In elimination style algorithms, data is not reused round to round. This is b/c of difficulties w/ dependence.

Input $\mathcal{X}$

for $t = 1, 2, \ldots$

Player chooses $x_t \in \mathcal{X}$ ← x_t chosen using $\{(x_s, y_s)\}_{s=1}^{t-1}$

Nature reveals $y_t \in \langle \theta^*, x_t \rangle + \varepsilon_t$, $\varepsilon_t \sim N(0,1)$ ← iid

Consider LS:

$$\hat{\theta}_t = \left(\sum_{s=1}^{t} x_s x_s^T \right)^{-1} \left(\sum_{s=1}^{t} x_s y_s \right) \qquad y_s = x_s^T \theta_* + \varepsilon_s$$

$$= \theta^* + \left(\sum_{s=1}^{t} x_s x_s^T \right)\left(\sum_{s=1}^{t} x_s \varepsilon_s \right)$$

$$\mathbb{P}\left(\langle z, \hat{\theta} - \theta^* \rangle > \varepsilon \right) \le e^{-\lambda \varepsilon} \mathbb{E}\left[\exp\left(\lambda \langle z, \hat{\theta} - \theta^* \rangle \right) \right]$$

$$= e^{-\lambda \varepsilon} \mathbb{E}\left[\exp\left(\lambda\, z^T \left(\sum_{s=1}^{t} x_s x_s^T \right)^{-1} \left(\sum_{s=1}^{t} x_s \varepsilon_s \right) \right) \right]$$

$W \in \mathbb{R}^t$

$$W_s = \left(\sum_{s=1}^{t} x_s x_s^T \right)^{-1} x_s \qquad \langle z, \hat{\theta} - \theta^* \rangle = \langle z, \sum_s W_s \varepsilon_s \rangle$$

Idea: Consider all possible $A = \sum x_s x_s^T$, $W_{s,A} = A^{-1} x_s$, union bound over all possible A

$$= e^{-\lambda \varepsilon} \mathbb{E}\left[\exp\left(\lambda \langle z, \sum_s W_s \varepsilon_s \rangle \right) \right]$$

$$= e^{-\lambda \varepsilon} \mathbb{E}\left[\prod_{s=1}^{t} \exp\left(\lambda \langle z, W_s \varepsilon_s \rangle \right) \right]$$

→ Cannot take $\mathbb{E}[\cdot]$ inside product b/c x_t depends on $\{\varepsilon_s\}_{s<t}$

x_s is independent of ε_s

La Peña formalized (cleaned-up literature) ~~on~~ self-normalized bounds.

Fix: Recall from last time: $\hat{\theta} = (\sum x_s x_s^T)^{-1} (\sum_s x_s y_s)$

$$\langle z, \hat{\theta} - \theta^* \rangle = \langle (\sum_s x_s x_s^T)^{-1} z, (\sum_s x_s x_s^T)(\hat{\theta} - \theta^*) \rangle$$

$$\leq \|z\|_{(\sum x_s x_s^T)^{-1}} \cdot \|\hat{\theta} - \theta^*\|_{(\sum x_s x_s^T)}$$

This process $\lesssim \sqrt{d} + \sqrt{2 \log(1/\delta)}$ (High prob)

"self-normalized" because $\mathrm{cov}(\hat{\theta} - \theta^*) \approx (\sum_s x_s x_s^T)^{-1}$

$\boxed{\text{Lemma}}$ Fix $\gamma > 0$. Let $y_t = \langle x_t, \theta^* \rangle + \varepsilon_t$ where $\varepsilon_t \sim N(0,1)$, x_t chosen based on $\{(x_s, y_s)\}_{s=1}^{t-1}$. If $\hat{\theta}_t = V_t(\gamma)^{-1} S_t$ where

$$V_t(\gamma) = \sum_{s=1}^{t} x_s x_s^T + \gamma I$$

$$S_t = \sum_{s=1}^{t} x_s y_s$$

then

$$\mathbb{P}\left(\exists t \in \mathbb{N} : \|\hat{\theta}_t - \theta^*\|_{V_t(\gamma)} > \sqrt{\gamma}\,\|\theta^*\|_2 + \sqrt{2\log(1/\delta) + \log\left(\frac{|V_t(\gamma)|}{\gamma^d}\right)}\right) \le \delta$$

Note: $\log\left(\frac{|V_t(\gamma)|}{\gamma^d}\right) \le d\log\left(\frac{d\gamma + TL^2}{d\,\gamma^{1/d}}\right)$

if $\|x_t\|_2 \le L$ for all t.

Now we have confidence bound for arbitrary sequence of meas. Opens doors to new algorithms.

Recall UCB from MAB. (MAB $x = e_i$ for $i=1,\dots,n$)

Idea: For each $x \in \mathcal{X}$ algorithm construct confidence bound around each $\langle x, \hat{\theta}\rangle$ and then pull arm w/ highest upper conf bound. Intuit: If UCB is very high then that arm either has very large reward OR, has not been sample enough times (and sampling shrinks conf bound).

Input $\mathcal{X}$

for $t = 1, 2, \ldots$

Construct confidence set C_t: $\theta_* \in C_t$ w.p. $\geq 1-\delta$ $\forall t$

$$UCB(x) = \max_{\theta \in C_t} \langle \theta, x \rangle$$

Play $x_t = \arg\max_{x \in \mathcal{X}} UCB(x)$

Nature reveal $y_t = \langle x_t, \theta^* \rangle + \varepsilon_t$.

Idea: Let $C_t = \{ \theta : \|\theta - \hat{\theta}_t\|_{V_t(\gamma)} \leq \beta_t \}$

where $\underline{\underline{\beta_t}} = \sqrt{\gamma}\, \|\theta^*\|_2 + \sqrt{2\log(1/\delta) + \log\left(\frac{|V_t(\gamma)|}{\gamma^d}\right)}$

Let $f(\lambda) = \log \left| \sum_x \lambda_x x x^T + \gamma I \right|$

$$g(\lambda) = \max_{x \in \mathcal{X}} \|x\|^2_{(\sum \lambda_x x x^T + \gamma I)^{-1}}$$

and $\lambda^* = \arg\max_\lambda f(\lambda)$. Then

$$\max_\lambda f(\lambda) = f(\lambda^*)$$

$$\min_{\lambda} g(\lambda) = g(\lambda^*)$$

$$g(\lambda^*) = Tr\left(\left(\sum_x \lambda_x^* x x^T + \gamma I\right)^{-1}\left(\sum_x \lambda_x^* x x^T\right)\right)$$

"effective dimension"

$= d$ if $\gamma = 0$, o.w. $< d$

Regret of UCB. $x_* = \arg\max_x \langle x, \theta^* \rangle$

$\boxed{\text{Theorem}}$ If $|\langle \theta^*, x_t \rangle| \le 1$, $\|x_t\| \le L$, $\forall t$ then

$$\hat{R}_T = \sum_{t=1}^{T} \langle x_* - x_t, \theta^* \rangle \le \sqrt{8 T \beta_T^2 \log \underbrace{\left(\frac{|V_T(\gamma)|}{\gamma^d}\right)}_{\simeq d}}$$

Remark Since $\beta_t \simeq \sqrt{d + \log(1/\delta)}$

$$\Rightarrow \hat{R}_T \simeq d\sqrt{T}$$

$\boxed{\text{Proof}}$ $x_t = \arg\max_x \max_{\theta \in C_t} \langle x, \theta \rangle = \arg\max_x UCB(x)$

Let $\hat{\theta}_t = \arg\max_{\theta \in C_t} \langle x_t, \theta \rangle$

$$\langle \theta_*, x_* \rangle \le UCB(x_*) \le UCB(x_t) = \langle x_t, \tilde{\theta}_t \rangle$$

$$\begin{aligned}
\langle \theta^*, x_* - x_t \rangle &= \langle \theta^*, x_t \rangle - \langle \theta^*, x_* \rangle \\
&= \langle \theta^*, x_t \rangle - \langle x_t, \tilde{\theta}_t \rangle \\
&= \langle \theta^* - \tilde{\theta}_t, x_t \rangle \\
&= \langle V_t(\gamma)(\theta^* - \tilde{\theta}_t), V_t(\gamma)^{-1} x_t \rangle \\
&\le \underbrace{\|\theta^* - \tilde{\theta}_t\|_{V_t(\gamma)}}_{\substack{\le \|\theta^* - \hat{\theta}_t\|_{V_t(\gamma)} + \|\hat{\theta}_t - \tilde{\theta}_t\|_{V_t(\gamma)} \\ \le 2\beta_t}} \cdot \|x_t\|_{V_t(\gamma)^{-1}} \\
&\le 2\beta_t \|x_t\|_{V_t(\gamma)^{-1}}
\end{aligned}$$

Recall $|\langle x_t, \theta^* \rangle| \le 1 \quad \forall t. \Rightarrow \langle \theta^*, x_t - x_* \rangle \le 2$

$$\begin{aligned}
\langle \theta^*, x_* - x_t \rangle &\le \min\{2\beta_t \|x_t\|_{V_t(\gamma)^{-1}}, 2\} \\
&\le 2\beta_T \min\{\|x_t\|_{V_t(\gamma)^{-1}}, 1\}
\end{aligned}$$

$$\hat{R}_T = \sum_{t=1}^{T} \langle \theta^*, x^* - x_t \rangle$$

$$\le \frac{T}{T} \sum_{t=1}^{T} 2 \beta_T \min\{ \|x_t\|_{V_t(\gamma)^{-1}}, 1 \}$$

$$\le T \cdot \left(\frac{1}{T} \sum_t \left(2 \beta_T \min\{ \|x_t\|_{V_t(\gamma)^{-1}}, 1 \}\right)^2 \right)^{1/2}$$

$$= \sqrt{T \beta_T^2 \sum_{t=1}^{T} \min\{1, \|x_t\|^2_{V_t(\gamma)^{-1}}\}}$$

$$\le \sqrt{T \beta_T^2 \cdot 2 \log\left(\frac{|V_t(\gamma)|}{\gamma^d} \right)}$$

$$|V_t(\gamma)| = |V_{t-1}(\gamma)| \cdot |I + x_t x_t^T| \quad \blacksquare$$