Lecture 11: Spectral Independence via Geometry of Polynomials

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 In the previous lecture we introduced the generating polynomial of a probability distribution $\mu : 2^{[n]} \to \mathbb{R}_{\geq 0}$,

$$g_\mu(z_1, \ldots, z_n) = \sum_S \mu(S) z^S,$$

where $z^S = \prod_{i \in S} z_i$.

Definition 11.1 (Real Stable Polynomial)**.** *We say a polynomial $p(z_1, \ldots, z_n)$ is* **real stable** *if $p(z_1, \ldots, z_n) \neq 0$ whenever $\Im(z_i) > 0$ for all i.*

There is a by now a rich theory of these polynomials. For example,

Theorem 11.2. *For any set of PSD matrices $A_1, \ldots, A_n \in \mathbb{R}^{n \times n}$ and a symmetric matrix $B \in \mathbb{R}^{n \times n}$ the polynomial $\det(B + z_1 A_1 + \cdots + z_n A_n)$ is real stable.*

So, for example the generating polynomial of the uniform distribution of spanning trees of any graph G is real stable.

We also know that this family is closed under many operators; if $p(z_1, \ldots, z_n)$ is real stable then so is,

Substituion $p(z_1, z_2, z_{i-1}, a, z_{i+1} \ldots, z_n)$ for any a with $\Im(a) \geq 0$, in particular for any $a \in \mathbb{R}$.

Differentiation $\partial_{z_i} p$ for any $1 \leq i \leq n$.

External Fields/Tilts $p(\lambda_1 z_1, \ldots, \lambda_n z_n)$ for any $\lambda_1, \ldots \lambda_n \geq 0$.

Inversion If z_i has degree d in p then $p(z_1, \ldots, z_{i-1}, -1/z_i, z_{i+1}, \ldots, z_n) z_i^d$.

Product If q is also stable then $p \cdot q$.

For a probability distribution μ and $\lambda : [n] \to \mathbb{R}_{\geq 0}$ we let $\lambda * \mu$ be the external-field/tilts with respect to λ. That is the distribution where the probability of any S is proportional to $\lambda^S \mu(S)$.

It is well-known fact that if g_μ is real stable then μ is negatively correlated. So, is 1-spectrally independent (if in addition it is homogeneous).

This definition has been generalized recently to the class of sector stable polynomials by Alimohammadi, Anari, Shiragur and Vuoung:

Definition 11.3 (Sector Stable Polynomials)**.** *Let*

$$S_\alpha := \{r \cdot e^{i\theta} : r > 0, |\theta| \leq \alpha \cdot \frac{\pi}{2}\}$$

be the sector around the positive real axis with aperture $\alpha\pi$. We say p is α-sector stable if $p(z_1, \ldots, z_n) \neq 0$ whenever $z_i \in S_\alpha$ for all i.

Note that many closure properties of real stable polynomial naturally generalize to sector stable polynomials, such as substitution to the closure of the sector, differentiation, external field and product.

They proved the following theorem:

Theorem 11.4. *If g_μ is α-sector stable then μ is $2/\alpha$-spectrally independent, namely for any $1 \leq i \leq n$,*

$$\sum_{j=1}^{n} |\mathbb{P}[j \in S | i \in S] - \mathbb{P}[j \in S | i \notin S]| \leq 2/\alpha.$$

Let me also emphasize the following theorem that we will prove in future:

Theorem 11.5. *Let μ be a probability distribution on $2^{[n]}$. Then, the following are equivalent,*

- *μ For every $\lambda : [n] \to \mathbb{R}_{\geq 0}$, the distribution $\lambda * \mu$ is η-spectrally independent.*
- *For every $\lambda : [n] \to \mathbb{R}_{\geq 0}$, the distribution $\lambda * \mu$ is η-entropically independent.*

Since sector stability is closed under external fields, combining these two theorems we obtain that α-sector stable polynomials are $2/\alpha$-entropic independent. This implies that the corresponding Markov chain has MLS constant $\Omega(n^{-2/\alpha})$).

11.1 Application

Lemma 11.6. *Given a graph $G = (V, E)$, the matching polynomial is*

$$\sum_{M \text{ matching}} \prod_{v \text{ saturated in } M} z_v.$$

The matching polynomial of any graph is 1-sector stable, i.e., it has no root in the right-half complex plane. This class of polynomials are also called Hurwitz stable.

Proof. The polynomial $p_1 = z_1, \dots, z_n$ is real stable. It is shown by Borcea and Brändén that if p is real stable then $(1 - \partial z_i \partial z_j)p$ is also real stable for any $i \neq j$. Now consider

$$p_2 = \left(\prod_{i \sim j} (1 - \partial z_i \partial z_j) \right) p_1$$

So, p_2 is real stable. But p_2 can be written equivalently as follows:

$$p_2 = \sum_{M \text{ matching}} (-1)^{|M|} \prod_{v \text{ saturated in } M} z_v,$$

i.e., it is the matching polynomial with "alternating" signs. So, p_2 has no roots in the upper-half complex plane. Now, it turns out that if a polynomial $p(z_1, \dots, z_n)$ is real stable then $p(e^{i\pi/2} z_1, \dots, e^{i\pi/2} z_n)$ is Hurwitz stable. So, $p_3 = p_2(\{e^{i\pi/2} z_v\}_{v \in V})$ is Hurwitz stable, i.e., 1-sector stable. But that is exactly the matching polynomial. □

So, this gives an algorithm to count matchings in all graphs; except there is a caveat one needs to be able to run the chain that is to compute the coefficient of a monomial z^S. For a set $S \subset V$, the coefficient of z^S is the number of perfect matchings in the induced graph $G[S]$. It turns out that such a quantity can be computed exactly for planar graphs, and therefore this gives an efficient algorithm the approximately count/sample matchings in planar graphs.

11.2 Some Tools in Complex Analysis

In this section we prove Theorem 11.4. The main idea of the proof is that if a polynomial has no roots close to the all-ones vector (at which we want to evaluate our polynomial) then it must be smooth, and therefore close to being independent.

To use this intuition we use the following lemma which says that if we have a smooth in the complex plane on the unit disc its derivate at 0 is bounded. To use this lemma we need to apply a map (a mobius function) which maps our sector to the disk and all-ones to 0. Then we apply the lemma. Finally we need to apply the inverse back to translate our findings to our own setting.

The main tool that we use is the following Schwartz-Pick lemma: Let $\mathbb{D}(0,)$ be the open unit disk in the complex plane around 0.

Lemma 11.7 (Schwarz-Pick)**.** . *Let* $f : \mathbb{D}(0,1) \to \mathbb{D}(0,1)$ *be a univariate holomorphic function. Then* $|f'(0)| \leq 1 - |f(0)|^2 \leq 1$.

Recall that in the previous lectures we observed that $\partial z_i \log g_\mu(1) = \mathbb{P}[i \in S]$ is the marginal probability of element i.

$$F_i(z_1, \dots, z_n) = \log \left(\frac{\partial_{z_i} g_\mu(z)}{(1 - z_i \partial_{z_i}) g_\mu(z)} \right)$$

Then,

$$\begin{aligned} \partial_{z_j} F_i(1) &= \partial_{z_j} \log \partial_{z_i} g_\mu(1) - \partial_{z_j} \log(1 - z_i \partial_{z_i}) g_\mu(1) \\ &= \mathbb{P}[j \in S | i \in S] - \mathbb{P}[j \in S | i \notin S] \end{aligned}$$

So, our main goal is to upper-bound gradient of F_i. In order to use our tool we need to do a change of basis.

Let $f(z)$ be the univariate polynomial,

$$f(z) = \psi(F_i(h_1(z), \dots, h_n(z)))$$

The idea is to choose $H : \mathbb{C} \to \mathbb{C}^n$ such that $H(\mathbb{D}(0,1)) \subseteq S_\alpha$ and that $\psi : \mathbb{C} \to \mathbb{C}$ maps back the sector to the unit disk. By the Pick's lemma and the chain rule we have

$$\begin{aligned} 1 - |f(0)|^2 \geq |f'(0)| &= |\psi'(F_i(H(0)))| \cdot \left| \sum_{j \neq i} \partial_{z_j} F_i(h_1(0), \dots, h_n(0)) \cdot h_j'(0) \right| \\ &= |\psi'(F_i(H(0)))| \cdot \left| \sum_{j \neq i} \Psi(i \to j) \cdot h_j'(0) \right| \end{aligned} \tag{11.1}$$

We start by construction h_i's. First, notice that the möbius function $z \to \frac{1+sz}{1-sz}$ maps $\mathbb{D}(0,1)$ to the Right half plane (and 0 gets mapped to the all-ones vector). To map to the sector it is enough to raise to an α factor, $z \to \left(\frac{1+sz}{1-sz}\right)^\alpha$. The parameter s will be chosen carefully to take the signs of the quantities $\Psi(i \to j)$ into account, namely we let

$$h_j(z) = \left(\frac{1 + s_j z}{1 - s_j z} \right)^\alpha$$

where $s_j = \text{sign}(\Psi(i \to j))$.

Now, observe that

$$h_j'(z) = \frac{2 s_j \alpha}{(1 - s_j z)^2} \cdot \left(\frac{1 + s_j z}{1 - s_j z} \right)^{\alpha - 1}$$

Now, plugging in, $z = 0$ we have $H(0) = 1$, $h'_j(0) = 2s_j\alpha$. Note that $s_j\Psi(i \to j) = |\Psi(i \to j)|$. Therefore, from (11.1) we get

$$\sum_{j \neq i} |\Psi(i \to j)| \leq \frac{1}{2\alpha} \cdot \frac{1 - |\psi(F_i(\mathbf{1}))|^2}{\psi'(F_i(\mathbf{1}))} \tag{11.2}$$

Claim 11.8. *For every $z_1, \ldots, z_n \in S_\alpha$ we have*

$$\frac{\partial_{z_i} g_\mu(z)}{(1 - z_i \partial_{z_i}) g_\mu(z)} \notin -S_\alpha.$$

In particular, the image of S_α under F_i is in the strip

$$T_\alpha := \{z \in \mathbb{C} : \Im(z) \leq \left(1 - \frac{\alpha}{2}\right)\pi\}$$

Proof. Let $y = \frac{\partial_{z_i} g_\mu(z)}{(1 - z_i \partial_{z_i}) g_\mu(z)}$ and suppose for contradiction that $y \in -S_\alpha$. Then, $\frac{-1}{y} \in S_\alpha$. By definition of y, we have

$$(1 - z_i \partial_{z_i}) g_\mu(z) - \frac{1}{y} \partial_{z_i} g_\mu(z) = 0.$$

But then since g_μ is multi-affine, $g_\mu(z_1, \ldots, z_{i-1}, \frac{-1}{y}, z_{i+1}, \ldots, z_n) = 0$. But this contradicts sector stability of g_μ. THis shows the first conclusion. Now, since $F_i(z) = \log \frac{\partial_{z_i} g_\mu(z)}{(1 - z_i \partial_{z_i})}$ and log maps any point outside of $-S_\alpha$ to the strip the conclusion follows. □

Having established this, first observe the map $z \to \frac{1/2}{1/2 - \alpha} \cdot z$ maps the strip T_α to T_1. So, $z \to \exp(\frac{1/2}{1/2 - \alpha} \cdot z)$ maps it to the Right half plane, i.e., S_1. Finally if we compose with the inverse of the Mobius function $\frac{1+z}{1-z}$ we get back the unit disk. Note that the inverse of the mobius transformation $z \to \frac{1+z}{1-z}$ is $z \to \frac{z-1}{2z} = g^{-1}(z)$. We let

$$\psi(z) = g^{-1}\left(\exp(\frac{1/2}{1/2 - \alpha} \cdot z)\right)$$

It can then be shown that

$$\psi'(z) = \frac{2\exp(\frac{1/2}{1/2-\alpha} \cdot z)}{(1 + \exp(\frac{1/2}{1/2-\alpha} \cdot z))^2} \cdot \frac{1/2}{1/2 - \alpha} = \frac{1/2}{1/2 - \alpha} \cdot \frac{1}{2}(1 - \psi(z)^2).$$

This together with (11.2) implies that

$$\sum_{j \neq i} \Psi(i \to j) \leq \frac{2}{\alpha} - 1$$

Adding I to the diagonal we get that $\lambda_{\max}(\Psi) \leq 2/\alpha$ as desired. This finishes the proof of Theorem 11.4.