6 Linear Bandits: Regret Minimization

This section is inspired by [Lattimore and Szepesvári, 2020].

Input: Finite set $\mathcal{X} \subset \mathbb{R}^d$, confidence level $\delta \in (0,1)$.
Let $\widehat{\mathcal{X}}_1 \leftarrow \mathcal{X}, \ell \leftarrow 1$
while $|\widehat{\mathcal{X}}_\ell| > 1$ **do**
  Let $\widehat{\lambda}_\ell \in \triangle_{\widehat{\mathcal{X}}_\ell}$ be a $\frac{d(d+1)}{2}$-sparse minimizer of $f(\lambda) = \max_{x \in \widehat{\mathcal{X}}_\ell} \|x\|^2_{(\sum_{x \in \widehat{\mathcal{X}}_\ell} \lambda_x x x^\top)^{-1}}$
  $\epsilon_\ell = 2^{-\ell}$, $\tau_\ell = 2d\epsilon_\ell^{-2} \log(4\ell^2|\mathcal{X}|/\delta)$
  Pull arm $x \in \mathcal{X}$ exactly $\lceil \widehat{\lambda}_{\ell,x} \tau_\ell \rceil$ times and construct the least squares estimator $\widehat{\theta}_\ell$ using only the observations of this round
  $\widehat{\mathcal{X}}_{\ell+1} \leftarrow \widehat{\mathcal{X}}_\ell \setminus \{x \in \widehat{\mathcal{X}}_\ell : \max_{x' \in \widehat{\mathcal{X}}_\ell} \langle x' - x, \widehat{\theta}_\ell \rangle > 2\epsilon_\ell\}$
  $\ell \leftarrow \ell + 1$
Output: $\widehat{\mathcal{X}}_\ell$

Recall: From standard MAB, the elimination alg. relied on just controlling $\hat{\theta}_i - \theta^*_i = \langle \hat{\theta} - \theta^*, e_i \rangle \quad \forall i \in [n]$

We showed $R_T \leq \sqrt{|\mathcal{X}| T \log(T)}$

Define $x^* = \arg\max_{x \in \mathcal{X}} \langle x, \theta^* \rangle$

$\boxed{\text{Lemma}}$ Assume $\max_{x \in \mathcal{X}} \langle x^* - x, \theta^* \rangle \leq 4$. W.p. $\geq 1-\delta$ $x^* \in \hat{\mathcal{X}}_\ell$ for all $\ell \geq 1$ and $\max_{x \in \mathcal{X}_\ell} \underbrace{\langle x^* - x, \theta^* \rangle}_{\Delta x} \leq 8\varepsilon_\ell$.

$\boxed{\text{Proof}}$ Fix $\mathcal{V} \subset \mathcal{X}$ and $x \in \mathcal{V}$. Define event

$$\mathcal{E}_{x,\ell}(\mathcal{V}) = \{ |\langle x, \hat{\theta}_\ell - \theta^* \rangle| \leq \varepsilon_\ell \}$$

where it is understood that $\hat{\theta}_\ell$ uses measurements from G-optimal design on $\underline{\underline{\mathcal{V}}}$ (i.e. $\hat{\mathcal{X}}_\ell = \mathcal{V}$)

$$\mathbb{P}\left(\bigcup_{\ell=1}^{\infty} \bigcup_{x \in \hat{\mathcal{X}}_\ell} \{ \mathcal{E}^c_{x,\ell}(\hat{\mathcal{X}}_\ell) \} \right) \leq \sum_{\ell=1}^{\infty} \mathbb{P}\left(\bigcup_{x \in \hat{\mathcal{X}}_\ell} \{ \mathcal{E}^c_{x,\ell}(\hat{\mathcal{X}}_\ell) \} \right)$$

$$= \sum_{\ell=1}^{\infty} \sum_{\mathcal{V} \subseteq \mathcal{X}} \mathbb{P}\left(\bigcup_{x \in \mathcal{V}} \{ \mathcal{E}_{x,\ell}^{c}(\mathcal{V}) \}, \ \mathcal{V} = \hat{\mathcal{X}}_\ell \right)$$

$$= \sum_{\ell=1}^{\infty} \sum_{\mathcal{V} \subseteq \mathcal{X}} \underbrace{\mathbb{P}\left(\bigcup_{x \in \mathcal{V}} \{ \mathcal{E}_{x,\ell}^{c}(\mathcal{V}) \} \,\Big|\, \mathcal{V} = \hat{\mathcal{X}}_\ell \right)}_{= \mathbb{P}\left(\bigcup_{x \in \mathcal{V}} \{ \mathcal{E}^{c} \} \right) \le \frac{\delta |\mathcal{V}|}{2\ell^2 |\mathcal{X}|} \le \frac{\delta}{2\ell^2}} \mathbb{P}(\mathcal{V} = \hat{\mathcal{X}}_\ell)$$

$$\le \sum_{\ell=1}^{\infty} \sum_{\mathcal{V} \subseteq \mathcal{X}} \frac{\delta}{2\ell^2} \mathbb{P}(\mathcal{V} = \hat{\mathcal{X}}_\ell)$$

$$= \sum_{\ell=1}^{\infty} \frac{\delta}{2\ell^2} \le \delta$$

$$\Rightarrow \mathbb{P}\left(\bigcap_{\ell=1}^{\infty} \bigcap_{x \in \hat{\mathcal{X}}_\ell} \{ |\langle x, \hat{\theta}_\ell - \theta^* \rangle| \le \varepsilon_\ell \} \right) \ge 1 - \delta.$$

Show $x^* \in \hat{\mathcal{X}}_\ell$ for all ℓ. $x^* \in \hat{\mathcal{X}}_1$ so assume in $\hat{\mathcal{X}}_\ell \ni x^*$

$$\langle x - x^*, \hat{\theta}_\ell \rangle = \langle x - x^*, \hat{\theta}_\ell - \theta^* \rangle + \langle x - x^*, \theta^* \rangle$$

$$= \underbrace{\langle x, \hat{\theta}_\ell - \theta^* \rangle}_{\le \varepsilon_\ell} - \underbrace{\langle x^*, \hat{\theta}_\ell - \theta^* \rangle}_{\ge -\varepsilon_\ell} + \underbrace{\langle x - x^*, \theta^* \rangle}_{\le 0}$$

$$\le 2\varepsilon_\ell.$$

$\Delta_x = \langle x^* - x, \theta^* \rangle$. Fix $\nu \geq 0$

$$\sum_{x \in \mathcal{X} \setminus x^*} \Delta_x T_x = \sum_{\substack{x \in \mathcal{X} \setminus x^* \\ \Delta_x \leq \nu}} \Delta_x T_x + \sum_{\substack{x \in \mathcal{X} \setminus x^* \\ \Delta_x > \nu}} \Delta_x T_x$$

From above Lemma we have $\max_{x \in \mathcal{X}_\ell} \langle x^* - x, \theta^* \rangle \leq 8\varepsilon_\ell = 8 \cdot 2^{-\ell}$

$$\leq \nu T + \sum_{\ell=1}^{\infty} \sum_{\substack{x \in \mathcal{X} \setminus x^* \\ \Delta_x > \nu}} \Delta_x \underbrace{\lceil \tau_\ell \hat{\lambda}_{\ell,x} \rceil}_{\neq 0 \text{ if } T_x^{(\ell)} \neq 0}$$

Consider $\ell : 8 \cdot 2^{-\ell} \leq \nu$
$\Rightarrow \ell \leq \log_2(8\nu^{-1})$

$$\leq \nu T + \sum_{\ell=1}^{\infty} 8\varepsilon_\ell \sum_{\substack{x \in \mathcal{X} \setminus x^* \\ \Delta_x > \nu}} (\tau_\ell \hat{\lambda}_{\ell,x} + 1) \mathbb{1}\{\hat{\lambda}_{\ell,x} > 0\}$$

$$= \nu T + \sum_{\ell=1}^{\log_2((\Delta \vee \nu)^{-1})} 8\varepsilon_\ell \left(\tau_\ell + \text{support}(\hat{\lambda}_\ell) \right)$$

$$\leq \nu T + d^2 + c \sum_{\ell=1}^{\log_2(8\Delta^{-1} \wedge 8\nu)} \varepsilon_\ell \frac{d \log(\ell^2 |\mathcal{X}| / \delta)}{\varepsilon_\ell^2}$$

$$\leq \nu T + d^2 + c\, d \log\left(|\mathcal{X}| \log(\Delta^{-1} \wedge \nu^{-1}) / \delta\right) \cdot \sum_{\ell}^{\log(\Delta^{-1} \wedge \nu^{-1})} \frac{1}{\varepsilon_\ell}$$

$$\leq \nu T + d^2 + c \frac{1}{(\nu \vee \Delta)^{-1}} d \log\left(|\mathcal{X}| \log(\Delta^{-1} \wedge \nu^{-1}) / \delta\right)$$

$\delta = 1/T$

$$\nu = 0 \Rightarrow R_T \le \frac{d \log(|\mathcal{X}| T)}{\Delta}$$

$$\nu \ge 0 \Rightarrow R_T \le c\sqrt{dT \log(|\mathcal{X}| T)}$$