Approximate Counting and Mixing Time of Markov chains **Fall 2024**

Problem Set 1

1) Use the coupling technique to show that the mixing time of the 1/2-lazy simple random walk on a cycle of length n is $O(n^2)$.

   Hint: Consider the following stochastic process: Let $X_0, X_1, \ldots$ be a martingale with $X_0 = 0$ and

$$X_i = \begin{cases} X_{i-1} + 1 & \text{w.p. } 1/2 \\ X_i = X_{i-1} - 1 & \text{otherwise.} \end{cases}$$

   . Show that for a constant $c > 0$, $\mathbb{P}[|X_{cn^2}| \geq n] \geq 1/2$.

2) Consider the Glauber dynamics to generate a u.r. independent set: Given a graph $G = (V, E)$ with **maximum degree 3**; let I be the current independent set; each time we choose a u.r. vertex v and we "randomize" it; if $N(v) \cap I \neq \emptyset$, i.e., v is blocked, we do nothing. Otherwise, we let $I = I + \{v\}$ w.p. 1/2 and we let $I = I - \{v\}$ otherwise. Use the coupling technique to show that this mixes in time polynomial in n. Feel free to assume that G has no triangles, i.e., three vertices u, v, w such that $\{u, v\}, \{v, w\}, \{w, u\} \in E$ if it simplifies your proof.

   Hint: Use path coupling with the following pre-metric: Consider a state I such that $v \notin I$ and $I_v = I + \{v\}$. define

$$d(I, I_v) = d(v) - \frac{3}{5}|B_I(v)|,$$

   where $B_I(v)$ is the set vertices neighbors u of v such that $u \notin I$ but u is blocked in I.