Finite Model Theory
Lecture 7: Ehrenfeucht-Fraisse Games

Spring 2025

Where Are We

- Classical model theory: concerned with satisfiability, implication.
- Finite model theory:
  - Satisfiability: undedicable (Trakhtenbrot); same for implication.
  - 0/1 laws: only makes sense on finite models
  - Model checking: only makes sense on finite models

Today: we start Expressiblity: what properties can we define?

Expressibility

Important techniques:

- Ehrenfeuched-Fraisse Games for FO
- Pebble Games for Infinitary logics (includes recursion)

The Expressibility Problem

Fix a property of structures, P. Is there a sentence φ that expresses P?

$$\boldsymbol{A} \vDash \varphi \text{ iff } P(\boldsymbol{A}) \text{ is true}$$

φ may be in FO, or some extension.

Example properties: $\mathrm{CONNECTIVITY}$, EVEN, TREE.

We may restrict the structure $\boldsymbol{A}$. Example: restrict to linear orders.

Example: EVEN

Find a sentence φ s.t. $\boldsymbol{A} \vDash \varphi$ iff the size of $\mathrm{Dom}(\boldsymbol{A})$ is even.

Example: EVEN

Find a sentence φ s.t. $\boldsymbol{A} \vDash \varphi$ iff the size of $\mathrm{Dom}(\boldsymbol{A})$ is even.

Impossible!
$\mu_n(\varphi) = 0$ when n = odd, $\mu_n(\varphi) = 1$ when n = even, violates 0/1-law.

Example: EVEN

Find a sentence φ s.t. $\boldsymbol{A} \models \varphi$ iff the size of $\mathrm{Dom}(\boldsymbol{A})$ is even.

Impossible!
$\mu_n(\varphi) = 0$ when $n = \text{odd}$, $\mu_n(\varphi) = 1$ when $n = \text{even}$, violates 0/1-law.

Now assume that the structure is a linear order $(L, <)$.

The 0/1 law no longer helps.

Elementary Equivalence

Attempt to prove P is not expressible:
Find $\boldsymbol{A}, \boldsymbol{B}$ s.t. $P(\boldsymbol{A}) = 1$, $P(\boldsymbol{B}) = 0$, and $\forall \varphi$, $\boldsymbol{A} \vDash \varphi$ iff $\boldsymbol{B} \vDash \varphi$.

Elementary Equivalence

Attempt to prove P is not expressible:
Find $\boldsymbol{A}, \boldsymbol{B}$ s.t. $P(\boldsymbol{A}) = 1$, $P(\boldsymbol{B}) = 0$, and $\forall \varphi$, $\boldsymbol{A} \models \varphi$ iff $\boldsymbol{B} \models \varphi$.

Definition

$\boldsymbol{A}$ and $\boldsymbol{B}$ are *elementary equivalent* if forall $\varphi \in \mathrm{FO}$, $\boldsymbol{A} \models \varphi$ iff $\boldsymbol{B} \models \varphi$.

We write $\boldsymbol{A} \equiv \boldsymbol{B}$.

Elementary Equivalence

Attempt to prove P is not expressible:
Find $\boldsymbol{A}, \boldsymbol{B}$ s.t. $P(\boldsymbol{A}) = 1$, $P(\boldsymbol{B}) = 0$, and $\forall \varphi$, $\boldsymbol{A} \vDash \varphi$ iff $\boldsymbol{B} \vDash \varphi$.

Definition

$\boldsymbol{A}$ and $\boldsymbol{B}$ are *elementary equivalent* if forall $\varphi \in \mathrm{FO}$, $\boldsymbol{A} \vDash \varphi$ iff $\boldsymbol{B} \vDash \varphi$.

We write $\boldsymbol{A} \equiv \boldsymbol{B}$.

Example: $\sigma = (0, \mathtt{succ})$. Then $(\mathbb{N}, 0, \mathtt{succ}) \equiv (\mathbb{N} \cup \mathbb{Z}, 0, \mathtt{succ})$.

If $P(\boldsymbol{A}) = 1$, $P(\boldsymbol{B}) = 0$ and $\boldsymbol{A} \equiv \boldsymbol{B}$, then P is not expressible in FO.

Elementary Equivalence

Attempt to prove P is not expressible:
Find $\boldsymbol{A}, \boldsymbol{B}$ s.t. $P(\boldsymbol{A}) = 1$, $P(\boldsymbol{B}) = 0$, and $\forall \varphi$, $\boldsymbol{A} \vDash \varphi$ iff $\boldsymbol{B} \vDash \varphi$.

Definition

$\boldsymbol{A}$ and $\boldsymbol{B}$ are *elementary equivalent* if forall $\varphi \in$ FO, $\boldsymbol{A} \vDash \varphi$ iff $\boldsymbol{B} \vDash \varphi$.

We write $\boldsymbol{A} \equiv \boldsymbol{B}$.

Example: $\sigma = (0, \mathtt{succ})$. Then $(\mathbb{N}, 0, \mathtt{succ}) \equiv (\mathbb{N} \cup \mathbb{Z}, 0, \mathtt{succ})$.

If $P(\boldsymbol{A}) = 1$, $P(\boldsymbol{B}) = 0$ and $\boldsymbol{A} \equiv \boldsymbol{B}$, then P is not expressible in FO.

Unfortunately, $\equiv$ does not help us over finite models: we explain next.

Isomorphism

Assume a *relational vocabulary* $\sigma = (R_1, \ldots, R_k, c_1, \ldots, c_m)$

$\boldsymbol{A} = (A, R_1^A, \ldots, R_k^A, c_1^A, \ldots, c_m^A)$, $\boldsymbol{B} = (B, R_1^B, \ldots, R_k^B, c_1^B, \ldots, c_m^B)$.

Isomorphism

Assume a *relational vocabulary* $\sigma = (R_1, \dots, R_k, c_1, \dots, c_m)$

$\boldsymbol{A} = (A, R_1^A, \dots, R_k^A, c_1^A, \dots, c_m^A)$, $\boldsymbol{B} = (B, R_1^B, \dots, R_k^B, c_1^B, \dots, c_m^B)$.

Definition

An *isomorphism* $f : \boldsymbol{A} \to \boldsymbol{B}$ is a bijection s.t f, f^{-1} are homomorphisms.

We write $\boldsymbol{A} \simeq \boldsymbol{B}$.

Isomorphism

Assume a *relational vocabulary* $\sigma = (R_1, \dots, R_k, c_1, \dots, c_m)$

$\boldsymbol{A} = (A, R_1^A, \dots, R_k^A, c_1^A, \dots, c_m^A)$, $\boldsymbol{B} = (B, R_1^B, \dots, R_k^B, c_1^B, \dots, c_m^B)$.

Definition

An *isomorphism* $f : \boldsymbol{A} \to \boldsymbol{B}$ is a bijection s.t f, f^{-1} are homomorphisms.

We write $\boldsymbol{A} \simeq \boldsymbol{B}$.

$$(a_1, \dots, a_k) \in R_i^A \text{ iff } (f(a_1), \dots, f(a_k)) \in R_i^B,\ i = 1, k$$
$$f(c_j^A) = c_j^B,\ j = 1, m.$$

Isomorphism

Assume a *relational vocabulary* $\sigma = (R_1, \ldots, R_k, c_1, \ldots, c_m)$

$\boldsymbol{A} = (A, R_1^A, \ldots, R_k^A, c_1^A, \ldots, c_m^A)$, $\boldsymbol{B} = (B, R_1^B, \ldots, R_k^B, c_1^B, \ldots, c_m^B)$.

Definition

An *isomorphism* $f : \boldsymbol{A} \to \boldsymbol{B}$ is a bijection s.t f, f^{-1} are homomorphisms.

We write $\boldsymbol{A} \simeq \boldsymbol{B}$.

$$(a_1, \ldots, a_k) \in R_i^A \text{ iff } (f(a_1), \ldots, f(a_k)) \in R_i^B,\ i = 1, k$$
$$f(c_j^A) = c_j^B,\ j = 1, m.$$

Fact

If $\boldsymbol{A} \simeq \boldsymbol{B}$ then $\boldsymbol{A} \equiv \boldsymbol{B}$.

We want $P(\boldsymbol{A}) = 1$, $P(\boldsymbol{B}) = 0$, (which implies $\boldsymbol{A} \not\simeq \boldsymbol{B}$), and $\boldsymbol{A} \equiv \boldsymbol{B}$.

Elementary Equivalence for Finite Structures

Theorem

If $\boldsymbol{A}, \boldsymbol{B}$ are finite and $\boldsymbol{A} \equiv \boldsymbol{B}$ then $\boldsymbol{A} \simeq \boldsymbol{B}$.

In words: if two finite structures satisfy the same FO sentences, then they are the same (up to isomorphism)

Elementary Equivalence for Finite Structures

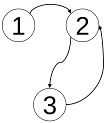
Theorem

If $\boldsymbol{A}, \boldsymbol{B}$ are finite and $\boldsymbol{A} \equiv \boldsymbol{B}$ then $\boldsymbol{A} \simeq \boldsymbol{B}$.

In words: if two finite structures satisfy the same FO sentences, then they are the same (up to isomorphism)

Proof. There exists a sentence $\varphi^{\boldsymbol{A}}$ that uniquely identifies $\boldsymbol{A}$.

$\boldsymbol{A} =$



$$\begin{aligned}\varphi^{\boldsymbol{A}} = \exists x \exists y \exists z (x \neq y) \wedge (x \neq z) \wedge (y \neq z) \\ \wedge \forall u (u = x) \vee (u = y) \vee (u = z) \\ \wedge \neg E(x,x) \wedge E(x,y) \wedge \neg E(x,z) \\ \wedge \neg E(y,x) \wedge \neg E(y,y) \wedge E(y,z) \\ \wedge \neg E(z,x) \wedge E(z,y) \wedge \neg E(z,z)\end{aligned}$$

Ehrenfeucht-Fraisse Games

Substructure and Induced Substructure

$\sigma = (R_1, \ldots, R_k, c_1, \ldots, c_m)$.

Substructure and Induced Substructure

$\sigma = (R_1, \ldots, R_k, c_1, \ldots, c_m)$.

$\boldsymbol{A}$ is a substructure of $\boldsymbol{B}$, $\boldsymbol{A} \subseteq \boldsymbol{B}$ if:

- $A \subseteq B$, where
  $A = \mathrm{Dom}(\boldsymbol{A})$, $B = \mathrm{Dom}(\boldsymbol{B})$
- $R_i^A \subseteq R_i^B$, $i = 1, k$.
- $c_j^A = c_j^B$, $j = 1, m$.

Substructure and Induced Substructure

$\sigma = (R_1, \ldots, R_k, c_1, \ldots, c_m)$.

$\boldsymbol{A}$ is a substructure of $\boldsymbol{B}$, $\boldsymbol{A} \subseteq \boldsymbol{B}$ if:

- $A \subseteq B$, where $A = \mathrm{Dom}(\boldsymbol{A})$, $B = \mathrm{Dom}(\boldsymbol{B})$
- $R_i^A \subseteq R_i^B$, $i = 1, k$.
- $c_j^A = c_j^B$, $j = 1, m$.

It is an induced substructure, $\boldsymbol{A} = \boldsymbol{B}|_A$ if, in addition:

- $R_i^B \cap A^{\mathrm{arity}(R_i)} \subseteq R_i^A$, $i = 1, k$.

Substructure and Induced Substructure

$\sigma = (R_1, \ldots, R_k, c_1, \ldots, c_m)$.

$\boldsymbol{A}$ is a substructure of $\boldsymbol{B}$, $\boldsymbol{A} \subseteq \boldsymbol{B}$ if:

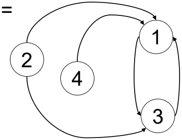
- $A \subseteq B$, where $A = \mathrm{Dom}(\boldsymbol{A})$, $B = \mathrm{Dom}(\boldsymbol{B})$
- $R_i^A \subseteq R_i^B$, $i = 1, k$.
- $c_j^A = c_j^B$, $j = 1, m$.

Which are induced subgraphs of $\boldsymbol{B}$?

It is an induced substructure, $\boldsymbol{A} = \boldsymbol{B}|_A$ if, in addition:

- $R_i^B \cap A^{\mathrm{arity}(R_i)} \subseteq R_i^A$, $i = 1, k$.

B=



Substructure and Induced Substructure

$\sigma = (R_1, \ldots, R_k, c_1, \ldots, c_m)$.

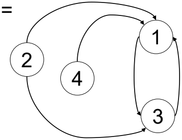
$\boldsymbol{A}$ is a substructure of $\boldsymbol{B}$, $\boldsymbol{A} \subseteq \boldsymbol{B}$ if:

- $A \subseteq B$, where
  $A = \text{Dom}(\boldsymbol{A})$, $B = \text{Dom}(\boldsymbol{B})$
- $R_i^A \subseteq R_i^B$, $i = 1, k$.
- $c_j^A = c_j^B$, $j = 1, m$.

It is an induced substructure, $\boldsymbol{A} = \boldsymbol{B}|_A$ if, in addition:

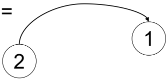
- $R_i^B \cap A^{\text{arity}(R_i)} \subseteq R_i^A$, $i = 1, k$.

B=



Which are induced subgraphs of $\boldsymbol{B}$?

A_1=



Substructure and Induced Substructure

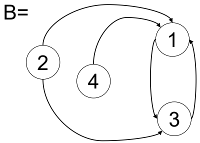
$\sigma = (R_1, \ldots, R_k, c_1, \ldots, c_m)$.

$\boldsymbol{A}$ is a substructure of $\boldsymbol{B}$, $\boldsymbol{A} \subseteq \boldsymbol{B}$ if:

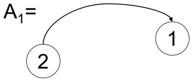
- $A \subseteq B$, where $A = \mathrm{Dom}(\boldsymbol{A})$, $B = \mathrm{Dom}(\boldsymbol{B})$
- $R_i^A \subseteq R_i^B$, $i = 1, k$.
- $c_j^A = c_j^B$, $j = 1, m$.

It is an induced substructure, $\boldsymbol{A} = \boldsymbol{B}|_A$ if, in addition:

- $R_i^B \cap A^{\mathrm{arity}(R_i)} \subseteq R_i^A$, $i = 1, k$.



Which are induced subgraphs of $\boldsymbol{B}$?



induced

Substructure and Induced Substructure

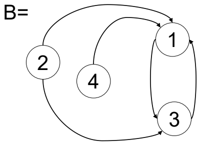
$\sigma = (R_1, \ldots, R_k, c_1, \ldots, c_m)$.

$\boldsymbol{A}$ is a substructure of $\boldsymbol{B}$, $\boldsymbol{A} \subseteq \boldsymbol{B}$ if:

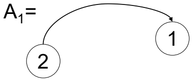
- $A \subseteq B$, where
  $A = \mathrm{Dom}(\boldsymbol{A})$, $B = \mathrm{Dom}(\boldsymbol{B})$
- $R_i^A \subseteq R_i^B$, $i = 1, k$.
- $c_j^A = c_j^B$, $j = 1, m$.

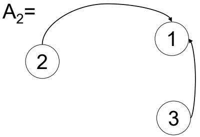
It is an induced substructure, $\boldsymbol{A} = \boldsymbol{B}|_A$ if, in addition:

- $R_i^B \cap A^{\mathrm{arity}(R_i)} \subseteq R_i^A$, $i = 1, k$.



Which are induced subgraphs of $\boldsymbol{B}$?





induced

Substructure and Induced Substructure

$\sigma = (R_1, \ldots, R_k, c_1, \ldots, c_m)$.

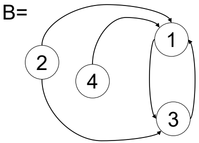
$\boldsymbol{A}$ is a substructure of $\boldsymbol{B}$, $\boldsymbol{A} \subseteq \boldsymbol{B}$ if:

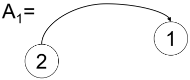
- $A \subseteq B$, where $A = \mathrm{Dom}(\boldsymbol{A})$, $B = \mathrm{Dom}(\boldsymbol{B})$
- $R_i^A \subseteq R_i^B$, $i = 1, k$.
- $c_j^A = c_j^B$, $j = 1, m$.

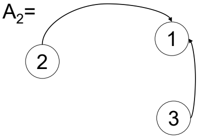
It is an induced substructure, $\boldsymbol{A} = \boldsymbol{B}|_A$ if, in addition:

- $R_i^B \cap A^{\mathrm{arity}(R_i)} \subseteq R_i^A$, $i = 1, k$.

Which are induced subgraphs of $\boldsymbol{B}$?







induced

substructure

Substructure and Induced Substructure

$\sigma = (R_1, \ldots, R_k, c_1, \ldots, c_m)$.

$\boldsymbol{A}$ is a substructure of $\boldsymbol{B}$, $\boldsymbol{A} \subseteq \boldsymbol{B}$ if:

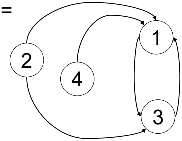
- $A \subseteq B$, where $A = \mathrm{Dom}(\boldsymbol{A})$, $B = \mathrm{Dom}(\boldsymbol{B})$
- $R_i^A \subseteq R_i^B$, $i = 1, k$.
- $c_j^A = c_j^B$, $j = 1, m$.

It is an induced substructure, $\boldsymbol{A} = \boldsymbol{B}|_A$ if, in addition:

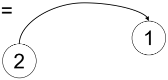
- $R_i^B \cap A^{\mathrm{arity}(R_i)} \subseteq R_i^A$, $i = 1, k$.

Which are induced subgraphs of $\boldsymbol{B}$?

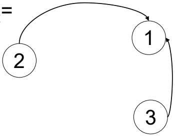
B=



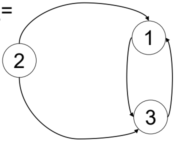
A$_1$=



A$_2$=



A$_3$=



induced

substructure

Substructure and Induced Substructure

$\sigma = (R_1, \dots, R_k, c_1, \dots, c_m)$.

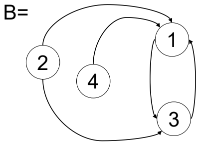
$\boldsymbol{A}$ is a substructure of $\boldsymbol{B}$, $\boldsymbol{A} \subseteq \boldsymbol{B}$ if:

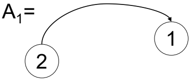
- $A \subseteq B$, where
  $A = \mathrm{Dom}(\boldsymbol{A})$, $B = \mathrm{Dom}(\boldsymbol{B})$
- $R_i^A \subseteq R_i^B$, $i = 1, k$.
- $c_j^A = c_j^B$, $j = 1, m$.

It is an induced substructure, $\boldsymbol{A} = \boldsymbol{B}|_A$ if, in addition:

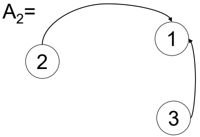
- $R_i^B \cap A^{\mathrm{arity}(R_i)} \subseteq R_i^A$, $i = 1, k$.

Which are induced subgraphs of $\boldsymbol{B}$?

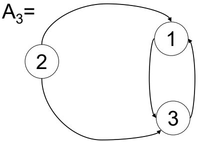




induced



substructure



induced

Partial Isomorphism

Let $\boldsymbol{A}, \boldsymbol{B}$ be two σ-structures, $\boldsymbol{a} = (a_1, \ldots, a_k) \in A^k$, $\boldsymbol{b} = (b_1, \ldots, b_k) \in B^k$.

Partial Isomorphism

Let $\boldsymbol{A}, \boldsymbol{B}$ be two σ-structures, $\boldsymbol{a} = (a_1, \ldots, a_k) \in A^k$, $\boldsymbol{b} = (b_1, \ldots, b_k) \in B^k$.

Definition

The pair $\boldsymbol{a}, \boldsymbol{b}$ is a partial isomorphism from $\boldsymbol{A}$ to $\boldsymbol{B}$ if the function $a_i \mapsto b_i$, $i = 1, k$ (maps $\boldsymbol{a}$ to $\boldsymbol{b}$), and $c_i^A \mapsto c_i^B$ (maps named constants to named constants) is an isomorphism between the substructures induced by $\boldsymbol{a}, \boldsymbol{b}$.

We write $\boldsymbol{a} \simeq \boldsymbol{b}$.

Partial Isomorphism

Let $\boldsymbol{A}, \boldsymbol{B}$ be two σ-structures, $\boldsymbol{a} = (a_1, \ldots, a_k) \in A^k$, $\boldsymbol{b} = (b_1, \ldots, b_k) \in B^k$.

Definition

The pair $\boldsymbol{a}, \boldsymbol{b}$ is a partial isomorphism from $\boldsymbol{A}$ to $\boldsymbol{B}$ if the function $a_i \mapsto b_i$, $i = 1, k$ (maps $\boldsymbol{a}$ to $\boldsymbol{b}$), and $c_i^A \mapsto c_i^B$ (maps named constants to named constants) is an isomorphism between the substructures induced by $\boldsymbol{a}, \boldsymbol{b}$.

We write $\boldsymbol{a} \simeq \boldsymbol{b}$.

In words:

- Forall i, j, $a_i = a_j$ iff $b_i = b_j$. (Equality is preserved.)
- Forall i, j, $a_i = c_j^A$ iff $b_i = c_j^B$. (Constants are preserved.)
- $(t_{i_1}, \ldots, t_{i_n}) \in R_j^A$ iff $(t'_{i_1}, \ldots, t'_{i_n}) \in R_j^B$, when the partial isomorphism maps $t_{i_1} \mapsto t'_{i_1}$, $t_{i_2} \mapsto t'_{i_2}$, etc.

Ehrenfeucht-Fraisse Games

There are two players, spoiler and duplicator.
They play on two structures $\boldsymbol{A}, \boldsymbol{B}$ in k rounds

Ehrenfeucht-Fraisse Games

There are two players, spoiler and duplicator.
They play on two structures $\boldsymbol{A}, \boldsymbol{B}$ in k rounds

Round i ($i = 1, k$):

- Spoiler places his pebble i on an element $a_i \in A$ or $b_i \in B$.

Ehrenfeucht-Fraisse Games

There are two players, spoiler and duplicator.
They play on two structures $\boldsymbol{A}, \boldsymbol{B}$ in k rounds

Round i $(i = 1, k)$:

- Spoiler places his pebble i on an element $a_i \in A$ or $b_i \in B$.
- Duplicator places her pebble i on an element $b_i \in B$ or $a_i \in A$.

Ehrenfeucht-Fraisse Games

There are two players, spoiler and duplicator.
They play on two structures $\boldsymbol{A}, \boldsymbol{B}$ in k rounds

Round i ($i = 1, k$):

- Spoiler places his pebble i on an element $a_i \in A$ or $b_i \in B$.
- Duplicator places her pebble i on an element $b_i \in B$ or $a_i \in A$.

Let $\boldsymbol{a} = (a_1, \ldots, a_k)$, $\boldsymbol{b} = (b_1, \ldots, b_k)$ be the pebbles at the game end.
Duplicator wins if $\boldsymbol{a}, \boldsymbol{b}$ forms a partial isomorphism; otherwise Spoiler wins.

Ehrenfeucht-Fraisse Games

There are two players, spoiler and duplicator.
They play on two structures $\boldsymbol{A}, \boldsymbol{B}$ in k rounds

Round i ($i = 1, k$):

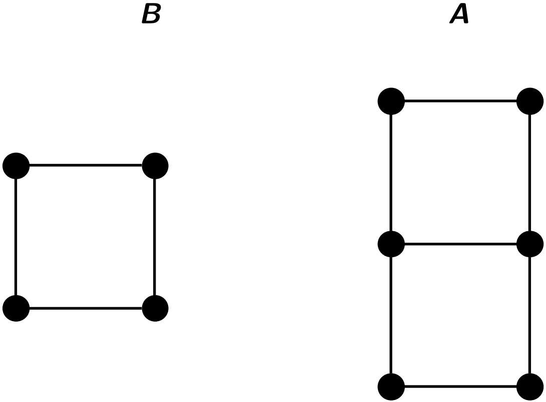
- Spoiler places his pebble i on an element $a_i \in A$ or $b_i \in B$.
- Duplicator places her pebble i on an element $b_i \in B$ or $a_i \in A$.

Let $\boldsymbol{a} = (a_1, \ldots, a_k)$, $\boldsymbol{b} = (b_1, \ldots, b_k)$ be the pebbles at the game end.
Duplicator wins if $\boldsymbol{a}, \boldsymbol{b}$ forms a partial isomorphism; otherwise Spoiler wins.

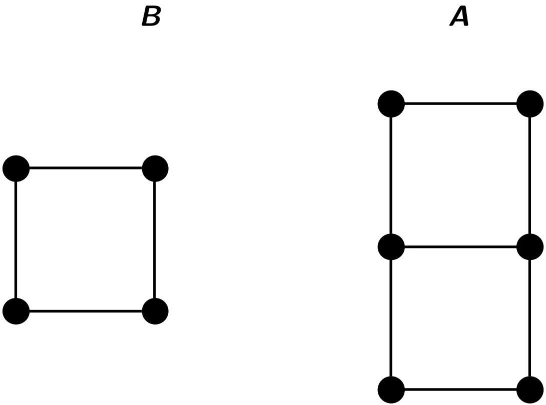
Definition

We write $\boldsymbol{A} \sim_k \boldsymbol{B}$ if the duplicator has a winning strategy for k rounds.

Example

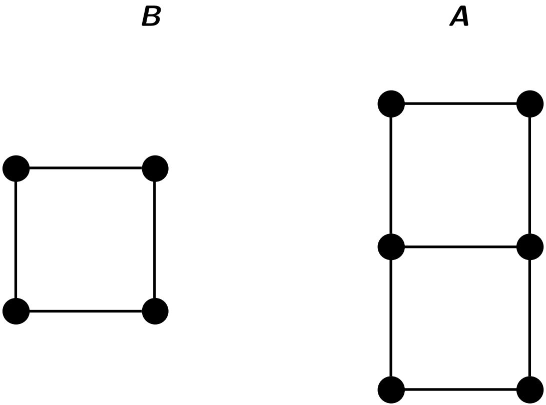


Example



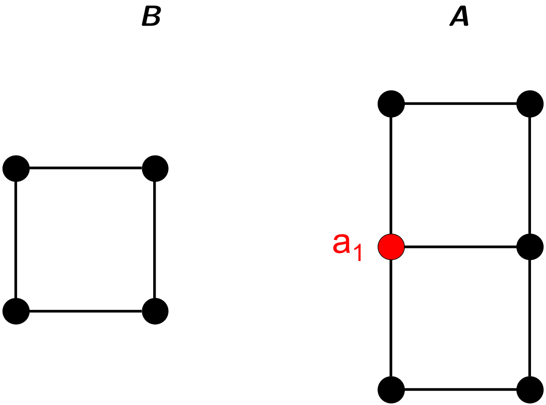
Duplicator wins in 2 rounds: $\boldsymbol{A} \sim_2 \boldsymbol{B}$.

Example



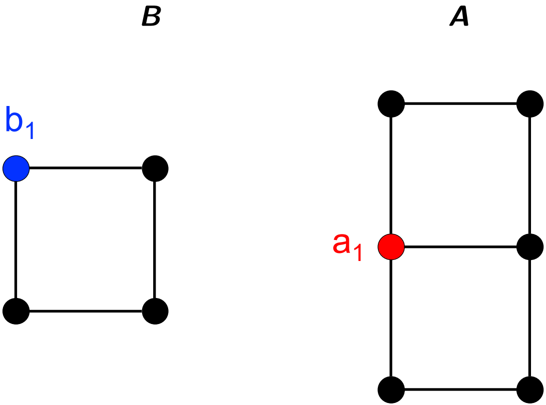
Duplicator wins in 2 rounds: $\boldsymbol{A} \sim_2 \boldsymbol{B}$.

Example



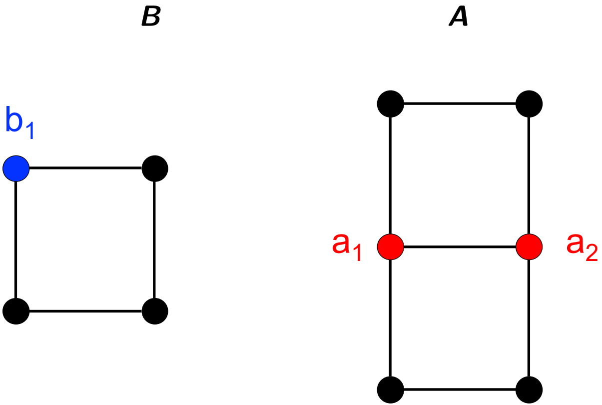
Duplicator wins in 2 rounds: $\boldsymbol{A} \sim_2 \boldsymbol{B}$.

Example



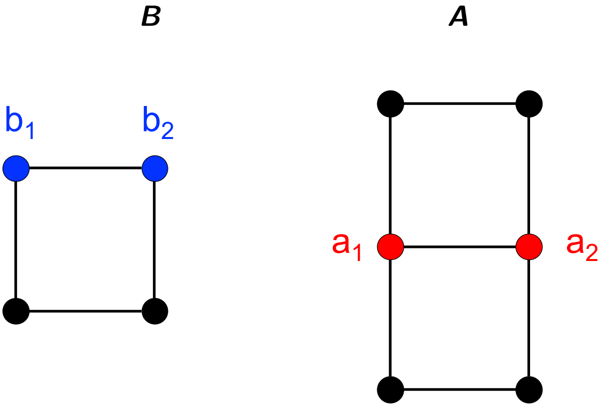
Duplicator wins in 2 rounds: $\boldsymbol{A} \sim_2 \boldsymbol{B}$.

Example



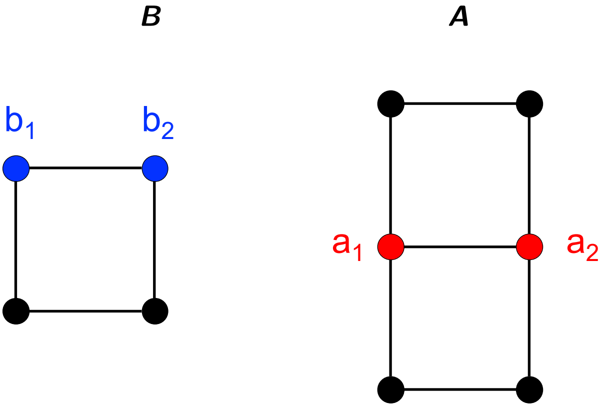
Duplicator wins in 2 rounds: $\boldsymbol{A} \sim_2 \boldsymbol{B}$.

Example



Duplicator wins in 2 rounds: $\boldsymbol{A} \sim_2 \boldsymbol{B}$.

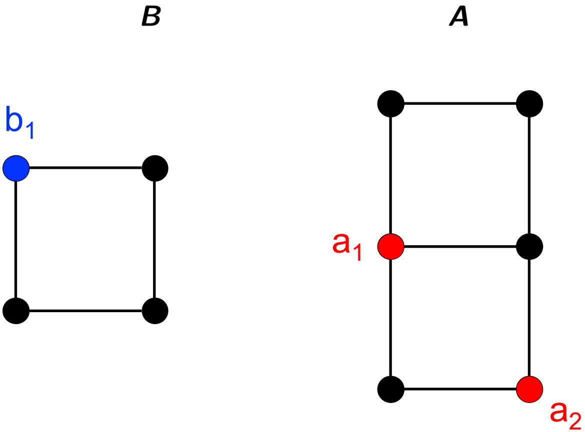
Example



Duplicator wins in 2 rounds: $\boldsymbol{A} \sim_2 \boldsymbol{B}$.

But in 3 or more rounds, spoiler wins: $\boldsymbol{A} \not\sim_3 \boldsymbol{B}$

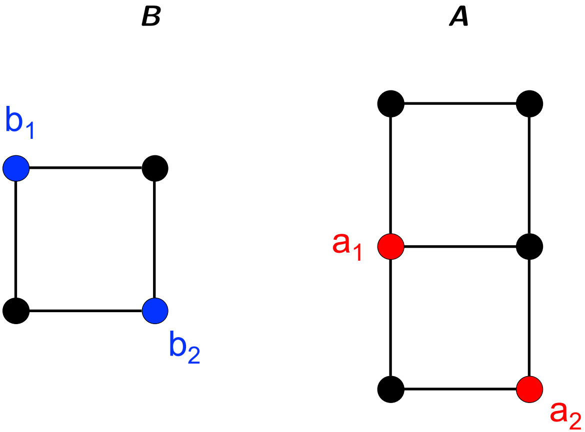
Example



Duplicator wins in 2 rounds: $\boldsymbol{A} \sim_2 \boldsymbol{B}$.

But in 3 or more rounds, spoiler wins: $\boldsymbol{A} \not\sim_3 \boldsymbol{B}$

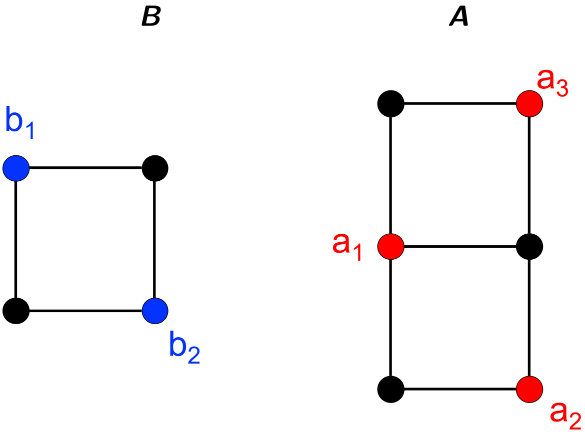
Example



Duplicator wins in 2 rounds: $\boldsymbol{A} \sim_2 \boldsymbol{B}$.

But in 3 or more rounds, spoiler wins: $\boldsymbol{A} \not\sim_3 \boldsymbol{B}$

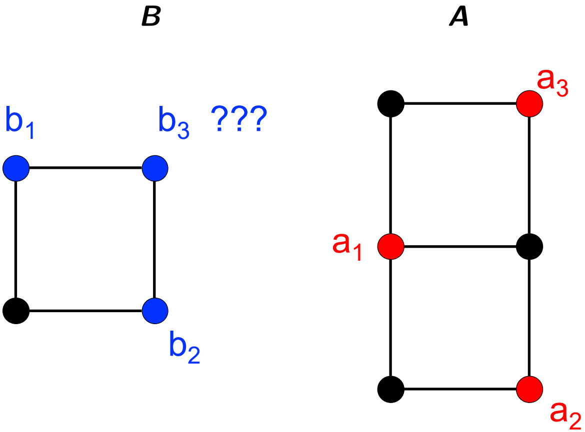
Example



Duplicator wins in 2 rounds: $\boldsymbol{A} \sim_2 \boldsymbol{B}$.

But in 3 or more rounds, spoiler wins: $\boldsymbol{A} \not\sim_3 \boldsymbol{B}$

Example



Duplicator wins in 2 rounds: $\boldsymbol{A} \sim_2 \boldsymbol{B}$.

But in 3 or more rounds, spoiler wins: $\boldsymbol{A} \not\sim_3 \boldsymbol{B}$

Quantifier Rank

The *quantifier rank* of a formula φ is defined inductively[1]:

$$\begin{aligned} qr(\boldsymbol{F}) =& qr(t_1 = t_2) = qr(R(t_1, \ldots, t_m)) = 0 \\ qr(\varphi \to \psi) =& \max(qr(\varphi), qr(\psi)) \\ qr(\forall x(\varphi)) =& 1 + qr(\varphi) \end{aligned}$$

$\mathrm{FO}[k] \stackrel{\text{def}}{=} \mathrm{FO}$ restricted to formulas with $qr \le k$.

[1] The *number* of quantifiers can be exponentially larger than $qr(\varphi)$ why?

Quantifier Rank

The *quantifier rank* of a formula φ is defined inductively[1]:

$$\begin{aligned} qr(\boldsymbol{F}) = & qr(t_1 = t_2) = qr(R(t_1, \ldots, t_m)) = 0 \\ qr(\varphi \to \psi) = & \max(qr(\varphi), qr(\psi)) \\ qr(\forall x(\varphi)) = & 1 + qr(\varphi) \end{aligned}$$

$\mathrm{FO}[k] \stackrel{\text{def}}{=} \mathrm{FO}$ restricted to formulas with $qr \leq k$.

Examples: $\varphi = (\exists x \forall y E(x,y)) \wedge (\forall u \exists v \neg(E(v,u)))$. $\qquad qr(\varphi) =$???

[1] The *number* of quantifiers can be exponentially larger than $qr(\varphi)$ why?

Quantifier Rank

The *quantifier rank* of a formula φ is defined inductively[1]:

$$\begin{aligned} qr(\boldsymbol{F}) =& qr(t_1 = t_2) = qr(R(t_1, \ldots, t_m)) = 0 \\ qr(\varphi \to \psi) =& \max(qr(\varphi), qr(\psi)) \\ qr(\forall x(\varphi)) =& 1 + qr(\varphi) \end{aligned}$$

$\mathrm{FO}[k] \stackrel{\text{def}}{=}$ FO restricted to formulas with $qr \leq k$.

Examples: $\varphi = (\exists x \forall y E(x,y)) \wedge (\forall u \exists v \neg(E(v,u)))$. $\qquad qr(\varphi) =$ 2

[1] The *number* of quantifiers can be exponentially larger than $qr(\varphi)$ why?

Quantifier Rank

The *quantifier rank* of a formula φ is defined inductively[1]:

$$\begin{aligned} qr(\boldsymbol{F}) &= qr(t_1 = t_2) = qr(R(t_1, \ldots, t_m)) = 0 \\ qr(\varphi \to \psi) &= \max(qr(\varphi), qr(\psi)) \\ qr(\forall x(\varphi)) &= 1 + qr(\varphi) \end{aligned}$$

$\mathrm{FO}[k] \stackrel{\mathrm{def}}{=}$ FO restricted to formulas with $qr \le k$.

Examples: $\varphi = (\exists x \forall y E(x,y)) \wedge (\forall u \exists v \neg(E(v,u)))$. $\quad qr(\varphi) =$ 2

Definition

$\boldsymbol{A} \equiv_k \boldsymbol{B}$ if every sentence $\varphi \in \mathrm{FO}[k]$ has the same truth value in $\boldsymbol{A}$ and $\boldsymbol{B}$.

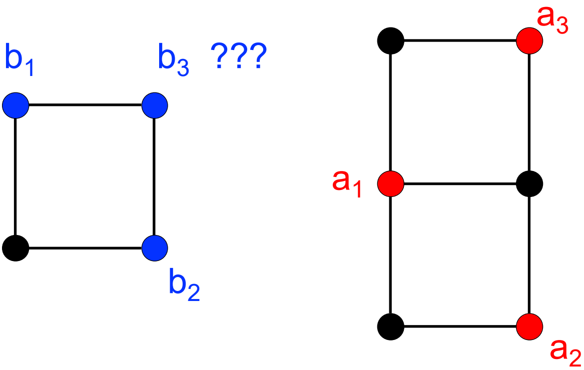
[1] The *number* of quantifiers can be exponentially larger than $qr(\varphi)$ why?

Ehrenfeucht-Fraisse Games: Main Result

Theorem (Ehrenfeucht-Fraisse)

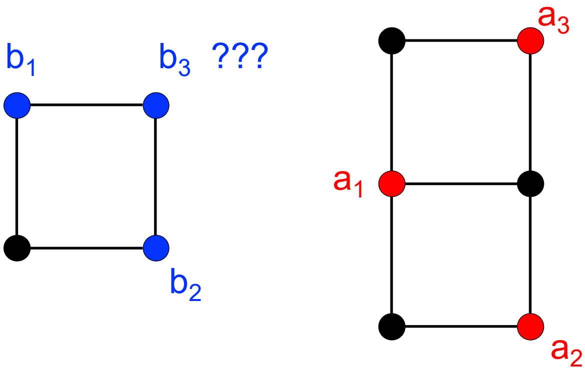
$\boldsymbol{A} \equiv_k \boldsymbol{B}$ *iff* $\boldsymbol{A} \sim_k \boldsymbol{B}$.

Example



Since $\boldsymbol{A} \not\sim_3 \boldsymbol{B}$, we have $\boldsymbol{A} \not\equiv_3 \boldsymbol{B}$.

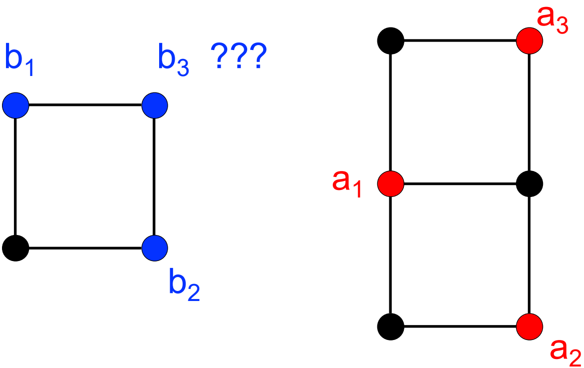
Example



Since $\boldsymbol{A} \not\sim_3 \boldsymbol{B}$, we have $\boldsymbol{A} \not\equiv_3 \boldsymbol{B}$.

Give example of $\varphi \in \mathrm{FO}[3]$ s.t. $\boldsymbol{A} \nvDash \varphi$, $\boldsymbol{B} \vDash \varphi$.

Example



Since $\boldsymbol{A} \not\sim_3 \boldsymbol{B}$, we have $\boldsymbol{A} \not\equiv_3 \boldsymbol{B}$.

Give example of $\varphi \in \mathrm{FO}[3]$ s.t. $\boldsymbol{A} \nvDash \varphi$, $\boldsymbol{B} \vDash \varphi$.

$\exists x_1 \exists x_2 \exists x_3((x_1 \neq x_2) \wedge (x_1 \neq x_3) \wedge (x_2 \neq x_3) \wedge \neg E(x_1, x_2) \wedge \neg E(x_2, x_3) \wedge \neg E(x_1, x_3))$

Proof of the Ehrenfeucht-Fraisse Theorem (Part 1)

Part 1: If $\boxed{\boldsymbol{A} \sim_k \boldsymbol{B}}$ then $\boxed{\boldsymbol{A} \equiv_k \boldsymbol{B}}$. Induction on k.

[2]Meaning: extend σ with new a constant symbol c, and set $c^A \stackrel{\text{def}}{=} a_1$, $c^B \stackrel{\text{def}}{=} b_1$.

Proof of the Ehrenfeucht-Fraisse Theorem (Part 1)

Part 1: If $\boxed{\boldsymbol{A} \sim_k \boldsymbol{B}}$ then $\boxed{\boldsymbol{A} \equiv_k \boldsymbol{B}}$. Induction on k.

Assume $\boldsymbol{A} \sim_k \boldsymbol{B}$, and let $\varphi \in \mathrm{FO}[k]$. Assume wlog $\varphi = \exists x \psi(x)$.

[2]Meaning: extend σ with new a constant symbol c, and set $c^A \stackrel{\text{def}}{=} a_1$, $c^B \stackrel{\text{def}}{=} b_1$.

Proof of the Ehrenfeucht-Fraisse Theorem (Part 1)

Part 1: If $\boxed{\boldsymbol{A} \sim_k \boldsymbol{B}}$ then $\boxed{\boldsymbol{A} \equiv_k \boldsymbol{B}}$. Induction on k.

Assume $\boldsymbol{A} \sim_k \boldsymbol{B}$, and let $\varphi \in \mathrm{FO}[k]$. Assume wlog $\varphi = \exists x \psi(x)$.

Assume $\boxed{\boldsymbol{A} \vDash \varphi}$.

[2] Meaning: extend σ with new a constant symbol c, and set $c^A \stackrel{\text{def}}{=} a_1$, $c^B \stackrel{\text{def}}{=} b_1$.

Proof of the Ehrenfeucht-Fraisse Theorem (Part 1)

Part 1: If $\boxed{\boldsymbol{A} \sim_k \boldsymbol{B}}$ then $\boxed{\boldsymbol{A} \equiv_k \boldsymbol{B}}$. Induction on k.

Assume $\boldsymbol{A} \sim_k \boldsymbol{B}$, and let $\varphi \in \mathrm{FO}[k]$. Assume wlog $\varphi = \exists x \psi(x)$.

Assume $\boxed{\boldsymbol{A} \vDash \varphi}$. Then $\exists a_1 \in \mathrm{Dom}(\boldsymbol{A})$, $\boldsymbol{A} \vDash \psi(a_1)$.

[2] Meaning: extend σ with new a constant symbol c, and set $c^A \stackrel{\text{def}}{=} a_1$, $c^B \stackrel{\text{def}}{=} b_1$.

Proof of the Ehrenfeucht-Fraisse Theorem (Part 1)

Part 1: If $\boxed{\boldsymbol{A} \sim_k \boldsymbol{B}}$ then $\boxed{\boldsymbol{A} \equiv_k \boldsymbol{B}}$. Induction on k.

Assume $\boldsymbol{A} \sim_k \boldsymbol{B}$, and let $\varphi \in \mathrm{FO}[k]$. Assume wlog $\varphi = \exists x \psi(x)$.

Assume $\boxed{\boldsymbol{A} \models \varphi}$. Then $\exists a_1 \in \mathrm{Dom}(\boldsymbol{A})$, $\boldsymbol{A} \models \psi(a_1)$.

Start an EF game with spoiler choosing a_1.
Duplicator replies $b_1 \in \mathrm{Dom}(\boldsymbol{B})$, and[2] $(\boldsymbol{A}, a_1) \sim_{k-1} (\boldsymbol{B}, b_1)$.

[2]Meaning: extend σ with new a constant symbol c, and set $c^A \stackrel{\text{def}}{=} a_1$, $c^B \stackrel{\text{def}}{=} b_1$.

Proof of the Ehrenfeucht-Fraisse Theorem (Part 1)

Part 1: If $\boxed{\boldsymbol{A} \sim_k \boldsymbol{B}}$ then $\boxed{\boldsymbol{A} \equiv_k \boldsymbol{B}}$. Induction on k.

Assume $\boldsymbol{A} \sim_k \boldsymbol{B}$, and let $\varphi \in \mathrm{FO}[k]$. Assume wlog $\varphi = \exists x \psi(x)$.

Assume $\boxed{\boldsymbol{A} \vDash \varphi}$. Then $\exists a_1 \in \mathrm{Dom}(\boldsymbol{A})$, $\boldsymbol{A} \vDash \psi(a_1)$.

Start an EF game with spoiler choosing a_1.
Duplicator replies $b_1 \in \mathrm{Dom}(\boldsymbol{B})$, and[2] $(\boldsymbol{A}, a_1) \sim_{k-1} (\boldsymbol{B}, b_1)$.

By induction, $(\boldsymbol{A}, a_1) \equiv_{k-1} (\boldsymbol{B}, b_1)$, hence $\boldsymbol{B} \vDash \psi(b_1)$.

[2] Meaning: extend σ with new a constant symbol c, and set $c^A \stackrel{\text{def}}{=} a_1$, $c^B \stackrel{\text{def}}{=} b_1$.

Proof of the Ehrenfeucht-Fraisse Theorem (Part 1)

Part 1: If $\boxed{\boldsymbol{A} \sim_k \boldsymbol{B}}$ then $\boxed{\boldsymbol{A} \equiv_k \boldsymbol{B}}$. Induction on k.

Assume $\boldsymbol{A} \sim_k \boldsymbol{B}$, and let $\varphi \in \mathrm{FO}[k]$. Assume wlog $\varphi = \exists x \psi(x)$.

Assume $\boxed{\boldsymbol{A} \models \varphi}$. Then $\exists a_1 \in \mathrm{Dom}(\boldsymbol{A})$, $\boldsymbol{A} \models \psi(a_1)$.

Start an EF game with spoiler choosing a_1.
Duplicator replies $b_1 \in \mathrm{Dom}(\boldsymbol{B})$, and[2] $(\boldsymbol{A}, a_1) \sim_{k-1} (\boldsymbol{B}, b_1)$.

By induction, $(\boldsymbol{A}, a_1) \equiv_{k-1} (\boldsymbol{B}, b_1)$, hence $\boldsymbol{B} \models \psi(b_1)$. $\boxed{\boldsymbol{B} \models \varphi}$

[2] Meaning: extend σ with new a constant symbol c, and set $c^A \stackrel{\text{def}}{=} a_1$, $c^B \stackrel{\text{def}}{=} b_1$.

Proof of the Ehrenfeucht-Fraisse Theorem (Part 1)

Part 1: If $\boxed{\boldsymbol{A} \sim_k \boldsymbol{B}}$ then $\boxed{\boldsymbol{A} \equiv_k \boldsymbol{B}}$. Induction on k.

Assume $\boldsymbol{A} \sim_k \boldsymbol{B}$, and let $\varphi \in \text{FO}[k]$. Assume wlog $\varphi = \exists x \psi(x)$.

Assume $\boxed{\boldsymbol{A} \vDash \varphi}$. Then $\exists a_1 \in \text{Dom}(\boldsymbol{A})$, $\boldsymbol{A} \vDash \psi(a_1)$.

Start an EF game with spoiler choosing a_1.
Duplicator replies $b_1 \in \text{Dom}(\boldsymbol{B})$, and[2] $(\boldsymbol{A}, a_1) \sim_{k-1} (\boldsymbol{B}, b_1)$.

By induction, $(\boldsymbol{A}, a_1) \equiv_{k-1} (\boldsymbol{B}, b_1)$, hence $\boldsymbol{B} \vDash \psi(b_1)$. $\boxed{\boldsymbol{B} \vDash \varphi}$

If $\boxed{\boldsymbol{B} \vDash \varphi}$, then, similarly, $\boxed{\boldsymbol{A} \vDash \varphi}$ (duplicator plays on $\boldsymbol{B}$).

[2] Meaning: extend σ with new a constant symbol c, and set $c^A \stackrel{\text{def}}{=} a_1$, $c^B \stackrel{\text{def}}{=} b_1$.

Proof of the Ehrenfeucht-Fraisse Theorem (Part 1)

Part 1: If $\boxed{\boldsymbol{A} \sim_k \boldsymbol{B}}$ then $\boxed{\boldsymbol{A} \equiv_k \boldsymbol{B}}$. Induction on k.

Assume $\boldsymbol{A} \sim_k \boldsymbol{B}$, and let $\varphi \in \mathrm{FO}[k]$. Assume wlog $\varphi = \exists x \psi(x)$.

Assume $\boxed{\boldsymbol{A} \vDash \varphi}$. Then $\exists a_1 \in \mathrm{Dom}(\boldsymbol{A})$, $\boldsymbol{A} \vDash \psi(a_1)$.

Start an EF game with spoiler choosing a_1.
Duplicator replies $b_1 \in \mathrm{Dom}(\boldsymbol{B})$, and[2] $(\boldsymbol{A}, a_1) \sim_{k-1} (\boldsymbol{B}, b_1)$.

By induction, $(\boldsymbol{A}, a_1) \equiv_{k-1} (\boldsymbol{B}, b_1)$, hence $\boldsymbol{B} \vDash \psi(b_1)$. $\boxed{\boldsymbol{B} \vDash \varphi}$

If $\boxed{\boldsymbol{B} \vDash \varphi}$, then, similarly, $\boxed{\boldsymbol{A} \vDash \varphi}$ (duplicator plays on $\boldsymbol{B}$).

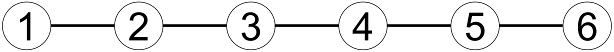
Next lecture: **Part 2:** If $\boxed{\boldsymbol{A} \equiv_k \boldsymbol{B}}$ then $\boxed{\boldsymbol{A} \sim_k \boldsymbol{B}}$

[2]Meaning: extend σ with new a constant symbol c, and set $c^A \stackrel{\text{def}}{=} a_1$, $c^B \stackrel{\text{def}}{=} b_1$.

EVEN of a Linear Order is Not Expressible in FO

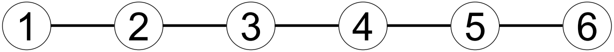
Linear Order

A finite, linear orders, over $\sigma = (<)$, is $\boldsymbol{L}_n \stackrel{\text{def}}{=} ([n], <)$

$\boldsymbol{L}_6 =$ 

Linear Order

A finite, linear orders, over $\sigma = (<)$, is $\boldsymbol{L}_n \stackrel{\text{def}}{=} ([n], <)$

$\boldsymbol{L}_6 =$ 

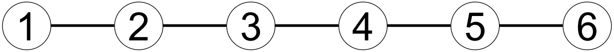
What we can say about $\boldsymbol{L}_n$:

- Check if x is the smallest element: $\texttt{isMin}(x) = \forall y(\neg y < x)$.

Linear Order

A finite, linear orders, over $\sigma = (<)$, is $\boldsymbol{L}_n \stackrel{\text{def}}{=} ([n], <)$

$\boldsymbol{L}_6 =$

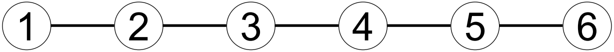


What we can say about $\boldsymbol{L}_n$:

- Check if x is the smallest element: $\texttt{isMin}(x) = \forall y(\neg y < x)$.
- Check if y is the successor of x:
  $\texttt{succ}(x, y) = x < y \wedge \neg\exists z(x < z \wedge z < y)$.

Linear Order

A finite, linear orders, over $\sigma = (<)$, is $\boldsymbol{L}_n \stackrel{\text{def}}{=} ([n], <)$

$\boldsymbol{L}_6 =$ 

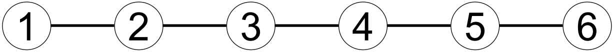
What we can say about $\boldsymbol{L}_n$:

- Check if x is the smallest element: $\texttt{isMin}(x) = \forall y(\neg y < x)$.
- Check if y is the successor of x:
  $\texttt{succ}(x, y) = x < y \wedge \neg\exists z(x < z \wedge z < y)$.

Can we check if n is EVEN?

Linear Order

A finite, linear orders, over $\sigma = (<)$, is $\boldsymbol{L}_n \stackrel{\text{def}}{=} ([n], <)$

$\boldsymbol{L}_6 =$ 

What we can say about $\boldsymbol{L}_n$:

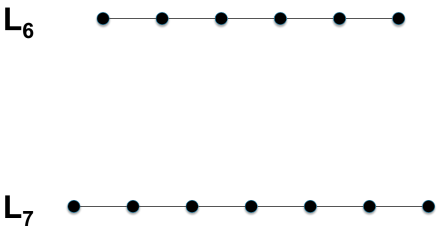
- Check if x is the smallest element: $\texttt{isMin}(x) = \forall y(\neg y < x)$.
- Check if y is the successor of x:
  $\texttt{succ}(x, y) = x < y \wedge \neg \exists z(x < z \wedge z < y)$.

Can we check if n is EVEN?

Will show: there is no sentence φ s.t. $\boldsymbol{L}_n \vDash \varphi$ iff n is even.

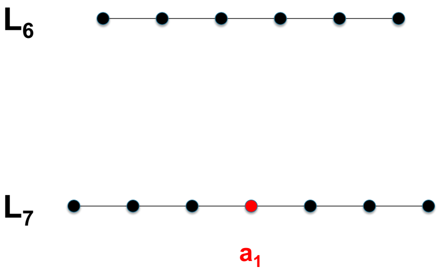
EF Games on Linear Orders

Show that $\boldsymbol{L}_6 \sim_2 \boldsymbol{L}_7$.



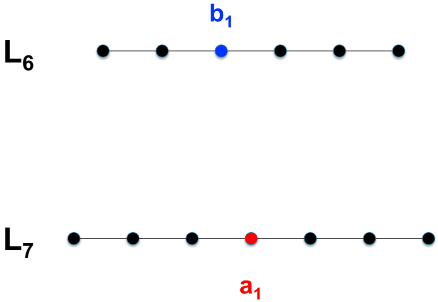
EF Games on Linear Orders

Show that $\boldsymbol{L}_6 \sim_2 \boldsymbol{L}_7$.



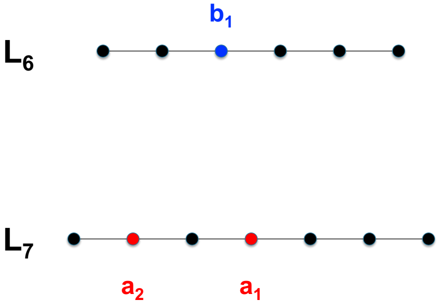
EF Games on Linear Orders

Show that $\boldsymbol{L}_6 \sim_2 \boldsymbol{L}_7$.



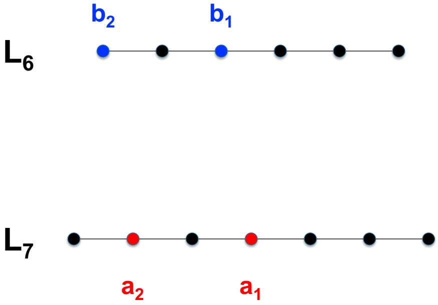
EF Games on Linear Orders

Show that $\boldsymbol{L}_6 \sim_2 \boldsymbol{L}_7$.



EF Games on Linear Orders

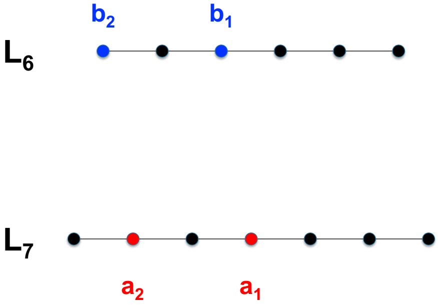
Show that $\boldsymbol{L}_6 \sim_2 \boldsymbol{L}_7$.



EF Games on Linear Orders

Show that $\boldsymbol{L}_6 \sim_2 \boldsymbol{L}_7$.

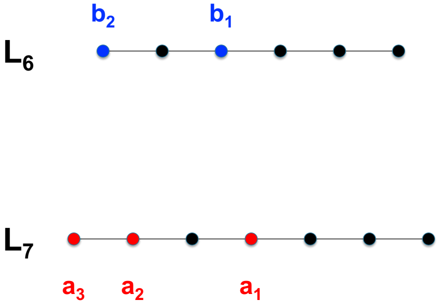
Show that $\boldsymbol{L}_6 \not\sim_3 \boldsymbol{L}_7$.



EF Games on Linear Orders

Show that $\boldsymbol{L}_6 \sim_2 \boldsymbol{L}_7$.

Show that $\boldsymbol{L}_6 \not\sim_3 \boldsymbol{L}_7$.



EVEN of Linear Orders Not Expressible in FO

Theorem

Let $m \neq n$. If $m, n \geq 2^k - 1$, then $\mathbf{L}_m \sim_k \mathbf{L}_n$; otherwise $\mathbf{L}_m \not\sim_k \mathbf{L}_n$.

EVEN of Linear Orders Not Expressible in FO

Theorem

Let $m \neq n$. If $m, n \geq 2^k - 1$, then $\boldsymbol{L}_m \sim_k \boldsymbol{L}_n$; otherwise $\boldsymbol{L}_m \not\sim_k \boldsymbol{L}_n$.

Corollary

EVEN is not expressible over linear orders.

Proof Suppose φ is s.t. $\boldsymbol{L}_n \vDash \varphi$ iff n is even.

EVEN of Linear Orders Not Expressible in FO

Theorem

Let $m \neq n$. If $m, n \geq 2^k - 1$, then $\mathbf{L}_m \sim_k \mathbf{L}_n$; otherwise $\mathbf{L}_m \not\sim_k \mathbf{L}_n$.

Corollary

EVEN is not expressible over linear orders.

Proof Suppose φ is s.t. $\mathbf{L}_n \models \varphi$ iff n is even.

Let $k = qr(\varphi)$.

EVEN of Linear Orders Not Expressible in FO

Theorem

Let $m \neq n$. If $m, n \geq 2^k - 1$, then $\mathbf{L}_m \sim_k \mathbf{L}_n$; otherwise $\mathbf{L}_m \not\sim_k \mathbf{L}_n$.

Corollary

EVEN is not expressible over linear orders.

Proof Suppose φ is s.t. $\mathbf{L}_n \models \varphi$ iff n is even.
Let $k = qr(\varphi)$. Choose $m \stackrel{\text{def}}{=} 2^k - 1$ and $n \stackrel{\text{def}}{=} 2^k$.

EVEN of Linear Orders Not Expressible in FO

Theorem

Let $m \neq n$. If $m, n \geq 2^k - 1$, then $\boldsymbol{L}_m \sim_k \boldsymbol{L}_n$; otherwise $\boldsymbol{L}_m \not\sim_k \boldsymbol{L}_n$.

Corollary

EVEN is not expressible over linear orders.

Proof Suppose φ is s.t. $\boldsymbol{L}_n \vDash \varphi$ iff n is even.
Let $k = qr(\varphi)$. Choose $m \stackrel{\text{def}}{=} 2^k - 1$ and $n \stackrel{\text{def}}{=} 2^k$.

$\boldsymbol{L}_n \vDash \varphi$ because n is even;

EVEN of Linear Orders Not Expressible in FO

Theorem

Let $m \neq n$. If $m, n \geq 2^k - 1$, then $\mathbf{L}_m \sim_k \mathbf{L}_n$; otherwise $\mathbf{L}_m \not\sim_k \mathbf{L}_n$.

Corollary

EVEN is not expressible over linear orders.

Proof Suppose φ is s.t. $\mathbf{L}_n \models \varphi$ iff n is even.
Let $k = qr(\varphi)$. Choose $m \stackrel{\text{def}}{=} 2^k - 1$ and $n \stackrel{\text{def}}{=} 2^k$.

$\mathbf{L}_n \models \varphi$ because n is even; $\mathbf{L}_m \sim_k \mathbf{L}_n$ by the Theorem;

EVEN of Linear Orders Not Expressible in FO

Theorem

Let $m \neq n$. If $m, n \geq 2^k - 1$, then $\boldsymbol{L}_m \sim_k \boldsymbol{L}_n$; otherwise $\boldsymbol{L}_m \not\sim_k \boldsymbol{L}_n$.

Corollary

EVEN is not expressible over linear orders.

Proof Suppose φ is s.t. $\boldsymbol{L}_n \models \varphi$ iff n is even.
Let $k = qr(\varphi)$. Choose $m \stackrel{\text{def}}{=} 2^k - 1$ and $n \stackrel{\text{def}}{=} 2^k$.

$\boldsymbol{L}_n \models \varphi$ because n is even; $\boldsymbol{L}_m \sim_k \boldsymbol{L}_n$ by the Theorem; then $\boldsymbol{L}_m \models \varphi$
Contradiction!

EVEN of Linear Orders Not Expressible in FO

Theorem

Let $m \neq n$. If $m, n \geq 2^k - 1$, then $\boldsymbol{L}_m \sim_k \boldsymbol{L}_n$; otherwise $\boldsymbol{L}_m \not\sim_k \boldsymbol{L}_n$.

Corollary

EVEN is not expressible over linear orders.

Proof Suppose φ is s.t. $\boldsymbol{L}_n \models \varphi$ iff n is even.
Let $k = qr(\varphi)$. Choose $m \stackrel{\text{def}}{=} 2^k - 1$ and $n \stackrel{\text{def}}{=} 2^k$.

$\boldsymbol{L}_n \models \varphi$ because n is even; $\boldsymbol{L}_m \sim_k \boldsymbol{L}_n$ by the Theorem; then $\boldsymbol{L}_m \models \varphi$
Contradiction!

It remains to prove the theorem

EF Games on Linear Orders

Let $L_m = ([m], <)$. Denote:[3]

$$L_m^{<a} \stackrel{\text{def}}{=} \{x \in L_m \mid x < a\} \qquad\qquad L_m^{>a} \stackrel{\text{def}}{=} \{x \in L_m \mid x > a\}$$

[3] Isomorphic to linear orders: $\boldsymbol{L}_m^{<a} \simeq \boldsymbol{L}_{a-1}$, $\boldsymbol{L}_m^{>a} \simeq L_{m-a}$.

EF Games on Linear Orders

Let $L_m = ([m], <)$. Denote:[3]

$$L_m^{<a} \stackrel{\text{def}}{=} \{x \in L_m \mid x < a\} \qquad\qquad L_m^{>a} \stackrel{\text{def}}{=} \{x \in L_m \mid x > a\}$$

Lemma

If $L_m^{<a} \sim_k L_n^{<b}$ and $L_m^{>a} \sim_k L_n^{>b}$, then $L_m \sim_k L_n$.

[3] Isomorphic to linear orders: $\boldsymbol{L}_m^{<a} \simeq \boldsymbol{L}_{a-1}$, $\boldsymbol{L}_m^{>a} \simeq L_{m-a}$.

EF Games on Linear Orders

Let $L_m = ([m], <)$. Denote:[3]

$$L_m^{<a} \stackrel{\text{def}}{=} \{x \in L_m \mid x < a\} \qquad L_m^{>a} \stackrel{\text{def}}{=} \{x \in L_m \mid x > a\}$$

Lemma

If $L_m^{<a} \sim_k L_n^{<b}$ and $L_m^{>a} \sim_k L_n^{>b}$, then $L_m \sim_k L_n$.

Proof.

- If spoiler plays on $L_m^{<a}$ then duplicator answers in $L_n^{<b}$.

[3] Isomorphic to linear orders: $\boldsymbol{L}_m^{<a} \simeq \boldsymbol{L}_{a-1}$, $\boldsymbol{L}_m^{>a} \simeq L_{m-a}$.

EF Games on Linear Orders

Let $L_m = ([m], <)$. Denote:[3]

$$L_m^{<a} \stackrel{\text{def}}{=} \{x \in L_m \mid x < a\} \qquad L_m^{>a} \stackrel{\text{def}}{=} \{x \in L_m \mid x > a\}$$

Lemma

If $L_m^{<a} \sim_k L_n^{<b}$ and $L_m^{>a} \sim_k L_n^{>b}$, then $L_m \sim_k L_n$.

Proof.

- If spoiler plays on $L_m^{<a}$ then duplicator answers in $L_n^{<b}$.
- If spoiler plays on $L_m^{>a}$ then duplicator answers in $L_n^{>b}$.

[3] Isomorphic to linear orders: $\boldsymbol{L}_m^{<a} \simeq \boldsymbol{L}_{a-1}$, $\boldsymbol{L}_m^{>a} \simeq L_{m-a}$.

EF Games on Linear Orders

Let $L_m = ([m], <)$. Denote:[3]

$$L_m^{<a} \stackrel{\text{def}}{=} \{x \in L_m \mid x < a\} \qquad L_m^{>a} \stackrel{\text{def}}{=} \{x \in L_m \mid x > a\}$$

Lemma

If $L_m^{<a} \sim_k L_n^{<b}$ and $L_m^{>a} \sim_k L_n^{>b}$, then $L_m \sim_k L_n$.

Proof.

- If spoiler plays on $L_m^{<a}$ then duplicator answers in $L_n^{<b}$.
- If spoiler plays on $L_m^{>a}$ then duplicator answers in $L_n^{>b}$.
- If spoiler plays on a then duplicator answers on b.

[3] Isomorphic to linear orders: $\boldsymbol{L}_m^{<a} \simeq \boldsymbol{L}_{a-1}$, $\boldsymbol{L}_m^{>a} \simeq L_{m-a}$.

EF Games on Linear Orders

Let $L_m = ([m], <)$. Denote:[3]

$$L_m^{<a} \stackrel{\text{def}}{=} \{x \in L_m \mid x < a\} \qquad\qquad L_m^{>a} \stackrel{\text{def}}{=} \{x \in L_m \mid x > a\}$$

Lemma

If $L_m^{<a} \sim_k L_n^{<b}$ and $L_m^{>a} \sim_k L_n^{>b}$, then $L_m \sim_k L_n$.

Proof.

- If spoiler plays on $L_m^{<a}$ then duplicator answers in $L_n^{<b}$.
- If spoiler plays on $L_m^{>a}$ then duplicator answers in $L_n^{>b}$.
- If spoiler plays on a then duplicator answers on b.
- If spoiler plays on the other structure, duplicator answers similarly.

[3] Isomorphic to linear orders: $\boldsymbol{L}_m^{<a} \simeq \boldsymbol{L}_{a-1}$, $\boldsymbol{L}_m^{>a} \simeq L_{m-a}$.

EF Games on Linear Orders

Theorem

Let $m \neq n$. If $m, n \geq 2^k - 1$, then $\mathbf{L}_m \sim_k \mathbf{L}_n$; otherwise, $\mathbf{L}_m \not\sim_k \mathbf{L}_n$.

EF Games on Linear Orders

Theorem

Let $m \neq n$. If $m, n \geq 2^k - 1$, then $\mathbf{L}_m \sim_k \mathbf{L}_n$; otherwise, $\mathbf{L}_m \not\sim_k \mathbf{L}_n$.

Proof. Induction on k. Assume $m, n \geq 2^k - 1$. Spoiler plays $a \in \boldsymbol{L}_m$.
Case 1: $|\boldsymbol{L}_m^{<a}| < 2^{k-1} - 1$ What does duplicator do?

EF Games on Linear Orders

Theorem

Let $m \neq n$. If $m, n \geq 2^k - 1$, then $\mathbf{L}_m \sim_k \mathbf{L}_n$; otherwise, $\mathbf{L}_m \not\sim_k \mathbf{L}_n$.

Proof. Induction on k. Assume $m, n \geq 2^k - 1$. Spoiler plays $a \in \mathbf{L}_m$.

Case 1: $|\mathbf{L}_m^{<a}| < 2^{k-1} - 1$ Choose $b = a$, thus $\boxed{\mathbf{L}_m^{<a} \sim_{k-1} \mathbf{L}_n^{<b}}$

EF Games on Linear Orders

Theorem

Let $m \neq n$. If $m, n \geq 2^k - 1$, then $\mathbf{L}_m \sim_k \mathbf{L}_n$; otherwise, $\mathbf{L}_m \not\sim_k \mathbf{L}_n$.

Proof. Induction on k. Assume $m, n \geq 2^k - 1$. Spoiler plays $a \in \mathbf{L}_m$.

Case 1: $|\mathbf{L}_m^{<a}| < 2^{k-1} - 1$ Choose $b = a$, thus $\boxed{\mathbf{L}_m^{<a} \sim_{k-1} \mathbf{L}_n^{<b}}$

Since $|\mathbf{L}_m^{>a}|, |\mathbf{L}_n^{>b}| > 2^{k-1} - 1$, $\boxed{\mathbf{L}_m^{>a} \sim_{k-1} \mathbf{L}_n^{>b}}$;

EF Games on Linear Orders

Theorem

Let $m \neq n$. If $m, n \geq 2^k - 1$, then $\mathbf{L}_m \sim_k \mathbf{L}_n$; otherwise, $\mathbf{L}_m \not\sim_k \mathbf{L}_n$.

Proof. Induction on k. Assume $m, n \geq 2^k - 1$. Spoiler plays $a \in \mathbf{L}_m$.

Case 1: $|\mathbf{L}_m^{<a}| < 2^{k-1} - 1$ Choose $b = a$, thus $\boxed{\mathbf{L}_m^{<a} \sim_{k-1} \mathbf{L}_n^{<b}}$

Since $|\mathbf{L}_m^{>a}|, |\mathbf{L}_n^{>b}| > 2^{k-1} - 1$, $\boxed{\mathbf{L}_m^{>a} \sim_{k-1} \mathbf{L}_n^{>b}}$; Lemma implies: $\boxed{\mathbf{L}_m \sim_k \mathbf{L}_n}$

EF Games on Linear Orders

Theorem

Let $m \neq n$. If $m, n \geq 2^k - 1$, then $\mathbf{L}_m \sim_k \mathbf{L}_n$; otherwise, $\mathbf{L}_m \not\sim_k \mathbf{L}_n$.

Proof. Induction on k. Assume $m, n \geq 2^k - 1$. Spoiler plays $a \in \mathbf{L}_m$.

Case 1: $|\mathbf{L}_m^{<a}| < 2^{k-1} - 1$ Choose $b = a$, thus $\boxed{\mathbf{L}_m^{<a} \sim_{k-1} \mathbf{L}_n^{<b}}$

Since $|\mathbf{L}_m^{>a}|, |\mathbf{L}_n^{>b}| > 2^{k-1} - 1$, $\boxed{\mathbf{L}_m^{>a} \sim_{k-1} \mathbf{L}_n^{>b}}$; Lemma implies: $\boxed{\mathbf{L}_m \sim_k \mathbf{L}_n}$

Case 2: $|\mathbf{L}_m^{>a}| < 2^{k-1} - 1$ Symmetric: choose b s.t. $\mathbf{L}_m^{>a} \simeq \mathbf{L}_m^{>b}$, etc.

EF Games on Linear Orders

Theorem

Let $m \neq n$. If $m, n \geq 2^k - 1$, then $\boldsymbol{L}_m \sim_k \boldsymbol{L}_n$; otherwise, $\boldsymbol{L}_m \not\sim_k \boldsymbol{L}_n$.

Proof. Induction on k. Assume $m, n \geq 2^k - 1$. Spoiler plays $a \in \boldsymbol{L}_m$.

Case 1: $|\boldsymbol{L}_m^{<a}| < 2^{k-1} - 1$ Choose $b = a$, thus $\boxed{\boldsymbol{L}_m^{<a} \sim_{k-1} \boldsymbol{L}_n^{<b}}$

Since $|\boldsymbol{L}_m^{>a}|, |\boldsymbol{L}_n^{>b}| > 2^{k-1} - 1$, $\boxed{\boldsymbol{L}_m^{>a} \sim_{k-1} \boldsymbol{L}_n^{>b}}$; Lemma implies: $\boxed{\boldsymbol{L}_m \sim_k \boldsymbol{L}_n}$

Case 2: $|\boldsymbol{L}_m^{>a}| < 2^{k-1} - 1$ Symmetric: choose b s.t. $\boldsymbol{L}_m^{>a} \simeq \boldsymbol{L}_m^{>b}$, etc.

Case 3: both $|\boldsymbol{L}_m^{<a}|, |\boldsymbol{L}_m^{>a}| \geq 2^{k-1} - 1$ (Is this possible?)

EF Games on Linear Orders

Theorem

Let $m \neq n$. If $m, n \geq 2^k - 1$, then $\mathbf{L}_m \sim_k \mathbf{L}_n$; otherwise, $\mathbf{L}_m \not\sim_k \mathbf{L}_n$.

Proof. Induction on k. Assume $m, n \geq 2^k - 1$. Spoiler plays $a \in \mathbf{L}_m$.

Case 1: $|\mathbf{L}_m^{<a}| < 2^{k-1} - 1$ Choose $b = a$, thus $\boxed{\mathbf{L}_m^{<a} \sim_{k-1} \mathbf{L}_n^{<b}}$

Since $|\mathbf{L}_m^{>a}|, |\mathbf{L}_n^{>b}| > 2^{k-1} - 1$, $\boxed{\mathbf{L}_m^{>a} \sim_{k-1} \mathbf{L}_n^{>b}}$; Lemma implies: $\boxed{\mathbf{L}_m \sim_k \mathbf{L}_n}$

Case 2: $|\mathbf{L}_m^{>a}| < 2^{k-1} - 1$ Symmetric: choose b s.t. $\mathbf{L}_m^{>a} \simeq \mathbf{L}_m^{>b}$, etc.

Case 3: both $|\mathbf{L}_m^{<a}|, |\mathbf{L}_m^{>a}| \geq 2^{k-1} - 1$ (Is this possible?)

Choose any b s.t. $|\mathbf{L}_n^{<b}|, |\mathbf{L}_n^{>b}| \geq 2^{k-1} - 1$:

EF Games on Linear Orders

Theorem

Let $m \neq n$. If $m, n \geq 2^k - 1$, then $\mathbf{L}_m \sim_k \mathbf{L}_n$; otherwise, $\mathbf{L}_m \not\sim_k \mathbf{L}_n$.

Proof. Induction on k. Assume $m, n \geq 2^k - 1$. Spoiler plays $a \in \mathbf{L}_m$.

Case 1: $|\mathbf{L}_m^{<a}| < 2^{k-1} - 1$ Choose $b = a$, thus $\boxed{\mathbf{L}_m^{<a} \sim_{k-1} \mathbf{L}_n^{<b}}$

Since $|\mathbf{L}_m^{>a}|, |\mathbf{L}_n^{>b}| > 2^{k-1} - 1$, $\boxed{\mathbf{L}_m^{>a} \sim_{k-1} \mathbf{L}_n^{>b}}$; Lemma implies: $\boxed{\mathbf{L}_m \sim_k \mathbf{L}_n}$

Case 2: $|\mathbf{L}_m^{>a}| < 2^{k-1} - 1$ Symmetric: choose b s.t. $\mathbf{L}_m^{>a} \simeq \mathbf{L}_m^{>b}$, etc.

Case 3: both $|\mathbf{L}_m^{<a}|, |\mathbf{L}_m^{>a}| \geq 2^{k-1} - 1$ (Is this possible?)

Choose any b s.t. $|\mathbf{L}_n^{<b}|, |\mathbf{L}_n^{>b}| \geq 2^{k-1} - 1$: $\boxed{\mathbf{L}_m^{<a} \sim_{k-1} \mathbf{L}_n^{<b}}$ and $\boxed{\mathbf{L}_m^{>a} \sim_{k-1} \mathbf{L}_n^{>b}}$;

EF Games on Linear Orders

Theorem

Let $m \neq n$. If $m, n \geq 2^k - 1$, then $\mathbf{L}_m \sim_k \mathbf{L}_n$; otherwise, $\mathbf{L}_m \not\sim_k \mathbf{L}_n$.

Proof. Induction on k. Assume $m, n \geq 2^k - 1$. Spoiler plays $a \in \mathbf{L}_m$.

Case 1: $|\mathbf{L}_m^{<a}| < 2^{k-1} - 1$ Choose $b = a$, thus $\boxed{\mathbf{L}_m^{<a} \sim_{k-1} \mathbf{L}_n^{<b}}$

Since $|\mathbf{L}_m^{>a}|, |\mathbf{L}_n^{>b}| > 2^{k-1} - 1$, $\boxed{\mathbf{L}_m^{>a} \sim_{k-1} \mathbf{L}_n^{>b}}$; Lemma implies: $\boxed{\mathbf{L}_m \sim_k \mathbf{L}_n}$

Case 2: $|\mathbf{L}_m^{>a}| < 2^{k-1} - 1$ Symmetric: choose b s.t. $\mathbf{L}_m^{>a} \simeq \mathbf{L}_m^{>b}$, etc.

Case 3: both $|\mathbf{L}_m^{<a}|, |\mathbf{L}_m^{>a}| \geq 2^{k-1} - 1$ (Is this possible?)

Choose any b s.t. $|\mathbf{L}_n^{<b}|, |\mathbf{L}_n^{>b}| \geq 2^{k-1} - 1$: $\boxed{\mathbf{L}_m^{<a} \sim_{k-1} \mathbf{L}_n^{<b}}$ and $\boxed{\mathbf{L}_m^{>a} \sim_{k-1} \mathbf{L}_n^{>b}}$;

Lemma implies: $\boxed{\mathbf{L}_m \sim_k \mathbf{L}_n}$.

Discussion

- Prove the converse at home: if $m < 2^k - 1$ and $m \neq n$, then duplicator has a winning strategy.
- More precisely: there exists a sentence $\varphi \in FO[k]$ s.t. $L_m \models \varphi$ and $L_n \not\models \varphi$. What is φ?
- The *Ehrenfeucht-Fraisse method* for showing inexpressibility in FO is this. For each $k > 0$ construct two structures $\boldsymbol{A}_k, \boldsymbol{B}_k$ then:
  - Prove: $\boldsymbol{A}_k \sim_k \boldsymbol{B}_k$.
  - Prove: $\boldsymbol{A}_k$ has the property, $\boldsymbol{B}_k$ does not.
- Proving $\sim_k$: difficult in general. We will discuss a sufficient condition: Hanf's lemma.