Exponential-Time Hypothesis (ETH)

The worst-case complexity of 3-SAT on $n$ variables formula $F$ is $\geq 2^{\delta n}$ for some $\delta > 0$.

Could have at most $m = O(n^2)$ clauses.

If graph $G$ has $N = O(n^{3/2})$ nodes, then $2^{\Omega(n)} = 2^{2^{O(n^{1/2})}}$.

Space-Saving Lemma: Let $\varepsilon > 0$, $k > 3$ be a constant.

There is a $2^{ckn}$ poly-time that takes a $k$-CNF formula $F$ and produces $k$-CNFs $F_1, \ldots, F_{2^{ckn}}$ such that $F \equiv v F_i$ and each $F_i$ has $(k^{1/3})^{\Omega(n)}$ clauses.

See ETH $\Rightarrow$ ETH for sparse formulas ($O(n)$ clauses).

Proof: Suppose ETH for sparse formulas is false.

For any $\varepsilon > 0$, we can solve sparse formulas in $2^{\varepsilon n}$ time.

Choose some $\varepsilon'' > 0$ and we'll solve several formulas in $2^{\varepsilon'' n}$ time.

Set $\varepsilon' = \varepsilon''/3$, $\varepsilon = \varepsilon''/3$.

$2^{\varepsilon''^{1/3} n}$ poly-time to create $F_1, \ldots, F_{2^{\varepsilon''^{1/3} n}}$.

Use our sparse alg. in $2^{\varepsilon''^{1/3} n} = 2^{\varepsilon'' n/3}$ time for formula...
poly(2^{\frac{2n}{3}}) \text{ this to decide sat is all of } F_1, \ldots, F_{2^n n} \text{. This contradicts ETH.}

Cor (1) ETH \implies \exists \text{ constant } S \geq 0 \text{ s.t. } \exists \text{ CNF}

formulas in } n \text{ vars requires time } \geq 2^{S^{n^2} \cdot \text{poly}(n)} \text{ from }

(1) \text{ ETH is equivalent to each of the following:}

(a) \text{ IS, VERTEX-COVER, DOMINATING SET on graphs with } n \text{ edges require time } \geq 2^{cn^{2/3}} \text{ for } c > 0

(b) \text{ CLIQUE on } n \text{-vertex graphs}

(c) \text{ SUBSET-SUM on } n \text{-bit integers}

(d) \text{ HAMILTONIAN-PATH, CIRCUIT-}

(e) \text{ 2-COLOR, R-COLOR on } n \text{-edge graphs}

Proof of Sparsified Lemma

Key idea: (chiral) flower

\text{ a collection of chiral-clauses, whose intersection has size } 4. \text{ heart H+}

\text{ Threshold } \Theta_p \text{ increases}

C_1 \ldots C_s \geq \Theta_p
\[ P = \{ C_i : i \in S \} \]

If any subformula \( \phi \) of \( P \) or satisfies all clauses in \( P \)

Reduce if formula contains clause \( C \& D \), remove \( D \).

Function \( \text{Sparsity}(F') \)

\[ F' = \text{reduce}(F') \]

If there is some \( F \) in \( F' \) (for any \( i \))

Choose the clause \( \phi \) to avoid \( \phi \) is minimized

and \( h \) is maximized

\( H \) be heart of \( F' \)

\( P \) be set of petals of \( F' \)

\[ \text{Sparsity}(F' \cup H) \]

\[ \text{Sparsity}(F' \cup P) \]

else

Append formula \( F' \) to the list of output formulas

\[ F_r = \text{reduce}(F) \]

\[ F_u \]

\[ F_r = F_u \lor F_v \]

\[ F_u \lor F_v \]

\[ F_r \lor F_u \lor F_v \]
\[ \begin{align*} 
\beta_r &= 2 \\
\beta_j &= \sum_{i=1}^{j-1} \alpha \beta_i \beta_{j-i} \\
\theta_0 &= 2 \\
\theta_j &= \alpha \beta_j \\
\alpha &\geq 2 \\
\end{align*} \]

We say that \( F' \) is \( k' \)-saturated if for every \( j \leq k' \) no clause contains \( h+p=j \).

**Lemma**

Suppose that \( F' \) is \( k' \)-saturated.

(a) If \( |C'| = h < j \leq k' \) there are fewer than \( \theta_j \) \( j \)-clauses (counted w.r.t. \( C' \) only).

(b) For \( j \leq k' \) \( F' \) has fewer than \( 2\theta_j - 1 \) \( h+j \)-clauses.

2n literals among the \( j \)-clauses.

If not, two are 2n \( \theta_{j-1} \) literal occurrences.
\[ \exists \text{ one literal in at least } \Omega_{j-1} \text{-clause} \]

That would be a \((1, j-1)\)-clause contradiction, contradicting \( \text{ every leaf has at most } C_{x, j} \text{ \(j\)-clause} \)

\[ C_{x, j} = \frac{2\Omega_{j-1}}{j} \]

Proof: Every leaf is \(k\)-spanned

**Lemma:** If a new clause \( C' \) with \( |C'| = i < j \) eliminates any \( j\)-clause from \( F_j \), then it eliminates at most \( 2\Omega_{j-i-2} \) total \( j\)-clauses, both old and new.
Suppose that \( \Delta(2(i-1)) \) total \( j \)-clauses, and at least one new \( j \)-clause.

\( j \)-spans \( u \) added a \( j \)-clause.

not at the root

\( u \) first node where all \( j \)-clauses were cleared.

\( u \neq \emptyset \) parent of \( u' \)

\( u'' \) parent of \( u' \)

\( \Theta j-i-1 \)

\( C' \) cluster contains \( C \)

\( \emptyset \) Added \( \geq \Theta j-i \) new \( j \)-clauses.

\( \emptyset \) going from \( u'' \) to \( u' \)

\( \emptyset \) that contain \( C' \)

\( \emptyset \) \( \Theta j-i \) petal contains \( C' \)

\( \emptyset \) at node \( u'' \)

let \( H_{u''} \) be center of node

all these petals contain \( H_{u''} \cup C' \)

\( H_{u''} \cup C' \) is disjoint

\( h+i \) (cluster) domain

\( (h+i, j-i) \)

\( H_{u''} \) (domain)
# of new clauses on any path \( N_j \) \[ \geq \text{# of steps on a path} \]

\[ \leq \text{# of clauses at end} \quad 2n \Theta_{j-1} \]

\[ + \text{# of new clauses eliminated along path} \]

\[ N_j \leq N_{j-1} + \sum_{i=1}^{2^{j-1}} (2^{(2j-i-2)}N_i + 2n\Theta_i) \]

\[ \quad \text{end} \]

\[ N_j \leq \beta j n \]

\[ \Rightarrow \text{path of length } \leq \beta j n \]

# petal steps on a path is not too big

\[ (k-1)n/d \]

For \( j \) count # of petals
\[ \alpha = \left( \frac{k-1}{2} \right)^2 \log \left( \frac{32(k+1)^2}{c} \right) \]

Steps of size j-clauses:

\[ \frac{n}{\Delta} \]

\# of new j-clauses:

\[ \leq \beta \cdot n \]

Each petal step creates j-clause:

\[ \geq \Theta \text{ petal} \]

\[ \frac{\beta \cdot n}{\Theta} = \frac{n}{\Delta} \]

\[ \text{ETH} \rightarrow \text{polym} \]

\[ \text{SETH} \rightarrow \text{polym} \]

Fine-grained reductions:

\[ \frac{h}{2} \left( \binom{\frac{n}{h}}{i} \right) \leq 2 \frac{h}{2} \left( \frac{n}{h} \right)^{2i} \]

\[ h_2 \left( \frac{1}{b} \right) \]

Binary entropy:

\[ h_2 \left( \frac{1}{b} \right) = b \log \left( \frac{1}{b} \right) + c \]
\( R = \frac{-1}{\ln(1 - P)} \)