Approximate Counting and Mixing Time of Markov Chains **Fall 2024**

Lecture 13: Matroids

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Disclaimer: *These notes have not been subjected to the usual scrutiny reserved for formal publications.*

In this lecture we discuss applications of the spectral independence machinery to sampling bases of matroids. A *matroid* $M = ([n], \mathcal{I})$ is defined on a ground set of elements, say $[n] = \{1, \ldots, n\}$ and a family of *independent sets* $\mathcal{I} \subseteq 2^{[n]}$ that satisfies the following properties:

Downward Closed: If $A \in \mathcal{I}$ then for any $B \subseteq A$, we have $B \in \mathcal{I}$.

Exchange Property: If $A, B \in \mathcal{I}$ and $|A| > |B|$ then there is an element $i \in B \setminus A$ such that $A \cup \{i\} \in \mathcal{I}$.

It follows from the exchange property that all *maximal* independent sets of a matroid M have the same size. Any maximal independent set of a matroid is called a *base* of the matroid.

Given any $S \subseteq [n]$, the rank of S, $r(S)$ is defined as follows:

$$r(S) = \max_{I \subseteq S, I \in \mathcal{I}} |I|,$$

i.e., it is the size of the largest independent set in S. So, if S is an independent set, then $r(S) = |S|$.

Matroids were defined and studied by Whitney around one hundred years ago in order to generalize the notion of linear independence in vector spaces. It is not hard to see that for any set of vectors $v_1, \ldots, v_n$ over a field F we can define a matroid, where any sets $A \subseteq [n]$ is an independent set if the corresponding set of vectors are linearly independent. The notion of rank in this case is the same as the rank of the vector space defined by of $v_1, \ldots, v_n$. Such a matroid is called a *linear* matroid.

Another famous example of matroids is the *graphic matroid*. Here, $[n]$ is the set of edges of a graph G and a set of edges form an independent set if they *do not* induce a cycle. It is not hard to see that graphic matroids are special cases of linear matroids. If G is connected, then bases of its graphic matroid are exactly spanning trees of G.

Bases Generating Polynomial. Given a matroid $M = ([n], \mathcal{I})$ of rank r, the *bases generating polynomial* of M is defined as follows:

$$g_M(z_1, \ldots, z_n) = \sum_{B\text{: base of } M} z^B,$$

where as usual, $z^B = \prod_{i \in B} z_i$. One of the goals of this course is to study properties of this polynomial.

Kruskal's Algorithm. Given a matroid M and a weight function $w : [n] \to \mathbb{R}_{\geq 0}$ we can run the following Greedy algorithm (which is analogue of the Kruskal's algorithm) to find the maximum weight base of M: Sort elements of M with respect to w and without loss of generality assume $w_1 \geq w_2 \geq \cdots \geq w_n$. Let $S = \emptyset$. For $i = 1 \to n$, if $S \cup \{i\} \in \mathcal{I}$ then set $S \leftarrow S \cup \{i\}$.

As we see in the next paragraph in fact we can optimize any convex function over the convex combination of bases of M.

Matroid Base Polytope. Given a matroid M, the *matroid base polytope*, $\mathcal{P}_{\mathcal{M}}$ is the convex hull of the indicator vectors of all bases of M. In other words, it is the Newton polytope of g_M. Edmonds proved a simple nice characterization of the matroid base polytope:

$$\begin{aligned} &\sum_{i \in S} x_i \leq r(S), && \forall S \subseteq [n] \\ &\sum_{i=1}^{n} x_i = r(M), && \\ &x_i \geq 0, && \forall 1 \leq i \leq n. \end{aligned} \tag{13.1}$$

Note that the above linear program has exponentially many constraints. But it has an efficient separation oracle, i.e., given any $x \in \mathbb{R}^n$ we can check in polynomial whether x is feasible and if not exhibit a violating constraint in polynomial time[1]. Because of that we can minimize any convex function over this polytope.

Although most of the matroids that we know can be represented by a linear matroid over a field F, it is proved that almost all matroids are not linear [**Nel18**]. Nonetheless, one can define and study many geometric structures based on this abstract structures. That is why Rota calls matroids *combinatorial geometries*.

Gelfand, Goresky, MacPherson and Serganova [GGMS87] proved the following characterization of the matroid base polytope:

Theorem 13.1 ([GGMS87])**.** *For any integer $1 \leq k \leq n$, given a k-homogeneous set system $\mathcal{B} \subseteq \in^{[\setminus]}$, i.e., $|S| = k$ for any $S \in \mathcal{B}$, $\mathcal{B}$ is the set of bases of a matroid iff every edge of the polytope conv$\{\mathbf{1}_B : B \in \mathcal{B}\}$ is parallel to $\mathbf{1}_i - \mathbf{1}_j$ for some $1 \leq i < j \leq n$.*

In other words, the above theorem shows that every edge of the matroid base poltyope $\mathcal{P}_{\mathcal{M}}$ is of the form $\mathbf{1}_i - \mathbf{1}_j$ corresponding to exchanging elements i, j between two bases of M, and vice versa, any homogeneous 0/1 polytope whose edges have this property corresponds to bases of a matroid.

Note that if we had defined matroids as the discrete objects which are representable over a field (i.e., only linear matroid case) then the above theorem would not be true. So, the above theorem shows that in a geometric sense matroids are the right generalization of linear vector spaces. We don't prove this theorem in this note and leave it as an exercise.

13.1 Bases Exchange Graph and the Mihail-Vazirani's conjecture

Given a matroid $M = ([n], \mathcal{I})$ consider the following simple walk on 1-skeleton of the matroid base polytope $\mathcal{P}_{\mathcal{M}}$: Construct a graph $G_M = (\mathcal{B}, \mathcal{E})$ with a vertex corresponding to each base of M and two bases B, B' are connected by an edge if there is an edge between them in $\mathcal{P}_{\mathcal{M}}$. By Theorem 13.1, B, B' are connected by an edge in G iff $|B \Delta B'| = |B \setminus B'| + |B' \setminus B| = 2$. Mihail and Vazirani conjectured that this graph has expansion 1 for any matroid M:

Conjecture 13.2 (Mihail-Vazriani'89)**.** *Let $G_M = (\mathcal{B}, \mathcal{E})$ be the bases exchange graph of a matroid M. For any $S \subseteq \mathcal{B}$,*

$$h(S) = \frac{|E(S, \overline{S})|}{|S|} \geq 1.$$

In this lecture we prove that the uniform distribution over the bases of any matroid is 1-spectrally independent.

[1] This follows from the fact that we can optimize any linear function over the bases of M.

Theorem 13.3. *For any matroid M, the uniform distribution π over all bases of M and all of its pinnings are 0-spectrally independent. Therefore, the corresponding Glauber-dynamics mixes in polynomial time.*

In particular, the above theorem implies that the following Markov chain mixes in polynomial time:

Definition 13.4 (Glauber Dynamics for Matroid Bases:)**.** *Given a base B: Choose an element $i \in B$ uniformly at random and delete it, $B - i$. Among all elements j, such that $B - i + j$ is a base choose one uniformly at random and go to $B - i + j$.*

Although we will not discuss here, but the proof of the above theorem also implies the Mihail-Vazirani's conjecture. We also remark that the general version of the Mihail-Vazirani's conjecture is still open:

Conjecture 13.5. *Let $V \subseteq \{0,1\}^n$. Let P be the convex-hull of all of the vectors in S. Let G be the 1-skeleton of P, i.e., its vertex set is V and its edge set correspond to edges of P. Then G is a 1-expander. i.e., for any $S \subseteq V$, $h(S) \geq 1$.*

Therefore, the 1-spectral independence of the uniform distribution distribution over bases of a matroid implies the special case of the above conjecture where every edge of the polytope P has ℓ_1 length exactly 2. The general version is still open.

Closure Properties of Matroids. Given a matroid $M = ([n], \mathcal{I})$ of rank r, it is closed under many operations.

- Contraction: For an element $1 \leq i \leq n$, M/i is the matroid on elements $[n] \setminus \{i\}$ with independent sets:
$$\{I : i \notin I, I \cup \{i\} \in \mathcal{I}\}.$$
For example, if M is a graphic matroid, then this operation exactly corresponds to edge contraction in graphs. This property in particular is very useful to our analysis in the next lecture as it shows that all pinnings of π also correspond to uniform distribution over bases of matroids.

- Deletion: For an element $1 \leq i \leq n$, $M \setminus i$ is the matroid on elements $[n] \setminus \{i\}$ with independent sets:
$$\{I : i \notin I, I \in \mathcal{I}\}.$$
For example, if M is a graphic matroid, then this operation exactly corresponds to edge deletion.

- Truncation: For an integer $1 \leq k \leq r$, the truncation of M to k, M_k is the matroid with elements $[n]$ and independent sets:
$$\{I : |I| \leq k, I \in \mathcal{I}\}.$$
For example, if M is a graphic matroid, then M_k has all forests of M with at most k edges. Note that the truncation of a graphic matroid is no longer a graphic matroid.

Negative Correlation. Given a matroid $M = ([n], \mathcal{I})$, let μ be the uniform distribution over the bases of M. In this course we study properties of this distribution. As we discussed in previous lectures, if M is a graphic matroid, then the uniform distribution over spanning trees is negatively correlated, namely for any pair of elements i, j (correspond to two distinct edges),
$$\mathbb{P}[i]\,\mathbb{P}[j] \geq \mathbb{P}[i,j]\,.$$

In fact as we said in the previous lecture the generating polynomial of the uniform distribution over spanning trees of a given graph is real stable (and we said it is 2-spectrally independent). Unfortunately, negative correlation and real stability properties don't generalize to all matroids.

A well-known counter example is the matroid $S8$. Here you can see a representation of this matroid of GF(2) where each element is a column of the following matrix:

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \end{bmatrix}$$

Note that the rank of this matrix is 4; so $\text{rank}(S8) = 4$. It is not hard to see that $|B_1| = 28$, $|B_8| = 20$, $|B_{1,8}| = 12$ and $|B| = 48$, where by B_1 we mean the set of bases that have column 1 and B is the set of all bases. The matroid is not negative correlated because

$$28 \cdot 20 \not\geq 12 \cdot 48.$$

13.2 Spectral Independence

Let M be a matroid of rank r.

We adopt the notation as before: For $i \in [n]$ we let π_i be the conditional distribution on $[n] - i$. As usual we let π^1 be the distribution on single elements, i.e., $\pi^1(i) = \frac{\mathbb{P}[i]}{r}$ which correspond to marginal probability of each element up to normalization.

Let $C \in \mathbb{R}^{[n]\times[n]}$ be the correlation matrix, i.e.,

$$C_\pi(i,j) = \frac{1}{r(r-1)} \mathbb{P}_{S\sim\pi}\left[i, j \in S\right]$$

with $C(i,i) = 0$ for all i. We start by proving basic facts:

Lemma 13.6. *The following holds for any matroid:*

- $\mathbb{E}_i \pi_i^1 = \pi^1$
- $\mathbb{E}_{i\sim\pi^1} C_{\pi_i} = C$
- $\mathbb{E}_i \pi_i \pi_i^T = C\Pi^{-1}C.$

where in the last identity Π *be the diagonal matrix with* $\pi^1(i)$ *on its* (i,i) *entry.*

Proof. We start with the first one:

$$\mathbb{E}_i \pi_i(j) = \sum_i \frac{\mathbb{P}\left[i\right]}{r} \cdot \frac{\mathbb{P}\left[j|i\right]}{r-1} = \sum_i \frac{\mathbb{P}\left[i,j\right]}{r(r-1)} = \frac{\mathbb{P}\left[j\right]}{r} = \pi^1(j).$$

To see the second one for $j \neq k$ we have

$$\begin{aligned}\sum_i \pi^1(i) C_{\pi_i}(j,k) &= \sum_i \pi^1(i) \frac{\mathbb{P}[j,k|i]}{(r-1)(r-2)} \\ &= \frac{1}{r(r-1)(r-2)} \sum_i \mathbb{P}[i]\,\mathbb{P}[j,k|i] \\ &= \sum_{i\neq j,k} \mathbb{P}[j,k,i] \\ &= \frac{\mathbb{P}[j,k]}{r(r-1)} = C_\pi(j,k).\end{aligned}$$

The second to last identity uses that every base has exactly r elements. The third conclusion can be proven as follows:

$$\begin{aligned}\mathbb{E}_{i\sim\pi^1} \pi_i^1 {\pi_i^1}^T(j,k) &= \sum_i \pi^1(i) \frac{\mathbb{P}[j|i]}{r-1} \cdot \frac{\mathbb{P}[k|i]}{r-1} \\ &= \frac{1}{r(r-1)^2} \sum_i \frac{\mathbb{P}[i,j]\,\mathbb{P}[k,i]}{\mathbb{P}[i]} \\ &= \frac{1}{r^2(r-1)^2} \sum_i \frac{\mathbb{P}[i,j]\,\mathbb{P}[k,i]}{\pi^1(i)} = C\Pi^{-1}C\end{aligned}$$

□

Lemma 13.7 (Oppenheim Trickledown Machinery)**.** *Suppose for every i we have*

$$C_{\pi_i} - \pi_i^1 {\pi_i^1}^T \preceq 0.$$

Then,

$$C_\pi - \pi^1 {\pi^1}^T \preceq 0$$

Proof. By the previous lemma we have

$$\mathbb{E}_{i\sim\pi^1} C_{\pi_i} - \pi_i^1 {\pi_i^1}^T = C_\pi - C_\pi \Pi^{-1} C_\pi \preceq 0.$$

Equivalently, we have

$$\Pi^{-1/2} C_\pi \Pi^{-1/2} \preceq (\Pi^{-1} C_\pi \Pi^{-1/2})^2$$

This implies that the matrix $P = \Pi^{-1} C_\pi$ has no eigenvalues in the interval $(0,1)$. Furthermore, observe for any i,j:

$$P(i,j) = \frac{\mathbb{P}[i,j]}{(r-1)\mathbb{P}[i]}$$

So, is a stochastic matrix on the vertex set $[n]$ and its largest eigenvalue is 1. Since the underlying graph is connected the second eigenvalue of P is < 1. But from above we know it is ≤ 0. This implies that

$$P - 1\pi^1 \preceq 0.$$

Or equivalently,

$$C_\pi - \pi^1 {\pi^1}^T \preceq 0$$

as desired. □

Lemma 13.8 (Base Case)**.** *If M is rank 2, then $C - \pi^1 {\pi^1}^T \preceq 0$.*

Proof. This lemma is the main property of the matroid that we use in the proof.

We say a pair of elements i, j are parallel if $rank(\{i, j\}) = 1$. The main observation is the following fact which can be proven by the exchange property.

Fact 13.9. *If i, j are parallel elements in a matroid M and j, k are parallel then i, k are parallel.*

Having that, it turns out that the matrix C is an "anti-block-diagonal" matrix (up to a normalization). Namely, for every parallel class we have an all-zero block matrix and every entry corresponding to two non-parallel elements is 1. Such a matrix can also be seen as an adjacency matrix of a complete multi-partite graph. It can be seen that this matrix has exactly one positive eigenvalue.

Similar to the previous lemma, $\Pi^{-1}C$ is the transition probability matrix of the simple random walk on a complete multi-partite graph (which has exactly one positive eigenvalue). Therefore, $\Pi^{-1}C - 1\pi^1 \preceq 0$. This implies $C - \pi^1 {\pi^1}^T \preceq 0$ as desired. □

Lemma 13.10. Π *is 1-spectrally independent.*

Proof. All we need to show is that the matrix $\mathcal{I}$ with entries

$$\mathcal{I}(i, j) = \mathbb{P}[j|i] - \mathbb{P}[j]$$

has $\lambda_{\max}(\mathcal{I}) \leq 0$. The main observation is that

$$\mathcal{I} = (r-1)\Pi^{-1}C - r \cdot \Pi^{-1}\pi^1 {\pi^1}^T \preceq (r-1)(\Pi^{-1}C - \Pi^{-1}\pi^1 {\pi^1}^T) \preceq 0$$

□

13.3 Lattices

Since Theorem 13.3 implies fast mixing of the Glauber dynamics for sampling bases of a given matroid, a central question that has puzzled researchers since then is sampling non-broken (circuit) bases (NBC bases) of a given matroid [BCT10].

Definition 13.11 (Lattice)**.** *A partially ordered set $(\mathcal{L}, <)$ is called a lattice if any pair of elements a, b have a least upper bound denoted by $a \vee b$ and a greatest lower bound denoted by $a \wedge b$. Here, we only consider finite lattices. A finite lattice has a minimum element that we denote by 0 and a maximum element that we denote by 1.*

We say $a \preceq b$ (b covers a) if $b > a$ but there is no element $c \in \mathcal{L}$ such that $a < c < b$.

Definition 13.12 (Ranked Lattices)**.** *We say a lattice is graded/ranked if it can be equipped with a function $r : \mathcal{L} \to \mathbb{N}$ such that if $a < b$ then $r(a) < r(b)$ and if b covers a, then $r(b) = r(a) + 1$.*

Definition 13.13 (Chains)**.** *We say a sequence of elements $a_1, \ldots, a_k \in \mathcal{L}$ form a chain if $a_1 < a_2 < \cdots < a_k$. Let $\mathcal{L}$ be a graded lattice with rank r. A sequence of elements $a_1, \ldots, a_{r-1}$ form a* maximal chain *if $0 \preceq a_1 \preceq \cdots \preceq a_{r-1} \preceq 1$.*

One of the most well-known examples of lattices is the lattice of flats of matroids.

Definition 13.14 (Flats)**.** *We say a set $F \subseteq [n]$ is a flat if for any element $i \notin F$ we have* $\mathrm{rank}(F+i) > \mathrm{rank}(F)$*.*

Lattice of flats is the lattice formed by all flats of the matroid with the relation $F < G$ if $F \subset G$. This is a ranked lattice with $r(F) = \mathrm{rank}(F)$. The minimum element is the $\emptyset$ and the maximum element is $[n]$.

Definition 13.15 (Möbius Function)**.** *Given a lattice $\mathcal{L}$ one can define a Möbius function μ on pair of elements $a \le b$ as follows:*

- *For any element a, let $\mu(a,a) = 1$.*
- *For any $a < b$ we have*
$$\mu(a,b) = -\sum_{a \le c < b} \mu(a,c).$$

To put it differently the second condition implies

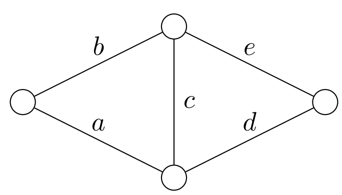
$$\sum_{a \le c \le b} \mu(a,c) = 0.$$

The characteristic polynomial of a ranked lattice $\mathcal{L}$ is defined as

$$\sum_a \mu(a) t^{r(a)}$$

where $\mu(a) = \mu(0,1)$.

Let us give a concrete example: Consider the following graph It has the following flats:



$$\begin{aligned}
&\emptyset\\
&\{a\},\{b\},\{c\},\{d\},\{e\}\\
&\{a,b,c\},\{c,d,e\},\{a,e\},\{a,d\},\{b,d\},\{b,e\}\\
&\{a,b,c,d,e\}.
\end{aligned}$$

Its characteristic polynomial is

$$t^4 - 5t^3 + 8t^2 - 4t$$

It turns out that this polynomial is exactly the same as the chromatic polynomial of G.

If we can evaluate the characteristic polynomial at -1 then we can evaluate the following quantities all of which are open:

- $\chi(-1)$ for the graphic matroid is the number of acyclic orientations of a graph [Sta73].
- $\chi(-1)$ for the co-graphic matroid is equal to the number of strongly connected orientations of the graph (see e.g., [GL19]).

- $\mu(0,1)$ is equal to the number of parking functions with respect to a unique source vertex [BCT10]
- $\chi(-1)$ for a linear matroid with vectors $v_1, \dots, v_n$ is equal to the number of regions defined by the intersection of the orthogonal hyperplanes (see e.g., [Sta07]).

Lemma 13.16. *Let $\mathcal{L}$ be lattice with Möbius function μ. Let c_i denote the number of chains $0 = a_0 < a_1 < \dots < a_i = 1$ in $\mathcal{L}$. Then,*

$$\mu(0,1) = -c_1 + c_2 - c_3 + \dots$$

Proof. We prove by induction. The statement obviously holds for $\mu(a,b)$ with $a \preceq b$. Now, by definition of the Möbius function,

$$\begin{aligned}\mu(a,b) &= -\sum_{a \le c < b} \mu(a,c)\\ &= -\sum_{a \le c < b} \sum_{i=1}^{\infty} (-1)^i c_i(a,c)\\ &= \sum_{i=1}^{\infty} (-1)^{i+1} \sum_{a \le c < b} c_i(a,c)\end{aligned}$$

Now, the main observation is that $\sum_{a \le c < b} c_i(a,c) = c_{i+1}(a,b)$. As, any i-chain from a to c can be extended to an $i+1$ chain from a to b just by appending b. $\square$

Adiprasito-Huh-Katz proved the coefficients of the characteristic polynomial of the lattice of flats of *any matroid* is a log-concave sequence, name the sequence $a_0, \dots, a_r$ satisfies

$$a_i^2 \ge a_{i-1} \cdot a_{i+1}.$$

A recent re-proof of this by Brändén and Leake uses geometry of polynomials, but it is unclear how to turn that proof into an spectral independence argument.

Definition 13.17 (E-labeling)**.** *Let $\mathcal{L}$ be a graded lattice. Let $cE(\mathcal{L})$ be the set of edges of $\mathcal{L}$ corresponding to all pairs a, b with $a \preceq b$. An E-labeling is a map $\lambda : \mathcal{E}(\mathcal{L}) \to \mathbb{N}$ such that if $a < b$ in $\mathcal{L}$ then there exists a unique maximal chain*

$$a = a_0 \preceq a_1 \preceq \dots \preceq a_k = b$$

such that $\lambda(a_0, a_1) \le \lambda(a_1, a_2) \le \dots \le \lambda(a_{k-1}, a_k)$.

Theorem 13.18. *Let λ be an E-labeling of P. Then, $(-1)^{r(a,b)}\mu(a,b)$ is equal to the number of strictly decreasing maximal chains from a to b, i.e.,*

$$(-1)^{r(a,b)}\mu(a,b) = |\{a = a_0 \preceq a_1 \dots \preceq a_k = b : \lambda(a_0, a_1) > \dots > \lambda(a_{k-1}, a_k)\}|.$$

Proof. Without loss o generality assume $a = 0, b = 1$. Let $S = \{r_1, \dots, r_{j-1}\} \subseteq [r-1]$ with $r_1 < \dots < r_{j-1}$. Let $\alpha_{\mathcal{L}}(S)$ be the number of chains $0 < a_1 < \dots < a_{j-1} < 1$ such that $r(a_i) = r_i$. The function $\alpha_{\mathcal{L}}$ is called the flag f-vector of $\mathcal{L}$.

Claim 13.19. *$\alpha_{\mathcal{L}}(S)$ is the number of maximal chains $0 = a_0 \preceq \dots \preceq a_r = 1$ such that*

$$\lambda(a_{i-1}, a_i) > \lambda(a_i, a_{i+1}) \implies i \in S, \forall 1 \le i < r.$$

The proof follows by defining a bijection between the chains counted in $\alpha_P(S)$ and the maximal chains defined above.

Now, let

$$\beta_{\mathcal{L}}(S) = \sum_{T \subseteq S} (-1)|S - T|\alpha_{\mathcal{L}}(T).$$

The function $\beta_{\mathcal{L}}$ is called the flag h-vector of $\mathcal{L}$. It follows from the above claim that $\beta_{\mathcal{L}}(T)$ is equal to the number of maximal chains $0 = a_0 \preceq \cdots \preceq a_r = 1$ such that $\lambda(x_{i-1}, x_i) > \lambda_(x_i, x_{i+1})$ if and only if $i \in T$. So, in particular, $\beta_{\mathcal{L}}([r-1])$ is equal to the number of strictly decreasing maximal chains. But, on the other hand,

$$\begin{aligned} \beta_{\mathcal{L}}([r-1]) &= \sum_{T \subseteq [r-1]} (-1)^{r-1-|T|}\alpha_{\mathcal{L}}(T) \\ &= (-1)^r \sum_{k \geq 1} (-1)^k c_k \end{aligned}$$

But the RHS is exactly the Mobius function $\mu(0,1)$ by the Halls' theorem as desired. ☐

Next, we explain a way to construct an E-labeling: Let $\mathcal{L}$ be the lattice of flats of a matroid. Fix an ordering on the elements say $1 < 2 < \cdots < n$ and for $a \preceq b$ let

$$\lambda(a, b) = -\max\{i : a \vee i = b\}.$$

We leave it as an exercise to show that this is a valid E-labeling.

Definition 13.20. *We say a maximal chain of flats* $\emptyset = F_0 \preceq \cdots \preceq F_r = [n]$ *is strictly increasing if* $\lambda(F_0, F_1) > \cdots > \lambda(F_{r-1}, F_r)$ *or in other words,*

$$\max\{F_1 - F_0\} < \max\{F_2 - F_1\} \cdots < \max\{F_r - F_{r-1}\}$$

So, the above theorem in particular implies that the number of strictly increasing maximal chains is exactly equal to the $\mu(0,1)$. Note that interestingly this definition is independent of the ordering on the underlying elements of the matroid.

Having this one can consider a family of Markov chains to count/sampling strictly increasing maximal chains:

Definition 13.21 (Glauber dynamics to sample strictly increasing maximal chains)**.** *Given a strictly increasing maximal chain* $F_0 \preceq \cdots \preceq\preceq F_r$*, choose* $1 \leq i \leq r-1$ *uniformly at random, remove* F_i *and choose a u.r. flat (of rank i) among all that gives a strictly increasing maximal chain.*

13.4 Non Broken Bases

Fix an ordering on elements of the matroid, say $1 < \cdots < n$. We say a set $C \subseteq [n]$ is a *circuit* iff $C \setminus \{i\} \in \mathcal{I}$ for any $i \in C$. A *broken circuit* is a set $C \setminus \{i\}$, where $C \subseteq [n]$ is a circuit and i is the **largest** element of C. An independent set $S \subseteq \mathcal{I}$ is a *non-broken* independent set (NBC independent set) if it contains no broken circuits. Similarly an NBC base is an NBC independent set of rank r.

Theorem 13.22. *The number of NBC bases of* M *is equal to the number of maximal increasing chains of the lattice of flats of* M*.*

Proof. In fact there is a simple bijection: Given a maximal strictly increasing chains $\emptyset = F_0 \preceq F_1 \preceq F_r = [n]$, $\cup_{i=1}^r \max_{j \in F_i - F_{i-1}}\{j\}$ is a non-broken base and vice versa we can construct a maximal strictly increasing chain given a non-broken base. $\square$

Given this the following conjecture is raised and still open:

Conjecture 13.23. *For any matroid M, and any total ordering of the elements of M, the down-up walk on the NBC bases of M mixes in polynomial time.*

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