Warm-up: 2 arms (A/B testing)

Input: 2 arms, $\mathfrak{Z} \in \mathbb{N}$

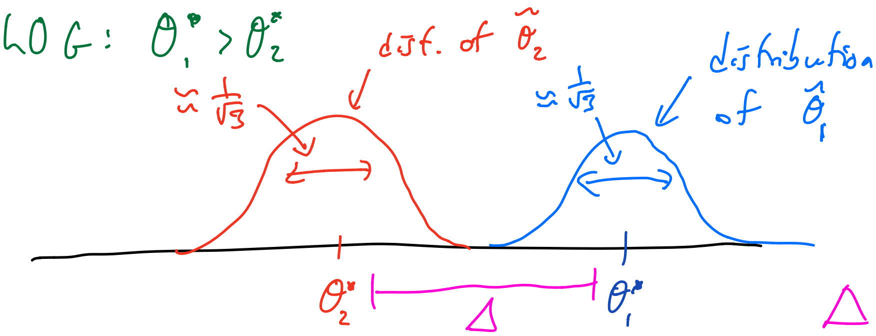
Pull each arm $\mathfrak{Z}$ times and observe $(X_{1,1}, \ldots X_{1,\mathfrak{Z}})$, $(X_{2,1}, \ldots, X_{2,\mathfrak{Z}})$

$$\hat{\theta}_i = \frac{1}{\mathfrak{Z}} \sum_{s=1}^{\mathfrak{Z}} X_{i,s}$$

For all $t > 2\mathfrak{Z}$ play $\arg\max_i \hat{\theta}_i$ (Assume $\mathrm{Var}(X_{i,1}) = O(1)$)

Goal: Analyze regret of this algorithm.

WLOG: $\theta_1^* > \theta_2^*$



$\Delta = \theta_1^* - \theta_2^*$

Want $\mathfrak{Z}$ to be large enough so that

$$\theta_1^* > \theta_2^* \implies \hat{\theta}_1 > \hat{\theta}_2.$$

Intuitively $\frac{1}{\sqrt{\mathfrak{Z}}} < \frac{\Delta}{2} \Rightarrow \mathfrak{Z} \gg 4\Delta^{-2}$

Goal: Construct a confidence interval s.t. $|\hat{\theta}_i - \theta_i^*| > \varepsilon$ w.p. $\leq 1-\delta$.

Let $X_1, \ldots, X_n$ be independent, identically distributed random variables. $\mathbb{E}[X_i] = 0$

$$\mathbb{P}\left(\frac{1}{n} \sum_{i=1}^{n} X_i > \varepsilon \right) \leq \delta.$$

<u>Chebychev's Inequality</u>

$$\mathbb{P}\left(\frac{1}{n} \sum_{i=1}^{n} X_i > \varepsilon \right) \leq \mathbb{P}\left(\left(\frac{1}{n} \sum_{i=1}^{n} X_i \right)^2 > \varepsilon^2 \right)$$

$$\leq \frac{\mathbb{E}\left[\left(\frac{1}{n} \sum_i^n X_i \right)^2 \right]}{\varepsilon^2} \leftarrow \text{Markov's}$$

$$= \frac{\mathbb{E}\left[\sum_i^n X_i^2 \right]}{n^2 \varepsilon^2}$$

$$= \frac{\mathrm{Var}(X_1)}{n \varepsilon^2} \qquad \left(X \sim \mathcal{N}(0, \sigma^2) \Rightarrow \frac{\sigma^2}{n \varepsilon^2} \right)$$

<u>Markov's Inequality</u> For any positive R.V. Z we have $\mathbb{P}(Z > \varepsilon) \leq \frac{\mathbb{E}[Z]}{t}$

(Proof hint. $\mathbb{E}[Z] = \int_0^\infty \mathbb{P}(Z > t)\, dt$)

Chernoff Bound Technique

Fix $\lambda > 0$.

$$\mathbb{P}\left(\frac{1}{n}\sum_{i=1}^{n} X_i > \varepsilon\right) = \mathbb{P}\left(\lambda \sum_{i=1}^{n} X_i > \lambda n \varepsilon\right)$$

$$= \mathbb{P}\left(\exp\left(\lambda \sum_{i=1}^{n} X_i\right) > \exp(\lambda n \varepsilon)\right)$$

$$\leq \frac{\mathbb{E}\left[\exp\left(\lambda \sum_i X_i\right)\right]}{\exp(\lambda n \varepsilon)} \quad \leftarrow \text{Markov}$$

By assumption $X_1 \perp X_2 \implies \mathbb{E}[g(X_1)g(X_2)] = \mathbb{E}[g(X_1)]\,\mathbb{E}[g(X_2)]$

$$= e^{-\lambda n \varepsilon}\, \mathbb{E}\left[\prod_{i=1}^{n} \exp(\lambda X_i)\right]$$

$$= e^{-\lambda n \varepsilon} \prod_{i=1}^{n} \mathbb{E}[\exp(\lambda X_i)]$$

$$= \exp(-\lambda n \varepsilon)\exp\left(n \log\left(\mathbb{E}[\exp(\lambda X_1)]\right)\right)$$

$$= \exp\Big(-\lambda n \varepsilon + n \underbrace{\log\left(\mathbb{E}[\exp(\lambda X_1)]\right)}_{=:\varphi_X(\lambda)}\Big)$$

$$\implies \mathbb{P}\left(\frac{1}{n}\sum_{i=1}^{n} X_i > \varepsilon\right) \leq \exp\left(-n\left(\sup_{\lambda>0} \lambda\varepsilon - \varphi_X(\lambda)\right)\right)$$

Ex. $X_i \sim \mathcal{N}(0, \sigma^2)$

$$\mathbb{E}[\exp(\lambda X)] = \int e^{\lambda x} \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{x^2}{2\sigma^2}} dx = e^{\lambda^2\sigma^2/2}$$

$$\varphi_X(\lambda) = \lambda^2\sigma^2/2$$

$$\sup_\lambda \lambda\varepsilon - \varphi_X(\lambda) = \sup_\lambda \lambda\varepsilon - \lambda^2\sigma^2/2 \quad \left(\lambda = \frac{\varepsilon}{\sigma^2}\right)$$

$$= \frac{\varepsilon^2}{2\sigma^2}$$

$$\Rightarrow \quad \mathbb{P}\left(\frac{1}{n}\sum_{i=1}^n X_i > \varepsilon\right) \le \exp\left(\frac{-n\varepsilon^2}{2\sigma^2}\right) = \delta$$

w.p. $\ge 1-\delta$ $\quad \frac{1}{n}\sum_{i=1}^n X_i \le \sqrt{2\sigma^2 \log(1/\delta)}$

Repeat w/ $-X_i \Rightarrow \mathbb{P}\left(\frac{1}{n}\sum_{i=1}^n X_i < -\varepsilon\right) \le \exp\left(\frac{-n\varepsilon^2}{2\sigma^2}\right)$

$$\mathbb{P}\left(\left\{\frac{1}{n}\sum_{i=1}^n X_i > \varepsilon\right\} \cup \left\{\frac{1}{n}\sum_i X_i < -\varepsilon\right\}\right)$$

$$\le \mathbb{P}\left(\frac{1}{n}\sum_i X_i > \varepsilon\right) + \mathbb{P}\left(\frac{1}{n}\sum_i X_i < -\varepsilon\right)$$

$$\le \delta$$

$\Rightarrow$ w.p. $\ge 1-\delta$ $\quad \left|\frac{1}{n}\sum_{i=1}^n X_i\right| \le \sqrt{2\sigma^2 \log(2/\delta)}$

$X_i \sim \text{Bernoulli}(p)$ $\qquad X_i = \begin{cases} 1 & \text{w.p. } p \\ 0 & \text{w.p. } 1-p \end{cases}$

$$\mathbb{E}[\exp(\lambda X)] = pe^{\lambda} + (1-p)e^{0}$$
$$= 1 - p + pe^{\lambda}$$

$$\sup_{\lambda} \lambda\varepsilon - \varphi_X(\lambda) \leq \begin{cases} \frac{\varepsilon^2}{4p} & p < 1/2 \\ \frac{\varepsilon^2}{4(1-p)} & p > 1/2 \\ 2\varepsilon^2 & p \in [0,1] \end{cases}$$

Hoeffding's Lemma For a mean-0 R.V. X s.t. $X \in [a,b]$ almost surely, we have $\log(\mathbb{E}[\exp(\lambda X)]) \leq \frac{\lambda^2(b-a)^2}{8}$

Definition A R.V. X is σ^2-sub-Gaussian if
$$\log(\mathbb{E}[\exp(\lambda X)]) \leq \frac{\lambda^2\sigma^2}{2}.$$

Corollary If $X_1, \ldots, X_n$ are mean-0 σ^2-sub-Gaussian iid R.V. then $\mathbb{P}\left(\frac{1}{n}\sum_{i=1}^{n} X_i > \varepsilon\right) \leq \exp\left(-\frac{n\varepsilon^2}{2\sigma^2}\right).$

Suppose $X_{i,s} \sim \text{Bernoulli}(\theta_i^*)$ then (by above)

we have $\mathcal{E}_i = \left\{ \left| \frac{1}{S} \sum_{s=1}^{S} (X_{i,s} - \theta_i^*) \right| \leq \sqrt{\frac{\log(4/\delta)}{2S}} \right\}$

w.p. $\geq 1 - \delta/2$ for each i (separately)

w.p. $\geq 1 - \delta$ for both $i \in \{1, 2\}$ simultaneously.

$$\mathbb{P}\left(\bigcup_{i=1}^{2} \mathcal{E}_i^c \right) \leq \delta.$$

Rule is play $\arg\max_i \hat{\theta}_i$. So on $\mathcal{E}_1 \cap \mathcal{E}_2$

$$\hat{\theta}_1 \geq \theta_1^* - \sqrt{\frac{\log(4/\delta)}{2S}}$$

$$= \theta_2^* + \Delta - \sqrt{\frac{\log(4/\delta)}{2S}} \qquad \theta_1^* - \theta_2^* = \Delta > 0$$

$$\geq \hat{\theta}_2 + \Delta - 2\sqrt{\frac{\log(4/\delta)}{2S}}$$

$$\Rightarrow \theta_1^* > \theta_2^* \Rightarrow \hat{\theta}_1 > \hat{\theta}_2 \quad \text{if} \quad S \geq \frac{2\log(4/\delta)}{\Delta^2}$$

$\uparrow$

$$R_T = \Delta \mathbb{E}[T_2]$$

$$= \Delta \cdot \xi$$

$$\leq \frac{2\log(4/\delta)}{\Delta}$$

(choose minimal ξ)

$$R_T \leq T\Delta$$

(Trivial)

$$R_T \leq \min\left\{\Delta T, \frac{2\log(4/\delta)}{\Delta}\right\}$$

$$\leq O\left(\sqrt{T}\right)$$

(worst case Δ)