-clauses & at least one new j-clause

not at the root



j-spanned

added a j clause

first node where all these eliminated clauses were present

$u' \neq r$

u'' parent of u'

at most $\Theta_{j-i}-1$ clauses contain C'

$\therefore$ Added $\geq \Theta_{j-i}$ new j clauses going from u'' to u' that contain C'

$\geq \Theta_{j-i}$ petals containing C' at node u''

size i

~~Let~~ Let $H_{u''}$ be heart of node u''

all these petals contain $H_{u''} \cup C'$ $\therefore$

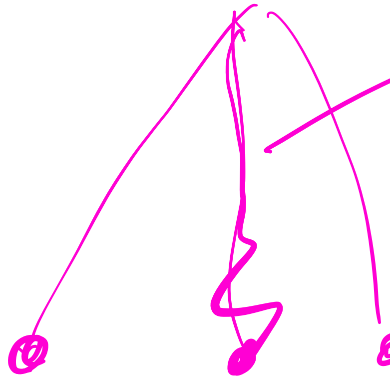
h disjoint

$h+i$

$h+j'$ clause size

$(h+i, j-i')$ flower

QED



\# of new clauses of size $\le j$ on any path
N_j any path
$\ge$ # of steps on a path

$\le$ # of j-clauses at end $2n\Theta_{j-1}$

+ # of new clauses eliminated along path

$$N_j \le N_{j-1} + \sum_{i=1}^{j-1} (2\Theta_{j-i} - 2) N_i + 2n\Theta_{j-1}$$

at end

...

$N_j \le \beta_j n$

$\Rightarrow$ Path of length $\le \beta_{k-1} n$

\# petal steps on a path is not too big

$(k-1)n/\alpha$.

Fix j count # of petal

steps of size j-clauses

n/α

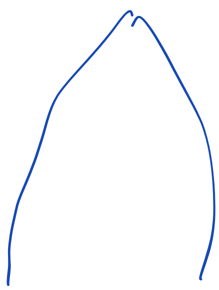
\# of new j-clauses

$\le \beta_j n$

each petal step creating j-clause creates $\ge \theta_j$ petals

$$\boxed{\alpha = \frac{(k-1)^2}{\varepsilon}\log\left(\frac{32(k-1)^2}{\varepsilon}\right)}$$

$$\frac{\beta_j n}{\theta_j} = n/\alpha .$$



ETH polytime

SETH polytime

Fine grained reduction

$$\sum_{i=0}^{\ell}\binom{n}{i} \le 2^{H_2(\ell/n)\cdot n}$$

$i \le \frac{\ell}{n}$

binary entropy

$$H_2(\gamma) = \gamma\log\left(\frac{1}{\gamma}\right) +$$

$(1-\gamma)\log_2\left(\frac{1}{1-\gamma}\right)$