Finite Model Theory
Lecture 5: Query Containment

Spring 2025

Announcements

- Good job on homework 1!
- Homework 2 is posted and due next Friday, 4/25.

Where We Are

- Finite models: we may write $\varphi \equiv \psi$ to mean $\varphi \equiv_{\text{fin}} \psi$.
- Trakhtenbrot: $\texttt{SAT}_{\text{fin}}(\varphi)$ is undecidable.
- Model checking $\boldsymbol{A} \models \varphi$
  - FO: data complexity AC^0, expression complexity PSPACE-complete
  - CQ: data complexity AC^0, expression complexity NP-complete
- We often use "database", "query" instead of "structure", "formula".

Today: Query Containment/Equivalence

Problem Definition

Query Equivalence

Definition (Equivalence)

Q_1, Q_2 are equivalent if $\forall \boldsymbol{D}$, $Q_1(\boldsymbol{D}) = Q_2(\boldsymbol{D})$. Notation: $Q_1 \equiv Q_2$.

Query Equivalence

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Example: $Q_1 \equiv Q_2$ where,

$$Q_1(x) = \exists y \exists z \exists u (E(x,y) \wedge E(y,x) \wedge E(y,u))$$
$$Q_2(x) = \exists y \exists z \exists v (E(x,y) \wedge E(y,x) \wedge E(x,v))$$

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It suffices to study equivalence of Boolean queries, because:
$Q_1(\boldsymbol{x}) \equiv Q_2(\boldsymbol{y})$ iff $(|\boldsymbol{x}| = |\boldsymbol{y}|)$ and for constants $\boldsymbol{c}$, $Q_1[\boldsymbol{c}/\boldsymbol{x}] \equiv Q_2[\boldsymbol{c}/\boldsymbol{y}]$.

Query Containment

Definition (Containment)

Q_1 is contained in Q_2 if $\forall \boldsymbol{D}$, $Q_1(\boldsymbol{D}) \subseteq Q_2(\boldsymbol{D})$. Notation: $Q_1 \subseteq Q_2$

Again, suffices if Q_1, Q_2 are Boolean. Then $Q_1 \subseteq Q_2$ same as $Q_1 \Rightarrow Q_2$.

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Example:

$$Q_1 = \exists x \exists y \exists z \exists u (E(x,y) \wedge E(y,z) \wedge E(z,u))$$
$$Q_2 = \exists x \exists y \exists z (E(x,y) \wedge E(y,z))$$

$$Q_1 \subseteq Q_2$$

Containment v.s. Equivalence

Fact

Equivalence and containment are (almost) the same problem:

$$\boxed{Q_1 \equiv Q_2} \text{ iff } \boxed{Q_1 \subseteq Q_2 \text{ and } Q_2 \subseteq Q_1}$$

$$\boxed{Q_1 \subseteq Q_2} \text{ iff}^1 \boxed{Q_1 \equiv Q_1 \wedge Q_2}$$

[1] Language must be closed under $\wedge$.

Containment for FO is Undecidable

Theorem

The problem Given Q_1, Q_2 in FO check whether $Q_1 \subseteq Q_2$ is undecidable.

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Proof By reduction from $\texttt{SAT}_{\text{fin}}$.

Let Φ be any sentence. (We want to check $\texttt{SAT}_{\text{fin}}(\Phi)$.)

Define $Q_1 \stackrel{\text{def}}{=} \Phi$ and $Q_2 \stackrel{\text{def}}{=} \texttt{FALSE}$. Then $Q_1 \subseteq Q_2$ iff $\neg\texttt{SAT}_{\text{fin}}(\Phi)$.

Containment for CQs

Containment for CQs

The containment problem for CQs is decidable; More precisely, NP-complete.

This is one of the oldest, most celebrated result in database theory. It is due to Chandra and Merlin, 1977.

Review: Equivalent Concepts

- Boolean Conjunctive Query:
  $Q = R(x, y, z) \wedge S(x, u) \wedge S(y, v) \wedge S(z, w) \wedge R(u, v, w)$

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- Boolean Conjunctive Query:
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- A Database $\boldsymbol{A}$ with domain $\texttt{Vars}(Q)$:

$\text{Dom}(\boldsymbol{A}) = \{x, y, z, u, v, w\}$

$R^A =$

x	y	z
u	v	w

$S^A =$

x	u
y	v
z	w

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If $Q(\boldsymbol{x})$ has head variables $\boldsymbol{x}$, then we add named constants $\boldsymbol{x}^A$ to $\boldsymbol{A}$.

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If $Q(\boldsymbol{x})$ has head variables $\boldsymbol{x}$, then we add named constants $\boldsymbol{x}^A$ to $\boldsymbol{A}$.

We call $\boldsymbol{A}$ the Canonical Database of Q, denoted $\boxed{\boldsymbol{D}_Q}$

Review: Homomorphisms

A homomorphisms between two structures $h : \boldsymbol{A} \to \boldsymbol{B}$
is a function $h : \text{Dom}(\boldsymbol{A}) \to \text{Dom}(\boldsymbol{B})$ s.t.

$$h(R^A) \subseteq h(R^B) \text{ for all relation names } R\text{, and}$$
$$h(c^A) = c^B \text{ for all constant names } c.$$

Notice that $\boldsymbol{A}$ or $\boldsymbol{B}$ can be a CQ Q, or a database $\boldsymbol{D}$.

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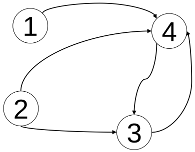
Example:

$Q = (E(x,y) \wedge E(y,z) \wedge E(x,z)$

$\boldsymbol{D} =$



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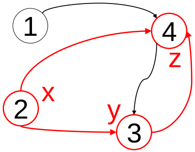
Example:

$Q = (E(x,y) \wedge E(y,z) \wedge E(x,z))$

$x \mapsto 2; y \mapsto 3; z \mapsto 4$

$\boldsymbol{D} =$ 

The Canonical Database

$\boldsymbol{D}_Q$ is just the query Q viewed as a database.

Main property: $\boxed{\boldsymbol{D}_Q \models Q}$

Proof: the identity function $Q \to \boldsymbol{D}_Q$ is a homomorphism.

Containment Theorem for Boolean CQs

Theorem (Chandra and Merlin, 1977)

The following are equivalent:

(1) Containment holds: $Q_1 \subseteq Q_2$. *(if* $\boldsymbol{D} \models Q_1$ *then* $\boldsymbol{D} \models Q_2$*)*

(2) $\boldsymbol{D}_{Q_1} \models Q_2$.

(3) There exists a homomorphism $h : Q_2 \to Q_1$

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Proof

(1) ⇒ (2) If $\boldsymbol{D}_{Q_1} \vDash Q_1$ and $Q_1 \subseteq Q_2$ then $\boldsymbol{D}_{Q_1} \vDash Q_2$

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Containment Theorem for Boolean CQs

Theorem (Chandra and Merlin, 1977)

The following are equivalent:

(1) Containment holds: $Q_1 \subseteq Q_2$. (if $\boldsymbol{D} \models Q_1$ then $\boldsymbol{D} \models Q_2$)

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(3) There exists a homomorphism $h : Q_2 \to Q_1$

Proof

(1) $\Rightarrow$ (2) If $\boldsymbol{D}_{Q_1} \models Q_1$ and $Q_1 \subseteq Q_2$ then $\boldsymbol{D}_{Q_1} \models Q_2$

(2) $\Rightarrow$ (3) If $\boldsymbol{D}_{Q_1} \models Q_2$ then $\exists h : Q_2 \to \boldsymbol{D}_{Q_1}$, hence $\exists h : Q_2 \to Q_1$.

(3) $\Rightarrow$ (1)

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(3) ⇒ (1) If $\boldsymbol{D} \vDash Q_1$, then $\exists g : Q_1 \rightarrow \boldsymbol{D}$;

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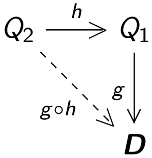
(2) ⇒ (3) If $\boldsymbol{D}_{Q_1} \vDash Q_2$ then $\exists h : Q_2 \to \boldsymbol{D}_{Q_1}$, hence $\exists h : Q_2 \to Q_1$.

(3) ⇒ (1) If $\boldsymbol{D} \vDash Q_1$, then $\exists g : Q_1 \to \boldsymbol{D}$; $g \circ h : Q_2 \to \boldsymbol{D}$ implies $\boldsymbol{D} \vDash Q_2$.



Example: Containment

Recall: $\boxed{Q_1 \subseteq Q_2}$ iff $\boxed{\exists h : Q_2 \to Q_1}$

$$
\begin{aligned}
Q_1 =& E(x, y) \wedge E(y, z) \wedge E(z, u) \\
Q_2 =& E(x, y) \wedge E(y, z)
\end{aligned}
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$Q_1 \subseteq Q_2$ why???

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$Q_1 \subseteq Q_2$ because of the homomorphism $x \mapsto x; y \mapsto y; z \mapsto z$.

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Notice: h must preserve head variables when present.

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\begin{aligned}
Q_1'(x) &= E(x,y) \wedge E(y,z) \wedge E(z,u) \\
Q_2'(x) &= E(x,y) \wedge E(y,z)
\end{aligned}
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$Q_1' \subseteq Q_2'$

Example: Containment

Recall: $\boxed{Q_1 \subseteq Q_2}$ iff $\boxed{\exists h : Q_2 \to Q_1}$

$$Q_1 = E(x,y) \wedge E(y,z) \wedge E(z,u)$$
$$Q_2 = E(x,y) \wedge E(y,z)$$

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$$Q_1''(u) = E(x,y) \wedge E(y,z) \wedge E(z,u)$$
$$Q_2''(y) = E(x,y) \wedge E(y,z)$$

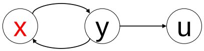
$Q_1'' \not\subseteq Q_2''$

$y \mapsto u; \quad x \mapsto z; \quad z \mapsto$????

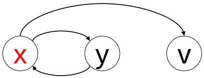
Example: Equivalence

Recall: $\boxed{Q_1 \equiv Q_2}$ iff $\boxed{Q_1 \subseteq Q_2 \text{ and } Q_2 \subseteq Q_1}$

$$Q_1(x) = E(x,y) \wedge E(y,x) \wedge E(y,u)$$



$$Q_2(x) = E(x,y) \wedge E(y,x) \wedge E(x,v)$$



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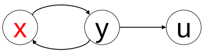

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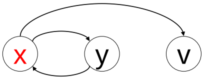

$h_1 : Q_2 \to Q_1$ $h_2 : Q_1 \to Q_2$

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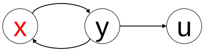

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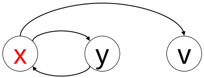

$h_1 : Q_2 \to Q_1$

$x \mapsto x,\ y \mapsto y,\ v \mapsto y$

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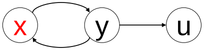

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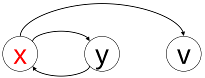

$h_1 : Q_2 \rightarrow Q_1$

$x \mapsto x,\ y \mapsto y,\ v \mapsto y$

$h_2 : Q_1 \rightarrow Q_2$

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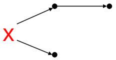
More Examples

Which pairs of queries are contained? Equivalent?

$Q_1(x) = E(x, y) \wedge E(y, z) \wedge E(x, w)$



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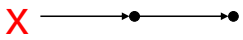
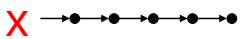
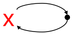


$Q_3(x) = E(x, u_1) \wedge E(u_1, u_2) \wedge \cdots \wedge E(u_4, u_5)$



$Q_4(x) = E(x, y) \wedge E(y, x)$



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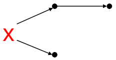

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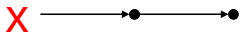

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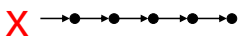
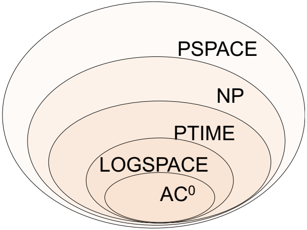


$Q_4(x) = E(x, y) \wedge E(y, x)$

$Q_4 \subseteq Q_3 \subsetneq Q_1 \equiv Q_2$

Complexity

What is the complexity of this problem?
Given Boolean Q_1, Q_2, check if $Q_1 \subseteq Q_2$



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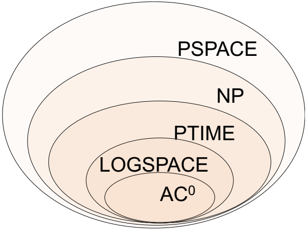

NP-complete!

Proof: same as the model checking for CQs: $Q_1 \subseteq Q_2$ iff $\boldsymbol{D}_{Q_1} \models Q_2$.

Containment Theorem for CQs – Revised

Theorem (Chandra and Merlin, 1977)

Let $Q_1(\boldsymbol{x}_1)$, $Q_2(\boldsymbol{x}_2)$ be CQ's with $|\boldsymbol{x}_1| = |\boldsymbol{x}_2|$. The following are equivalent:

(1) $Q_1 \subseteq Q_2$ $\qquad$ $(\forall \boldsymbol{D}, Q_1(\boldsymbol{D}) \subseteq Q_2(\boldsymbol{D}))$

(2) $\boldsymbol{x}_1^{\boldsymbol{D}_{Q_1}} \in Q_2(\boldsymbol{D}_{Q_1})$ $\qquad$ *($Q_2(\boldsymbol{D}_{Q_1})$ returns the canonical tuple $\boldsymbol{x}_1$)*

(3) $\exists h : Q_2(\boldsymbol{x}_2) \to Q_1(\boldsymbol{x}_1)$. $\qquad$ *(must map $\boldsymbol{x}_2 \mapsto \boldsymbol{x}_1$)*

Containment of UCQs

$Q(\boldsymbol{x}) = Q_1(\boldsymbol{x}) \vee Q_2(\boldsymbol{x}) \vee \cdots$

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The following are equivalent:

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Proof Assume w.l.o.g. Q, Q' are Boolean UCQs.

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$\boxed{\forall i \exists j, Q_i \subseteq Q'_j}$: Assume $\boldsymbol{D} \models Q$. Then $\exists i$, $\boldsymbol{D} \models Q_i$.

Let j be such that $Q_i \subseteq Q'_j$. Then $\boldsymbol{D} \models Q'_j$. We have shown $\boxed{\boldsymbol{D} \models Q'}$

Discussion

- Homomorphism criterion for checking containment of CQs/UCQs
- A simple, little known consequence: the same criterion can be adapted to implication of positive CNF:[2]

$$\varphi_1 = \forall x \forall y \forall z (E(x,y) \vee E(y,z))$$
$$\varphi_2 = \forall x \forall y \forall z \forall u (E(x,y) \vee E(y,z) \vee E(z,u))$$

  then $\varphi_1 \Rightarrow \varphi_2$

- The problem given CQs Q_1, Q_2, does $Q_1 \subseteq Q_2$ hold? is NP-complete.

[2] h must go in the opposite direction: $h : \varphi_1 \to \varphi_2$.

Adding Inequalities: $<, \leq, \neq$

Inequalities

Extend CQ with $<, \leq, \neq$. E.g. $Q(x, y, z) = R(x, y) \wedge R(x, z) \wedge y \neq z$.

The extend languages is denoted $\text{CQ}^{<}$, or $\text{CQ}^{\leq,\neq}$, or $\text{CQ}(\leq, \neq)$.

We assume $\text{Dom}(\boldsymbol{D})$ is densely ordered, e.g. $\mathbb{Q}$.

Problems: containment.

Homomorphism is Sufficient

Idea: treat $x < y$ as another relational predicate $R(x, y)$

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Review: Preorder

Definition

A relation $\leq$ on a set V is called a preorder if:

- It is reflexive: $x \leq x$.
- It is transitive: $x \leq y$, $y \leq z$ implies $x \leq z$.

Write $\boxed{a \equiv b}$ for $a \leq b$ and $b \leq a$.

The preorder is total if $\forall a, b \in V$, either $a \leq b$ or $b \leq a$ or $a \equiv b$.

Extending Q with a Total Preorder $\preceq$

Fix a total preorder $\preceq$ on $\mathrm{Vars}(Q) \cup \mathrm{Const}(Q)$,

$Q_{\preceq}$ denotes the extension of Q with $\preceq$.

Note $Q_{\preceq}$ may be inconsistent, i.e. $\equiv \texttt{False}$.

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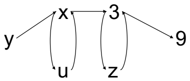

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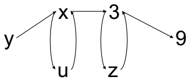

$$Q_\preceq = R(x, y, 3) \wedge S(y, z, u, 9) \wedge (y < x) \wedge (x = u) \wedge (x < 3) \wedge (3 = z) \wedge \cdots$$

A Necessary and Sufficient Condition

Theorem

Let Q, Q' be $CQ^{<,\leq,\neq}$ queries. The following conditions are equivalent:

(1) $Q \subseteq Q'$ $\quad$ *($\forall \boldsymbol{D}$, if $\boldsymbol{D} \vDash Q$ then $\boldsymbol{D} \vDash Q'$)*

(2) For any consistent total preorder $\leq$ on Q, $\exists h : Q' \to Q_{\leq}$.

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Proof:

(2) $\Rightarrow$ (1) If $\boldsymbol{D} \models Q$, then there exists a homomorphism:

$$h_0 : Q \to \boldsymbol{D}$$

This induces a total preorder $\leq$ on Q. Let h be a homomorphism:

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Their composition is a homomorphism $Q' \to \boldsymbol{D}$, proving $Q'(\boldsymbol{D}) = \texttt{true}$.

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(1) $\Rightarrow$ (2) follows from $\boldsymbol{D}_{Q_{\leq}} \models Q$, hence $\boldsymbol{D}_{Q_{\leq}} \models Q'$, and $\exists h : Q' \to Q_{\leq}$.

Example

$$Q = S(x,y) \wedge S(y,z) \wedge (x < z)$$

$$Q' = S(u,v) \wedge (u < v)$$

Lets prove that $Q \subseteq Q'$.

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5 consistent total preorders on Q:

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\begin{aligned}
Q_1 =& S(x,y) \wedge S(y,z) \wedge (y < x) \wedge (y < z)\\
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Q_3 =& S(x,y) \wedge S(y,z) \wedge (x < y) \wedge (y < z)\\
Q_4 =& S(x,y) \wedge S(y,z) \wedge (x < y) \wedge (y = z)\\
Q_5 =& S(x,y) \wedge S(y,z) \wedge (x < y) \wedge (z < y)
\end{aligned}
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In each case, either $(u,v) \mapsto (x,y)$ or $(u,v) \mapsto (y,z)$ is a homomorphism.

Complexity

Assume Q, Q' are CQ's that may contain $<, \leq, \neq$.

Theorem

The problem given Q, Q' determine whether $Q \subseteq Q'$ is Π_2^p-complete.

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Review: there exists Q s.t. Given Q', check $Q \subseteq Q'$ is NP-complete.
Reduction from 3CNF Φ.

$$\Phi = (X \vee \neg Y \vee Z) \wedge (\neg X \vee Y \vee \neg Z) \wedge (\neg X \vee \neg Y \vee Z) \wedge \cdots.$$

$$Q'_\Phi = B(x, z, y) \wedge C(y, x, z) \wedge C(z, x, y) \wedge \cdots$$

$Q = A(0,0,1), A(0,1,0), \ldots$ (all tuples except $A(0,0,0)$; similarly B, C, D)

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The problem given Q, Q' determine whether $Q \subseteq Q'$ is Π_2^p-complete.

Proof: Membership in Π_2^p follows from:

$Q \subseteq Q'$ iff for all extensions $Q_{\leq}$, there exists a homomorphisms $Q' \to Q_{\leq}$.

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- Q has 4 relations A, B, C, D each with 7 tuples.
- Q'_Φ has one atom/clause; e.g. $(X_i \vee \neg X_j \vee X_k)$ becomes $B(x_i, x_k, x_j)$.
- So far: $\exists X_1 \cdots \exists X_n \Phi$ iff $\exists h : Q'_\Phi \to Q$.

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- So far: $\exists X_1 \cdots \exists X_n \Phi$ iff $\exists h : Q'_\Phi \to Q$.

For each universal variable X_i, add the following atoms:

- Add $S(0, u_i, v_i) \wedge S(1, v_i, w_i) \wedge (u_i < w_i)$ to Q.
- Add $S(x_i, a_i, b_i) \wedge (a_i < b_i)$ to Q'_Φ.

$\boxed{Q \subseteq Q'_\Phi}$ holds iff both $x_i \mapsto 0$, $x_i \mapsto 1$ lead to a homomorphisms.

Summary

- The big question: what other extensions of CQ can we allow and still be able to decide containment?
- The following have been studied: inequalities, safe negation $\neg$, certain aggregates $\text{sum}, \text{min}, \text{max}, \text{count}$.
- Containment/equivalence for pure CQ/UCQ is very elegant. Extensions add significant difficulties.