Finite Model Theory
Lecture 3: Zero-One Law for FO

Spring 2025

Announcements

- Homework 1 is due tonight: submit on Canvas.

- Homework 2 to be posted today.

The Classics

- Gödel's Completeness Theorem.
- Church-Turing's and Trakhtenbrot's Theorems.
- Löwenheim-Skolem(-Tarski).
- Los-Vaught Test.
- The Compactness Theorem: will discuss next.

We will use these results to prove the 0/1 law

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  If Σ has an infinite model then it has models of any cardinality $\geq |\sigma|$.
- Los-Vaught Test. If Σ is κ-categorical then it is complete.
  If Σ is also r.e., then it is decidable.
- The Compactness Theorem: will discuss next.

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The Compactness Theorem

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Theorem (Compactness Theorem)

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Proof If $\Sigma \vDash \varphi$, then $\Sigma \vdash \varphi$ holds.
$\Sigma \vdash \varphi$ is a finite sequence $\varphi_1, \ldots, \varphi_n$. Exists finite $\Sigma_0 \subseteq \Sigma$ s.t. $\Sigma_0 \vdash \varphi$.
If $\Sigma_0 \vdash \varphi$ then $\Sigma_0 \vDash \varphi$ holds too.

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Theorem (Compactness Theorem (equivalent statement))

If every finite subset $\Sigma_0 \subseteq \Sigma$ is satisfiable, then Σ is satisfiable.

Proof Assume $\text{not}(\texttt{SAT}(\Sigma))$. Then $\Sigma \models \boldsymbol{F}$.
Then $\Sigma_0 \models \boldsymbol{F}$ for some finite $\Sigma_0 \subseteq \Sigma$. Then $\text{not}(\texttt{SAT}(\Sigma_0))$, contradiction.

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Proof For each node $x_i \in V(G)$, three propositional symbols r_i, g_i, b_i.

Let Σ consist of the following statements:

- For every i: exactly one of r_i, g_i, b_i is true.
- For every edge (i, j): $\neg(r_i \wedge r_j) \wedge \neg(g_i \wedge g_j) \wedge \neg(b_i \wedge b_j)$

Every finite subset of Σ is satisfiable, hence Σ is satisfiable.

Discussion of the Compactness Theorem

"If every finite subset of Σ is satisfiable then Σ is satisfiable."

- A deeper theorem than Gödel's completeness.
- Often proven independently, e.g. using ultraproducts.
- Some of the coolest applications: non-standard numbers, non-standard reals.

Proof of the 0/1 Law

Proof of the Zero-One Law: Plan

$\boxed{\text{Zero-one Law: } \lim_{n\to\infty} \mu_n(\varphi) \text{ is 0 or 1, for every } \varphi}$

Proof outline:

- Define the set of extension axioms, Σ; they have $\mu_n \to 1$.
- Compactness Theorem: Σ has a model.
- Löwenheim-Skolem Theorem: Σ has a countable model.
- Back-and-forth argument: all countable models of Σ are isomorphic.
- Los-Vaught: Σ is complete: if $\Sigma \models \varphi$ then $\mu_n(\varphi) \to 1$, otherwise $\to 0$.

Review: Probabilities

Assume, for simplicity, the language of graphs: $\sigma = (E)$.

We defined μ_n using counting: $\mu_n(\varphi) = \frac{\#_n\varphi}{\#_n\boldsymbol{T}}$.

E.g. $\mu_n(\forall x \exists y E(x,y)) = \frac{(2^n-1)^n}{2^{n^2}}$ Equivalently:

- Let G_n be obtained by including each edge (i,j), $i,j \in [n]$, randomly and independently with probability $1/2$.
- Then $\mu_n(\varphi) = \mathrm{Prob}(G_n \vDash \varphi)$.

Review: Tricks from Probability Theory

- If $\varphi_1, \varphi_2, \ldots$ are independent, then $\mu_n(\varphi_1 \wedge \varphi_2 \wedge \cdots) = \mu_n(\varphi_1) \cdot \mu_n(\varphi_2) \cdots$.

- To show that $\mu_n(\varphi_1 \wedge \varphi_2 \wedge \cdots)$ is "large", show that $\mu_n((\neg\varphi_1) \vee (\neg\varphi_2) \vee \cdots)$ is "small".

- To show that $\mu_n(\varphi_1 \vee \varphi_2 \vee \cdots)$ is "small", use the union bound: $\mu_n(\varphi_1 \vee \varphi_2 \vee \cdots) \leq \mu_n(\varphi_1) + \mu_n(\varphi_2) + \cdots$

The Extension Formulas and the Extension Axioms

For $k > 0$ denote $S_k = ([k] \times \{k\}) \cup (\{k\} \times [k])$ and $\Delta \subseteq S_k$.

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$$EA_{k,\Delta} = \forall x_1 \ldots \forall x_{k-1}(\bigwedge_{i<j<k} (x_i \neq x_j)) \to \exists x_k(\bigwedge_{i<k}(x_k \neq x_i) \wedge EF_{k,\Delta})$$

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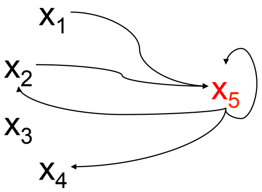

$$\begin{aligned}
&E(x_1, x_5) \wedge \neg E(x_5, x_1) \wedge \\
&E(x_2, x_5) \wedge E(x_5, x_2) \wedge \\
&\neg E(x_3, x_5) \wedge \neg E(x_5, x_3) \wedge \\
&\neg E(x_4, x_5) \wedge E(x_5, x_4) \wedge \\
&E(x_5, x_5)
\end{aligned}$$

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$$\leq n^{k-1} c^{n-k+1} \to 0$$

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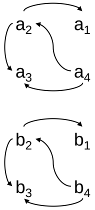

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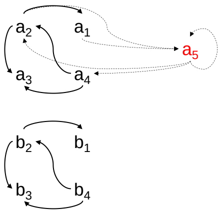

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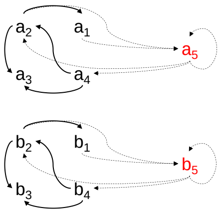

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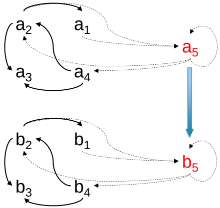

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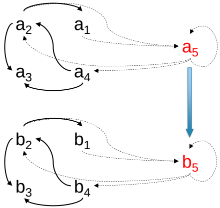

By the Los-Vaught test, Σ is complete.

Every $\mu_n(\varphi)$ converges to 0 or to 1

Let φ be any FO sentence

Σ is complete, hence either $\Sigma \models \varphi$ or $\Sigma \models \neg\varphi$.

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Case 2: $\Sigma \models \neg\varphi$. Then $\mu_n(\varphi) \to 0$ similarly.

Recap of the Proof

- Set of extension axioms, Σ; they have $\mu_n \to 1$.
- Compactness Theorem: Σ has a model.
- Löwenheim-Skolem Theorem: Σ has a countable model.
- Back-and-forth argument: all countable models of Σ are isomorphic.
- Los-Vaught: Σ is complete: if $\Sigma \models \varphi$ then $\mu_n(\varphi) \to 1$, otherwise $\to 0$.

Application of the 0/1 Law

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Contradiction! Hence "$|E|$ is even" is not expressible in FO.

Discussion

- The 0/1 law fails when σ has constants: $\mu(E(a,b)) = \frac{1}{2}$. HW1!
- Undirected Radom Graph (Rado Graph) has explicit construction.
  - Look it up in Libkin's book, or on wikipedia.
  - Give explicit construction for directed graph.
  - Give explicit construction for arbitrary σ.
- We assumed $\mu_n(E(i,j)) = \frac{1}{2}$. Same holds for any $\mu_n(E(i,j)) = c$.
- In the Erdös-Rényi random graph $G(n, p_n)$, p_n depends on n.

  Spencer, *The Strange Logic of Random Graphs*.

Finite Models

Problems over Finite Models

We will consider only finite modes $\boldsymbol{A}$ from now on.

We may consider a fragment of FO, or an extension of FO.

- Static Analysis: does φ have a finite model? does $\varphi \to \psi$ hold in all finite models? Does $\varphi \equiv \psi$ hold in all finite models?
- Model checking (a.k.a. query evaluation): check whether $\boldsymbol{A} \models \varphi$.
- Expressivity: given a property, can we express it using a sentence φ?

Restrictions and Extensions of FO

- Conjunctive queries, unions of conjunctive queries (next lecture)
- Restriction: FO^k; extensions $\mathrm{FO} + \mathrm{lfp}$, $L^{\omega}_{\infty\omega}$; restriction/extension: $L^{k}_{\infty\omega}$. (next week)
- Second order: SO, MSO, ESO, EMSO. (will discuss too)

Static Analysis

Statics Analysis

Goal: check some property of sentence(s) based only on the syntax.
Examples:

- $\texttt{SAT}_{\mathsf{fin}}(\varphi)$: undecidable by Trakhtenbrot's theorem.
- Finite validity: $\models_{\mathsf{fin}} \varphi$: undecidable why???
- Finite implication: $\models_{\mathsf{fin}} (\varphi \to \psi)$: undecidable.
- Finite equivalence: $\models_{\mathsf{fin}} (\varphi \equiv \psi)$: undecidable.

These problems may be decidable in fragments of FO: examples next

The Finite Model Property

Let $L \subseteq FO$ be a subset of FO.

Definition

We say that L has the finite model property, or it is *finitely controllable* if, $\forall \varphi \in L$: if φ has any model, then φ has a finite model.

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We say that L has the small model property if there exists a computable function $f : \mathbb{N} \to \mathbb{N}$ s.t.,
$\forall \varphi \in L$: if φ has any model then it has a finite model of size $\leq f(|\varphi|)$.

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Theorem

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To check $SAT(\varphi)$ enumerate all finite structures $\boldsymbol{A}$ AND all proofs $\vdash \psi$:

- If $\texttt{SAT}(\varphi)$ then some finite model will show up in the first list; YES
- If $\texttt{UNSAT}(\varphi)$ then $\neg\varphi$ will show up in the second list; NO

Application 1: Bernays-Schönfinkel

Let L be the set of sentences with quantifier prefix $\exists^*\forall^*$.
L is called the Bernays-Schönfinkel class.

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The set of $\exists^*\forall^*$ sentences has the small model property, hence it is decidable.

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What is the "small model" function $f(|\varphi|)$?

Application 2: FO^2

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Theorem

FO^2 has the small model property, with an exponential f. More precisely: for any sentence in $\varphi \in FO^2$, if φ is satisfiable then it has a model of size $2^{O(|\varphi|)}$. In particular, FO^2 is decidable.

This is a result by Grädel, Kolaitis, Vardi. We will not prove it.

Pay attention to Trakhtenbrot's proof: should require ≥ 3 variables.

Summary of Static Analysis

- Basically, every static analysis question is undecidable.
- Can be decidable in special cases:
  - FO^2 (but not FO^k for $k \geq 3$).
  - Bernays-Schönfinkel class $\exists^*\forall^*$; a.k.a. EPF. (essentially propositional formula)