CSE599d: Advanced Query Processing

Lecture 17: Entropic Inequalities

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Degree Sequencees

Overview

- AGM bound is $\max_{\boldsymbol{D}} |Q(\boldsymbol{D})|$; tight up to a “rounding factor” $1/2^n$.
- In practice, bound is too loose, because cardinalities are insufficient.
- More general statistics: degree sequences.

Local Statistics

Cardinality of each table: $|R|, |S|, \ldots$
Domain sizes of attributes: $|R.X|, |R.Y|, \ldots, |S.Z|, \ldots$

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Local statistics: a collection of frequency moments $\boxed{||\deg_R(*|X)||_p^p = \sum_i f_i^p}$

The Statistics: Examples

$R =$

X	Y	Z
1	a	10
1	b	20
1	b	30
2	a	40
2	b	50
3	b	60
3	c	70
4	d	80

Degree sequences:
$\text{deg}(*|X) = (3, 2, 2, 1)$

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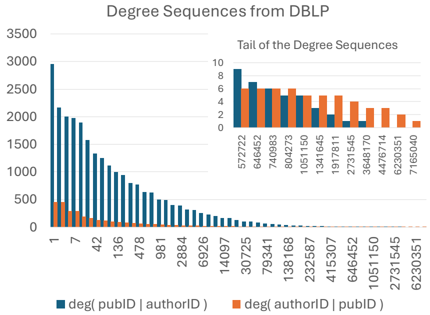
Local statistics: $p = 1, 2, \ldots, 10, \infty$ moments

Frequencies Matter



DBLP in 2024:

- $|\text{DBLP}| = 24 \cdot 10^6$
- Most prolific author: `H. Vincent Poor` 2951 pubs
- Most authors per paper: 450 [Srivastava et al., 2023]

Upper Bounds: Examples

$Q = R(X, Y) \bowtie S(Y, Z)$

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Many others Take the min!

Discussion

Pessimistic Cardinality Estimator:

- One-sided theoretical guarantee.[1]
- Uses local statistics on input tables.
- Easy to combine: just take their min.

How do we derive and compute these upper bounds?

[1] Why call it "estimator"? Term coined by [Cai et al., 2019].

Entropic Inequalities

Entropic Vectors

Random variable X with finite outcomes $X \in D$. Its entropy is:

$$\boxed{h(X) \stackrel{\text{def}}{=} -\sum_{x \in D} p(x) \log p(x)} \in \mathbb{R}_+$$

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$h(XY) = \log 4$

X	p
a	$3/4$
b	$1/4$

$h(X) \leq \log 2$

Y	p
p	$1/4$
q	$2/4$
m	$1/4$

$h(Y) \leq \log 3$

$\emptyset$	p
	1

$h(\emptyset) = 0$

Shannon Inequalities

Basic Shannon Inequalities

$$h(U \cup V) \geq h(U) \qquad \text{Monotonicity}$$

$$h(U) + h(V) \geq h(U \cup V) + h(U \cap V) \qquad \text{Submodularity}$$

A Shannon inequality is a consequence of these inequalities.

Example

$$h(XY) + h(YZ) + h(XZ) \geq 2h(XYZ)$$

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\underline{h(XY) + h(YZ)} &+ h(XZ) \\
&\geq h(XYZ) + \underline{h(Y) + h(XZ)} \\
&\geq h(XYZ) + h(XYZ) + h(\emptyset)
\end{aligned}
$$

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We use Shannon inequalities to derive upper bounds

Example

$$Q(X,Y,Z) = R(X,Y) \bowtie S(Y,Z) \bowtie T(Z,X)$$

$$\boxed{|Q| \leq (|R| \cdot |S| \cdot |T|)^{1/2}}$$

[Atserias et al., 2013, Friedgut and Kahn, 1998]

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Define the uniform distribution on the output Q:

Q

X	Y	Z	
a	3	m	1/5
a	2	q	1/5
b	2	q	1/5
d	3	m	1/5
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S

Y	Z	
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T

X	Z	
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$\log |R|$

$\geq h(XY)$

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$$|R| \cdot |S| \cdot |T| \geq |Q|^2$$

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Generalized Shearer's Inequality

Theorem If $\boldsymbol{w}$ is a fractional edge cover of the hyperedges $\boldsymbol{Y}_1, \ldots, \boldsymbol{Y}_m$, then

$$\boxed{w_1 h(\boldsymbol{Y}_1) + \cdots + w_m h(\boldsymbol{Y}_m) \geq h(\boldsymbol{X})}$$

[2] [Balister and Bollobás, 2012]

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Proof[2] Equivalently:

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where each variable is "covered $\geq k_0$ times".

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- Every variable remains covered $\geq k_0$ times.

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Theorem If $\boldsymbol{w}$ is a fractional edge cover of the hyperedges $\boldsymbol{Y}_1, \ldots, \boldsymbol{Y}_m$, then

$$\boxed{w_1 h(\boldsymbol{Y}_1) + \cdots + w_m h(\boldsymbol{Y}_m) \geq h(\boldsymbol{X})}$$

Proof[2] Equivalently:

$$\boxed{k_1 h(\boldsymbol{Y}_1) + \cdots + k_m h(\boldsymbol{Y}_m) \geq k_0 h(\boldsymbol{X})}$$

where each variable is "covered $\geq k_0$ times".

Repeatedly rewrite $h(\boldsymbol{Y}_i) + h(\boldsymbol{Y}_j) \to h(\boldsymbol{Y}_i \cup \boldsymbol{Y}_j) + h(\boldsymbol{Y}_i \cap \boldsymbol{Y}_j)$

- Every variable remains covered $\geq k_0$ times.
- $\sum_\ell |\boldsymbol{Y}_\ell|^2$ strictly increases (homework!).

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- Every variable remains covered $\geq k_0$ times.
- $\sum_\ell |\boldsymbol{Y}_\ell|^2$ strictly increases (homework!). When do we stop?
- Stop when $\boldsymbol{Y}_1 \supset \boldsymbol{Y}_2 \supset \cdots$ Then $\boldsymbol{Y}_1 = \boldsymbol{X}$ and $k_1 \geq k_0$:

$$\boxed{k_1 h(\boldsymbol{Y}_1) + k_2 h(\boldsymbol{Y}_2) + \cdots \geq k_1 h(\boldsymbol{Y}_1) \geq k_0 h(\boldsymbol{X})}$$

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Alternate Proof of the AGM Bound

$Q(\boldsymbol{X}) = R_1(\boldsymbol{Y}_1) \wedge \cdots \wedge R_m(\boldsymbol{Y}_m).$

For every fractional edge cover $\boldsymbol{w}$: $\boxed{|Q(\boldsymbol{D})| \leq |R_1^D|^{w_1} \cdots |R_m^D|^{w_m}}$

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Let h be the entropic vector of the uniform distribution on $Q(\boldsymbol{D})$.

$$\begin{aligned} w_1 \log |R_1^D| + \cdots + w_m \log |R_m^D| \geq \\ \geq w_1 h(\boldsymbol{Y}_1) + \cdots + w_m h(\boldsymbol{Y}_m) \end{aligned}$$

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\geq & w_1 h(\boldsymbol{Y}_1) + \cdots + w_m h(\boldsymbol{Y}_m) \\
\geq & h(\boldsymbol{X}) && \text{Shearer's inequality} \\
= & \log Q(\boldsymbol{D}) && \text{uniform distribution on } \boldsymbol{X}
\end{aligned}
$$

Discussion

- AGM/Shearer limited to cardinality statistics.
- More general statistics require Shannon inequalities.
- Everything gets harder: fractional edge cover no longer sufficient, order of the submodularity steps matters, tightness no longer guaranteed.
- Yet, lots of theory insights can make it work very well: next.

The LpBound

The Conditional Entropy

$$\boxed{h(\boldsymbol{V}|\boldsymbol{U}) \stackrel{\text{def}}{=} h(\boldsymbol{UV}) - h(\boldsymbol{U})}$$

May add $\boldsymbol{U}$ to $\boldsymbol{V}$: $h(\boldsymbol{UV}|\boldsymbol{U}) = h(\boldsymbol{V}|\boldsymbol{U})$

Semantics: $h(\boldsymbol{V}|\boldsymbol{U}) = \mathbb{E}_{\boldsymbol{u}}[h(\boldsymbol{V}|\boldsymbol{U} = \boldsymbol{u})]$

The Chain Rule: $\boxed{h(\boldsymbol{U}) + h(\boldsymbol{V}|\boldsymbol{U}) = h(\boldsymbol{UV})}$

Submodularity becomes: $\boxed{h(\boldsymbol{V}|\boldsymbol{U}) \geq h(\boldsymbol{V}|\boldsymbol{UW})}$

Connecting Statistics to the Entropy Vector

Relation $R(X, Y, Z)$:

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$$h(Y|X) \leq \log||\deg_R(Y|X)||_\infty$$

$$\mathbb{E}_x[h(Y|X=x)] \leq \mathbb{E}_x[\log(\deg(Y|X=x))]$$
$$\leq \max_x(\log(\deg(Y|X=x)))$$

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$$\mathbb{E}_x[h(Y|X=x)] \leq \mathbb{E}_x[\log(\text{deg}(Y|X=x))] \leq \max_x(\log(\text{deg}(Y|X=x)))$$

$$\boxed{h(X) + p \cdot h(Y|X) \leq \log||\text{deg}(Y|X)||_p^p}$$

[Zhang et al., 2025]

Equivalently: $\frac{1}{p}h(X) + h(Y|X) \leq \log||\text{deg}(Y|X)||_p$

The LpBound[3]

$Q(\mathbf{X}) = R_1(\mathbf{Y}_1) \wedge \cdots \wedge R_m(\mathbf{Y}_m).$

Collect various statistics of the form $||\text{deg}_{R_j^D}(\mathbf{Y}|\mathbf{X})||_p$, for $\mathbf{X}, \mathbf{Y} \subseteq \texttt{Vars}(R_j)$.

This includes cardinalities, domain sizes, max degrees.

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Prove a Shannon inequality of the form:

$$\sum_i w_i\Big(h(X_i) + p \cdot h(Y_i|X_i)\Big) \geq h(\boldsymbol{X})$$

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Prove a Shannon inequality of the form:

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It follows:

$$\boxed{\prod_i ||\text{deg}_{R_{j_i}^D}(\boldsymbol{Y}_i|\boldsymbol{X}_i)||_{p_i}^{w_i p_i} \geq Q(\boldsymbol{D})}$$

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Example 1

$Q(X, Y, Z) = R(X, Y) \wedge S(Y, Z) \wedge T(Z, X)$

Then $|Q| \leq \left(||\text{deg}_R(Y|X)||_2^2 \cdot ||\text{deg}_S(Z|Y)||_2^2 \cdot ||\text{deg}_T(X|Z)||_2^2\right)^{1/3}$.

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Proof: let h be the entropy vector of the uniform distribution on $Q(\boldsymbol{D})$:

$$\log ||\text{deg}_R(Y|X)||_2^2 + \log ||\text{deg}_S(Z|Y)||_2^2 + \log ||\text{deg}_T(X|Z)||_2^2 \geq$$

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&\geq h(XY) + h(Y|XZ) + h(YZ) + h(Z|XY) + h(XZ) + h(X|YZ) \\
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$$Q = R(X,Y) \wedge S(Y,Z) \wedge T(Z,U) \wedge \left(X + Z = U\right) \wedge \left(Y * U = X\right)$$

Assume $|R^D| = |S^D| = |T^D| = N$ $\qquad AGM(Q) = N^2$.

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&\geq h(XYZ) + h(YZU) + \underline{h(U|XZ)} + \underline{h(X|YU)} \\
&\geq h(XYZ) + h(YZU) + h(U|XYZ) + h(X|YZU) \\
&= 2h(XYZU) = \boxed{2\log|Q|}
\end{aligned}$$

Discussion: AGM v.s. LpBound

- Any Shannon inequality gives a bound on $Q(\boldsymbol{D})$; similar to AGM.
- Beyond Shearer inequalities: general Shannon inequalities are more difficult (next).
- AGM bound: tight up to a rounding factor $\frac{1}{2^n}$.
  LpBound: not tight in general (!!!)
  LpBound for simple conditionals: tight up to rounding factor $\frac{1}{2^{2^n-1}}$

Computing the Upper Bound

Motivation

n = number of variables in the query.

AGM bound:

- Optimal fractional edge cover: Primal Linear Program.
- Optimal vertex packing: Dual Linear Program, with n unknowns.

LpBound:

- Optimal Shannon inequality: Dual Linear Program.
- Optimal vector h: Primal Linear program, with 2^n unknowns.

Note: my choice of which LP to call "primal" or "dual" is inconsistent between AGM and LpBound.

The Primal Linear Program

$Q(\boldsymbol{X}) = \bigwedge_j R_j(\boldsymbol{X}_j)$, m atoms, n variables.

$\boxed{\text{Maximize } h(\boldsymbol{X})}$ where:

- There are 2^n unknowns: $h(\boldsymbol{U})$ for all $\boldsymbol{U} \subseteq \boldsymbol{X}$.

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- For each stats add the corresponding constraint:

$$h(\boldsymbol{X}_j) \leq \log |R_j|$$

$$\cdots$$

$$h(\boldsymbol{U}) + p h(\boldsymbol{V}|\boldsymbol{U}) \leq \log ||\text{deg}_{R_k}(\boldsymbol{V}|\boldsymbol{U})||_p^p$$

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$Q(\boldsymbol{X}) = \bigwedge_j R_j(\boldsymbol{X}_j)$, m atoms, n variables.

Maximize $h(\boldsymbol{X})$ where:

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$$h(\boldsymbol{U}) + ph(\boldsymbol{V}|\boldsymbol{U}) \leq \log ||\text{deg}_{R_k}(\boldsymbol{V}|\boldsymbol{U})||_p^p$$

- Add all Shannon inequalities as constraints:

$$-h(XY) - h(YZ) + h(XYZ) + h(Y) \leq 0$$

$$\ldots$$

Example

$R(X, Y) \wedge S(Y, Z) \wedge T(Z, X)$

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Primal:

Maximize $h(XYZ)$, where:

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Example

$R(X, Y) \wedge S(Y, Z) \wedge T(Z, X)$

Primal:

Maximize $h(XYZ)$, where:

$$c_1 : \quad h(XY) \leq \log |R|$$
$$c_2 : \quad h(YZ) \leq \log |S|$$
$$c_3 : \quad h(XZ) \leq \log |T|$$

Example

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Maximize $h(XYZ)$, where:

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c_1 &: & h(XY) &\leq \log|R| \\
c_2 &: & h(YZ) &\leq \log|S| \\
c_3 &: & h(XZ) &\leq \log|T| \\
\sigma_1 &: & -h(XY) - h(YZ) & \\
& & +h(XYZ) + h(Y) \leq 0 &
\end{aligned}
$$

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\sigma_2 :&\quad -h(Y) - h(XZ) \\
&\quad +h(XYZ) \leq 0 \\
&\quad \ldots
\end{aligned}
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Example

$R(X,Y) \wedge S(Y,Z) \wedge T(Z,X)$

Primal:

Maximize $h(XYZ)$, where:

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\begin{aligned}
{\color{red}c_1}: &\quad h(XY) \leq \log|R| \\
{\color{red}c_2}: &\quad h(YZ) \leq \log|S| \\
{\color{red}c_3}: &\quad h(XZ) \leq \log|T| \\
\sigma_1: &\quad -h(XY) - h(YZ) \\
&\quad +h(XYZ) + h(Y) \leq 0 \\
\sigma_2: &\quad -h(Y) - h(XZ) \\
&\quad +h(XYZ) \leq 0 \\
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Dual:

Minimize $c_1 \log|R| + c_2 \log|S| + c_3 \log|T|$

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Correctness: any feasible solution $c_1, c_2, c_3, \sigma_1, \ldots, \sigma_{18}$ of the dual defines a Shannon inequality $c_1 h(XY) + c_2 h(YZ) + c_3 h(XZ) \geq h(XYZ)$.

Correctness Proof

Theorem

Any feasible solution $c_1, c_2, c_3, \sigma_1, \ldots, \sigma_{18}$ *of the primal defines a Shannon inequality* $c_1 h(XY) + c_2 h(YZ) + c_3 h(XZ) \geq h(XYZ)$.

Proof: Multiply each inequality in the dual with its h-term and add them:

$$h(XYZ)(\sigma_1 + \cdots) + h(XY)(c_1 - \sigma_1 + \cdots) + \cdots \geq h(XYZ)$$

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e.g. $\sigma_1(-h(XY) - h(YZ) + h(XYZ) + h(Y))$

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Shannon inequalities $-h(XY) - h(YZ) + h(XYZ) + h(Y) \leq 0$ imply:

$$c_1 h(XY) + c_2 h(YZ) + c_3 h(XZ) \geq h(XYZ)$$

Discussion of the LpBound

- Primal: maximize $h(\boldsymbol{X})$, where h is a polymatroid[4]
  2^n unknowns $\boldsymbol{h} \in \mathbb{R}_+^{2^n}$.
  $\boldsymbol{h}$ is not a database instance in general.

- Special case: all stats are cardinalities (AGM bound)
  Shearer inequality; $\boldsymbol{h}$ is modular, defines product database.
  LP has n unknowns $h(X_1), \ldots, h(X_n)$: fractional vertex packing.

- Special case: all degree stats are simple.
  $\boldsymbol{h}$ is normal, defines normal database [Suciu, 2023].
  Solvable by an LP with n^2 unknowns [Im et al., 2025].

- Special case: Q is Berge acyclic, degree stats are simple.
  Solvable by an LP with $m + n$ unknowns.

[4] Satisfies all basic Shannon inequalities

Final Thoughts: Entropic Inequalities in Databases

- LpBound, (plus successor CorrBound): a practical cardinality estimator.
- Generic Join: matches the AGM bound. Practical, but incompatible with query engines.
- PandaExpress: matches the LpBound. Not yet practical. [Khamis et al., 2025].

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