Finite Model Theory
Lecture 11: Fixpoints and Infinitary Logic

Spring 2025

Announcements

- HW2 graded, posted.
- HW1 posted too (apparently it wasn't posted)
- HW3 due on Friday
- This week: fixpoints, infinitary logics, pebble games

Fixpoints

Tarski-Knaster

Recall: 2^U is the set of subsets of U.

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Theorem (Tarski-Knaster)

Every monotone function F has a least fixpoint, denoted $\mathrm{lfp}(F)$.

We consider only the finite case, discussed next.

The Kleene Sequence

Suppose U is finite, and $F : 2^U \to 2^U$ is monotone.

Definition

The Kleene Sequence is $X^0, X^1, \cdots, X^n, \cdots$, where $X^0 = \varnothing$ and $X^{i+1} = F(X^i)$

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Since U is finite, $\exists n \geq 0$ such that $X^n = X^{n+1}$. Then $\boxed{X^n = \mathrm{lfp}(F)}$

Datalog

Datalog

A datalog rule has the form:

$$P(\boldsymbol{x}_0) = R_1(\boldsymbol{x}_1) \wedge \cdots \wedge R_m(\boldsymbol{x}_m)$$

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Safety condition: every variable in the head must occur in the body.

A datalog program is a set of datalog rules.

Classic Datalog Example

Transitive closure:

$$\boxed{\begin{array}{l} T(x,y) = E(x,y). \\ T(x,y) = E(x,z) \wedge T(z,y). \end{array}}$$

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An equivalent, less commonly used notation:

$$T(x,y) = E(x,y) \vee \exists z(E(x,z) \wedge T(z,y))$$

Fixpoint Semantics of Datalog

Extensional Database Predicates, EDB: the input relations.

Intentional Database Predicates, IDB: the computed relations.

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Immediate Consequence Operator (ICO): new IDBs from current IDBs:

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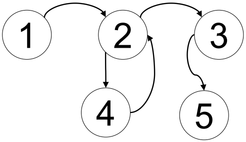
The semantics of a datalog program is $\text{lfp}(F)$

Use Kleene sequence: $\boldsymbol{P}^0 (:= \emptyset) \subset \boldsymbol{P}^1 \subset \boldsymbol{P}^2 \subset \cdots \subset \boldsymbol{P}^n = \boldsymbol{P}^{n+1} = \text{lfp}(F)$

Example: Reachability

$$\boxed{\begin{array}{l} P(4). \\ P(y) = P(x) \wedge E(x,y). \end{array}}$$



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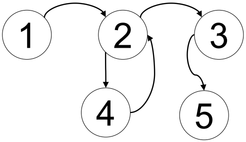

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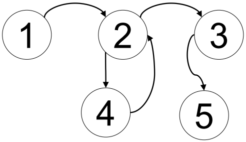

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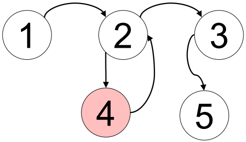

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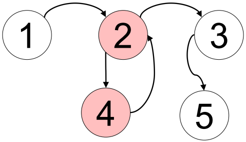

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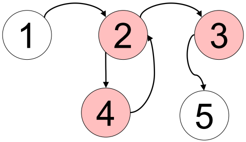

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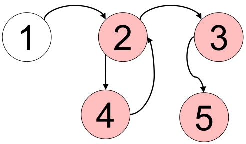

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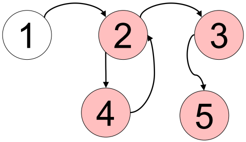

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Is there a path of even length from a to b?

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$$\boxed{\begin{array}{l} P(x,y) = E(x,z) \wedge E(z,y) \\ P(x,y) = P(x,z) \wedge E(z,u) \wedge E(u,y) \\ \text{Answer}() = P(a,b) \end{array}}$$

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Product graph $G \times G$:

- Nodes: $V \times V$
- Edges: $\{((x_1, x_2), (y_1, y_2)) \mid (x_1, y_1), (x_2, y_2) \in E\}$

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P(x1, x2, y1, y2) = E(x1, y1) \wedge E(x2, y2) \\
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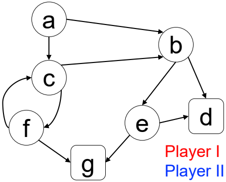
Win-Move Game

$T(x, y, z)$: x has children y, z; $L(x)$: x is a leaf.



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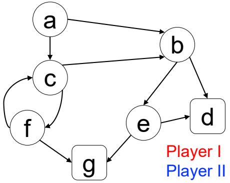
A pebble is placed on a starting node:

- Player I moves to a child,
- Player II moves to a child, etc
- Player who cannot move loses.



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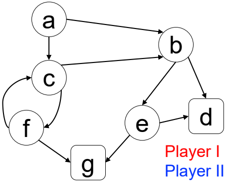

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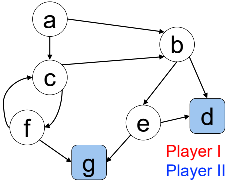

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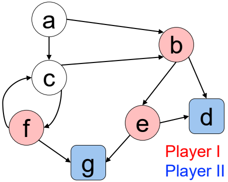

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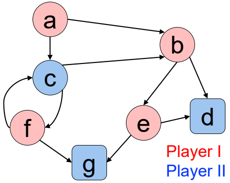

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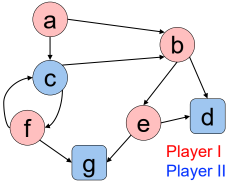

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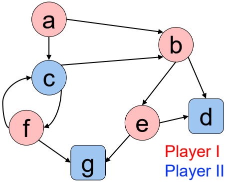

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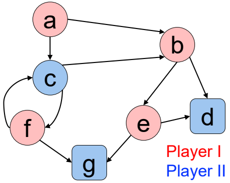

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Example: transitive closure of the complement graph:

$$\boxed{\begin{array}{l} T(x,y) = V(x) \wedge V(y) \wedge \neg E(x,y). \\ T(x,y) = T(x,z) \wedge \neg E(z,y). \end{array}}$$

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Example: the complement of the transitive closure:

$$\boxed{\begin{array}{l} T(x,y) = E(x,y). \\ T(x,y) = T(x,z) \wedge E(z,y). \\ \text{Answer}(x,y) = V(x) \wedge V(y) \wedge \neg T(x,y). \end{array}}$$

Discussion

- Datalog can express some cool queries. Try at home:
  - Same generation: if $G = (V, E)$ is a tree, find pairs of nodes x, y in the same generation (same distance to the root)
  - Check if G is a totally balanced tree.
- But it cannot express some trivial queries:
  - Is $|E|$ even?
  - Is $|A| \leq |B|$? (Homework)
- To prove inexpressibility results for datalog we will show that it is a subset of a much more powerful logic, $L^{\omega}_{\infty\omega}$, then describe a game for $L^{\omega}_{\infty\omega}$.
- First, we discuss Fixpoint Logic

Fixpoint Logic

Definition

Fixpoint Logic, LFP extends FO with a least fixpoint predicate:

$$\mathrm{lfp}_{P,\boldsymbol{x}}\varphi$$

where P is a fresh relational symbol that occurs positively, and φ is a formula that uses P has free variables $\boldsymbol{x}$.

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Example: transitive closure $T(u, v)$ becomes:

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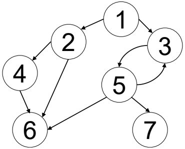
We will continue to use the more friendly notation, example:

$$\boxed{T(u,v)\ \texttt{where}\ T(x,y) = E(x,y) \vee \exists z(E(x,z) \wedge T(z,y))}$$

Example

What doses this formula compute?

$$\boxed{S(x) = (\forall y(E(x,y) \rightarrow S(y)))}$$



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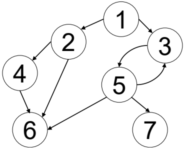

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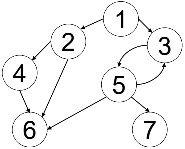

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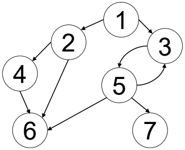

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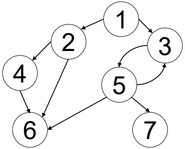

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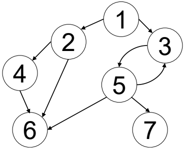

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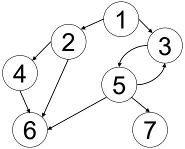

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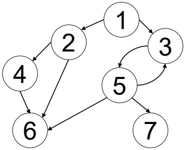

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G is acyclic iff $\forall x S(x)$

Example: The Win-Move Game

$G = (V, E)$ no more restriction to two children

$$\boxed{P1(x) = \exists y(E(x,y) \wedge \forall z(E(y,z) \to P1(z)))}$$

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Theorem (Kolaitis)

The win-move game is not expressible in stratified datalog.

Contrast with our previous solution on graphs with degree 2.

Unfolding the Recursion

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The ICO: $F(S) = \{x \mid \forall y(E(x, y) \to S(y))\}$

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$$\boxed{\mathrm{lfp}(S)(x) = \varphi^0(x) \vee \varphi^1(x) \vee \cdots \vee \varphi^n(x) \vee \cdots}$$

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- What are ∞ and ω? Will discuss next.

Ordinal Numbers

Review: Cardinal Numbers

A cardinal number is an equivalence class $|A|$ under bijection.

$\aleph_0 = |\mathbb{N}|$ is the smallest infinite cardinal number

$\mathfrak{c} = |\mathbb{R}|$ is the cardinal of the continuum.

Weird arithmetic: $\aleph_0 + \mathfrak{c} = \mathfrak{c}$, $\aleph_0 \times \mathfrak{c} = \mathfrak{c}$, $\mathfrak{c} \times \mathfrak{c} = \mathfrak{c}$, ...

Much larger cardinal numbers exists: $\aleph_0 < 2^{\aleph_0} < 2^{2^{\aleph_0}} < 2^{2^{2^{\aleph_0}}} < \cdots$

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Fact

$\leq$ is a well order iff its total and has no infinite descending chains:

$$x_0 > x_1 > x_2 > \dots$$

Ordinal Numbers

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A cardinal number is an equivalence class (under bijection) of a set.

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An ordinal number is an isomorphism class of a well ordered set.

Examples

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All these are countable ordinals, i.e. their cardinal number is $\aleph_0$

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Every set can be well ordered.

Every ordinal number has a unique cardinal number.

A cardinal number can have many ordinal numbers.

$L_{\infty\omega}$

$L^{\omega}_{\infty\omega}$

- Let α, β be ordinals. The infinitary logic $L_{\alpha\beta}$ is:
  - Atoms: $x_i = x_j$, $R(\cdots)$
  - Formulas: $\bigvee_{i\in I} \varphi_i$, and $\underbrace{(\ldots \exists x_j \ldots)}_{j\in J}\varphi$, and $\neg\varphi$,for $|I| < \alpha$, $|J| < \beta$.

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- $L^{\omega}_{\infty\omega} = \bigcup_{k \geq 0} L^k_{\infty\omega}$.

What is $\bigcup_{k \geq 0} FO^k$?

Discussion

Next time:

- The logic LFP is a subset of $L^{\omega}_{\infty\omega}$.
- Pebble games, which generalize EF games, characterize expressibility in $L^{\omega}_{\infty\omega}$.
- As an example, will show that EVEN is not expressible in $L^{\omega}_{\infty\omega}$, hence neither in LFP or Datalog.