Finite Model Theory
Lecture 2: Zero-One Law for FO

Spring 2025

Announcement: Grading

By default, the course is Credit/No-credit.

If you want this course for a PhD/BSMS requirement, you need a grade:

- Send me an email
- I will ask you to submit all the homework assignments

Review: Basic Concepts

Vocabulary σ, structure $\boldsymbol{A}$

Formula, sentence φ, set of sentences Σ

Definition of Truth: $\boldsymbol{A} \models \varphi$

Implication, Validity: $\Sigma \models \varphi$, $\models \varphi$, $\texttt{VAL}(\varphi)$

Satisfiability: $\texttt{SAT}(\varphi)$, $\texttt{SAT}(\Sigma)$

Review: What do these Sentences Say?

$$\exists x \exists y \exists z (x \neq y) \wedge (x \neq z) \wedge (y \neq z)$$

$$\exists x \exists y \exists z \forall u (u = x) \vee (u = y) \vee (u = z)$$

$$\begin{aligned}
\exists x \exists y \exists z & (x \neq y) \wedge (x \neq z) \wedge (y \neq z) \\
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$\exists x \exists y \exists z (x \neq y) \wedge (x \neq z) \wedge (y \neq z)$ There are at least 3 elements

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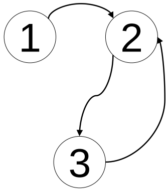
The graph is isomorphic to:



Infinity Axioms

We have seen examples where $\mathtt{SAT}(\Sigma)$ is true $\mathtt{SAT}_{\mathsf{fin}}(\Sigma)$ is false. E.g. $\Sigma = \{\varphi_2, \varphi_3, \dots\}$ where φ_n says "there are $\geq n$ elements"

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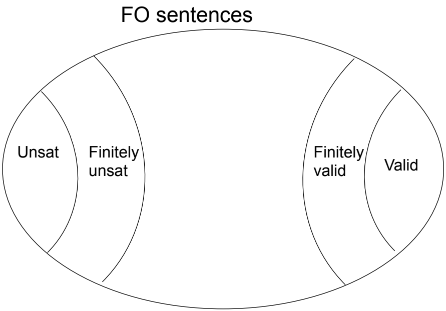
Examples:

- From the End-of-the-line example: $\varphi_1 \wedge \varphi_2 \wedge \varphi_3 \wedge \neg\varphi$.
- $\leq$ is a total order (3 axioms) and it is dense (1 axiom).
- A very short infinity axiom:
$$\forall x(\neg E(x,x) \wedge \exists u(E(x,u) \wedge \forall y(E(y,x) \Rightarrow E(y,u))))$$

The Sentence Map



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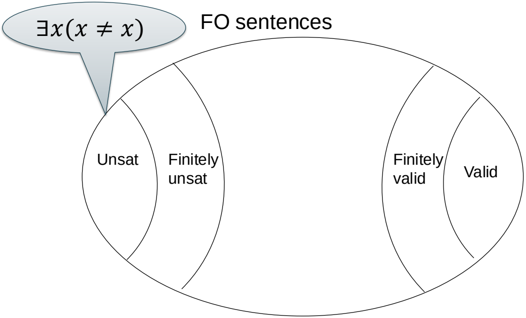

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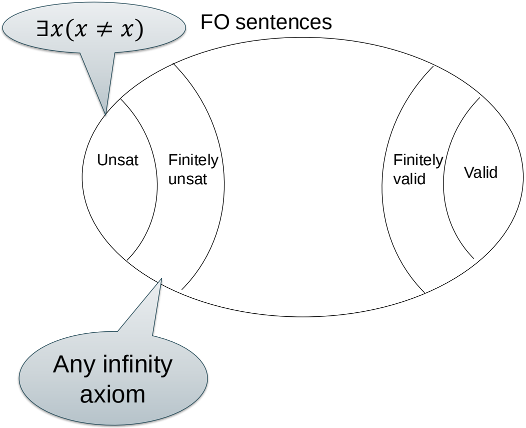

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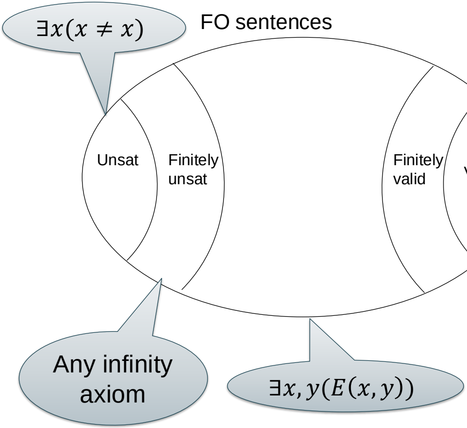

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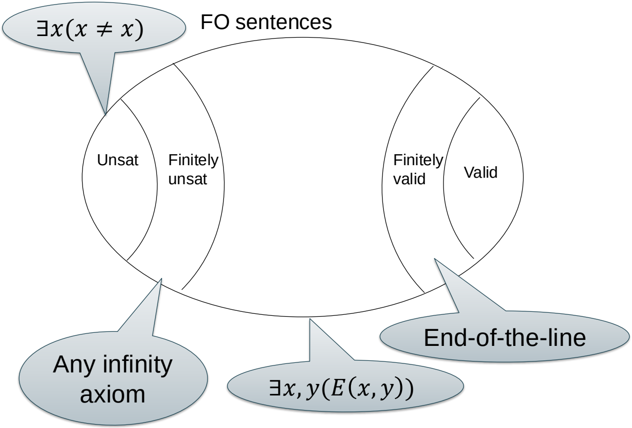

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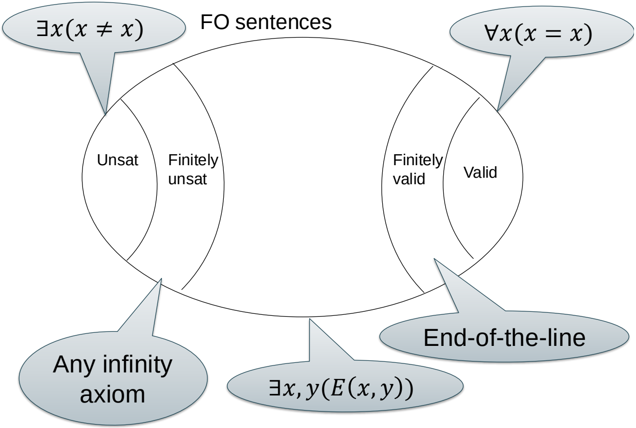

Statement of the 0/1 Law

The Zero-One Law for FO

- Some sentences are neither true (in all structures) nor false.
- The Zero-One Law says this: over *finite* structures, every sentence is true or false *with high probability*.
- Proven by Fagin in 1976 (part of his PhD thesis).

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Vocabulary σ has only relation symbols (no functions, no constants)

Recall: $[n] = \{1, 2, \ldots, n\}$ and $\boldsymbol{T}$ = "true".

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Theorem (Fagin'1976)

For every sentence φ, either $\lim_{n\to\infty}\mu_n(\varphi) = 0$ or $\lim_{n\to\infty}\mu_n(\varphi) = 1$.

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$\varphi = \exists x \exists y E(x, y)$ $\quad \#_n(\varphi) = 2^{n^2} - 1$ $\quad \mu_n = \frac{2^{n^2} - 1}{2^{n^2}} \to 1$

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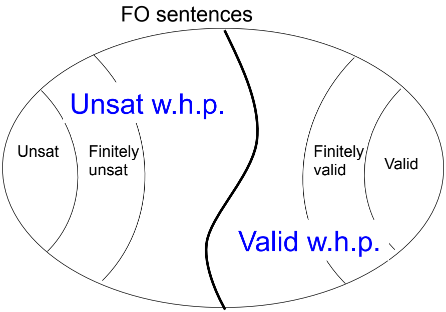
$\varphi = \forall x \exists y E(x, y)$ $\quad \#_n(\varphi) = (2^n - 1)^n$ $\quad \mu_n = \frac{(2^n - 1)^n}{2^{n^2}} \to 1$

The Sentence Map Revised



Discussion

Attempted proof: Derive the general formula $\#_n\varphi$, then compute $\lim \#_n\varphi/2^{n^2}$ and observe it is 0 or 1.

Issue: we don't know how to compute $\#_n\varphi$ in general.

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- $\#_n(\forall x \forall y(E(x,y) \to E(y,x))) =$????
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There exists φ where computing $\#_n\varphi$ given input n is $\#P_1$-complete.

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There exists φ where computing $\#_n\varphi$ given input n is $\#P_1$-complete.

We will prove the 0/1 law using classical model theory (following Fagin).

The Classics

- Gödel's Completeness Theorem
- Church-Turing's Undecidability Theorem
- Löwenheim-Skolem(-Tarski)
- Los-Vaught Test.
- The Compactness Theorem (maybe next time?)

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- Gödel's Completeness Theorem
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Gödel's Completeness Theorem

Overview

- Gödel was motivated by Hilbert's Entscheidungsproblem.
- Part of his PhD Thesis. (We need to raise the bar at UW!)
- In essence, proves that there exists semi-decision procedure for $\Sigma \models \varphi$.
- We can't do better. Church-Turing's theorem: $\Sigma \models \varphi$ is undecidable.

Axioms

There are dozens of choices for the axioms[1]. Recall $\neg\varphi$ is $\varphi \to \boldsymbol{F}$.

$$
\begin{aligned}
A_1 :& \varphi \to (\psi \to \varphi) \\
A_2 :& (\varphi \to (\psi \to \gamma)) \to ((\varphi \to \psi) \to (\varphi \to \gamma)) \\
A_3 :& \neg\neg\varphi \to \varphi \\
A_4 :& \forall x \varphi \to \varphi[t/x] && \text{for any term } t \\
A_5 :& (\forall x(\varphi \to \psi)) \to (\forall x(\varphi) \to \forall x(\psi)) \\
A_6 :& \varphi \to \forall x(\varphi) && x \notin \mathrm{FreeVars}(\varphi) \\
A_7 :& x = x \\
A_8 :& (x = y) \to (\varphi \to \varphi[y/x])
\end{aligned}
$$

These are axiom *schemas*: each A_i defines an infinite set of formulas.

[1] Fans of the Curry-Howard isomorphisms will recognize typed λ-calculus in A_1, A_2.

Deductions (a.k.a. Proofs)

Modus Ponens: if $\boxed{\varphi}$ and $\boxed{\varphi \to \psi}$ are true, then $\boxed{\psi}$ is true.

Let Σ be a set of formulas.

Definition (Deduction, or Proof)

A deduction $\Sigma \vdash \varphi$ is a sequence $\varphi_1, \varphi_2, \ldots, \varphi_n$ such that, for every i:

- φ_i is an instance of an Axiom $A_1 - A_8$, or
- $\varphi_i \in \Sigma$, or
- φ_i is obtained by modus ponens from two earlier formulas, or
- $\varphi_n = \varphi$.

Example of a Deduction

Recall the axioms:

Prove $\boxed{\varphi \to \varphi}$

$$A_1: \varphi \to (\psi \to \varphi)$$
$$A_2: (\varphi \to (\psi \to \gamma)) \to ((\varphi \to \psi) \to (\varphi \to \gamma))$$
$$A_3: \dots$$
$$\dots$$

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Prove at home $\boldsymbol{F} \to \varphi$ and $\varphi \to \psi, \psi \to \omega \vdash \varphi \to \omega$.

Soundness and Completeness

Theorem (Soundness)

If $\Sigma \vdash \varphi$ then $\Sigma \models \varphi$.

Simple proof by induction.

Theorem (Gödel's Completeness Theorem)

If $\Sigma \models \varphi$ then $\Sigma \vdash \varphi$.

Constructive proof, but we won't discuss it.

Discussion of Gödel's Theorem

- $\Sigma \vDash \varphi$ is semantics: it says something about truth.
- $\Sigma \vdash \varphi$ is syntactic: an application of rules.

We can decide if a deduction $\varphi_1, \varphi_2, \ldots, \varphi_n = \varphi$ is correct.

But it is undecidable if a deduction $\Sigma \vdash \varphi$ exists (Church-Turing).

Undecidability Theorem

Undecidability

Recall: $\texttt{VAL}(\varphi)$ means: $\models \varphi$

Theorem (Church-Turing)

$\texttt{VAL}$ is undecidable.

It follows that $\texttt{SAT}$ is undecidable, because $\texttt{VAL}(\varphi) = \neg\texttt{SAT}(\neg\varphi)$.

In English:
There is no algorithm to check $\models \varphi$ or $\vdash \varphi$. Same for $\Sigma \models \varphi$ or $\Sigma \vdash \varphi$.

Review: Decidability

A property P is decidable if there exists algorithm A such that:

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Fact

If P is both r.e. and co-r.e. then P is decidable.

Proof Enumerate both P and $\neg P$.

Validity is R.E. and Satisfiability is Co-R.E.

Assume Σ is r.e. (E.g. it may be finite.)

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Immediate consequence: SAT is co-r.e., because $\mathtt{SAT}(\varphi) = \neg\mathtt{VAL}(\neg\varphi)$.

Finite v.s. Classical Model Theory

$\mathtt{VAL}_{\mathsf{fin}}$, $\mathtt{SAT}_{\mathsf{fin}}$ differ from $\mathtt{VAL}$, $\mathtt{SAT}$.

Could $\mathtt{VAL}_{\mathsf{fin}}$, $\mathtt{SAT}_{\mathsf{fin}}$ be decidable?

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- In finite model theory $\mathtt{SAT}_{\text{fin}}$ is r.e. why?
  Enumerate all finite models $\boldsymbol{A}$, check $\boldsymbol{A} \models \varphi$

Trakhtenbrot's Undecidability Theorem

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$\mathtt{SAT}_{\mathrm{fin}}$ is r.e.

No axiomatization of the finite exists! WHY???

Discussion

- All proves of undecidability are by reduction from an undecidable problem.
- A simple proof of Church-Turing using the word problem is here `http://www.cis.upenn.edu/~val/CIS682/`
- I plan to give (later) a brute-force proof of Trakhtenbrot's thm by encoding a Turing Machine, since that is reused in descriptive complexity.

Löwenheim-Skolem-Tarski Theorem

Review: Cardinal Numbers

A cardinal number is an equivalence class $|A|$ under bijection.

$\aleph_0 = |\mathbb{N}|$ is the smallest infinite cardinal number

$\mathfrak{c} = |\mathbb{R}|$ is the cardinal of the continuum.

Weird arithmetic: $\aleph_0 + \mathfrak{c} = \mathfrak{c}$, $\aleph_0 \times \mathfrak{c} = \mathfrak{c}$, $\mathfrak{c} \times \mathfrak{c} = \mathfrak{c}$, ...

Much larger cardinal numbers exists: $\aleph_0 < 2^{\aleph_0} < 2^{2^{\aleph_0}} < 2^{2^{2^{\aleph_0}}} < \cdots$

Löwenheim-Skolem-Tarski Theorem

Suppose the vocabulary σ has is finite or countable.

Theorem (Löwenheim-Skolem)

If Σ admits an infinite model, then it admits a countable model.

An infinity axiom can say "the world is infinite" but cannot say which infinite.

Löwenheim-Skolem Theorem: Proof

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- Write each $\varphi \in \Sigma$ in prenex-normal form: $(\forall|\exists)^*\psi$, then "Skolemize":

$$\forall x \exists y \forall z \exists u(\varphi) \mapsto \forall x \forall z(\varphi[f_1(x)/y, f_2(x,z)/u])$$

  Let Σ' be the set of Skolemized sentences.

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- Choose countable $S \subseteq D$, and let $\bar{S}$ be its closure under all f's:

$$S \subseteq \bar{S}, \text{ and } c_1, \ldots, c_n \in \bar{S} \text{ implies } f(c_1, \ldots, c_n) \in \bar{S}$$

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$$S \subseteq \bar{S}, \text{ and } c_1, \ldots, c_n \in \bar{S} \text{ implies } f(c_1, \ldots, c_n) \in \bar{S}$$
- Then $\bar{S}$ is a countable model of Σ.

Discussion

- We assumed $|\sigma| \le \aleph_0$. If $|\sigma| = \kappa > \aleph_0$ then the theorem states that Σ has a model of cardinality κ (same proof).

- The upwards version is called: Löwenheim-Skolem-Tarski theorem and states that, for every $\kappa \ge |\sigma|$, Σ has a model of cardinality κ.
(Proof: simply increase σ by adding κ constant symbols to it.)

The Los-Vaught Test

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Theorem

If Σ is r.e. and is complete, then $\Sigma \models \varphi$ is decidable.

Proof: To check $\Sigma \models \varphi$, it suffices to check $\Sigma \vdash \varphi$ (Gödel's completeness).

To check $\Sigma \vdash \varphi$, enumerate all deductions from Σ: $\varphi_1, \varphi_2, \ldots$

Either φ or $\neg\varphi$ will show up.

The Los-Vaught Test

Call Σ $\aleph_0$-categorical if any two countable models of Σ are isomorphic.

Observation: if $\boldsymbol{D}_1, \boldsymbol{D}_2$ are *isomorphic* then $\boldsymbol{D}_1 \vDash \varphi$ iff $\boldsymbol{D}_2 \vDash \varphi$.

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- $\Sigma \cup \{\varphi\}$ has a model $\boldsymbol{D}_1$; assume it is countable why can we?
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- $\Sigma \cup \{\varphi\}$ has a model $\boldsymbol{D}_1$; assume it is countable why can we?
- $\Sigma \cup \{\neg\varphi\}$ has a model $\boldsymbol{D}_2$; assume it is countable.
- Then $\boldsymbol{D}_1, \boldsymbol{D}_2$ are isomorphic.
- Contradiction because $\boldsymbol{D}_1 \models \varphi$ and $\boldsymbol{D}_2 \models \neg\varphi$.

Application of the Los-Vaught Test

The theory of dense linear orders without endpoints is complete.

$$\forall x \forall y \neg((x < y) \wedge (y < x))$$
$$\forall x \forall y((x < y) \vee (x = y) \vee (y < x))$$
$$\forall x \forall y \forall z((x < y) \wedge (y < z) \rightarrow (x < z))$$

Dense: $\forall x \forall y (x < y \rightarrow \exists v (x < v < y))$

W/o Endpoints: $\forall x \exists u \exists w (u < x < w)$

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(Note: linear order is not complete: e.g. it may be dense or not.)

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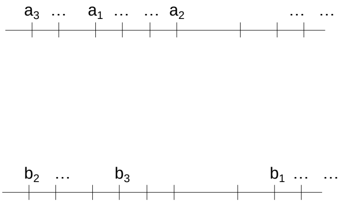
Proof: we apply the Los-Vaught test. Let $\boldsymbol{A}, \boldsymbol{B}$ be countable models. We prove isomorphism, $\boldsymbol{A} \cong \boldsymbol{B}$, using the Back and Forth argument.

The Back-and-Forth argument

$\boldsymbol{A} = (\{a_1, a_2, \ldots\}, <)$, $\boldsymbol{B} = (\{b_1, b_2, \ldots\}, <)$ are total orders w/o endpoints.

Construct inductively A_i, B_i s.t. $(A_i, <) \cong (B_i, <)$.

- Add a_i and matching $b \in B$ s.t. $(A_{i-1} \cup \{a_i\}, <) \cong (B_{i-1} \cup \{b\}, <)$.



- Add b_i and any matching $a \in A$.

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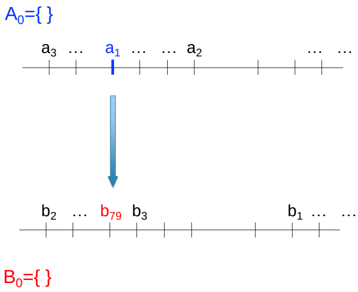

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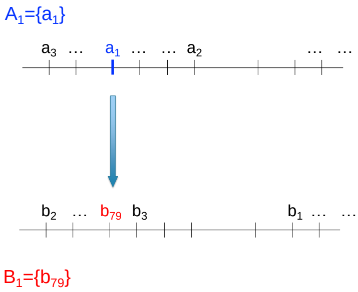

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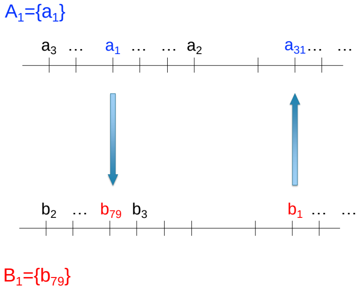

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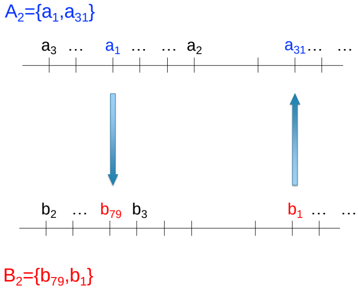

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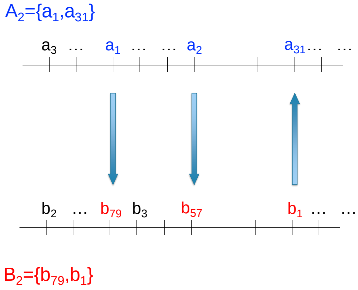

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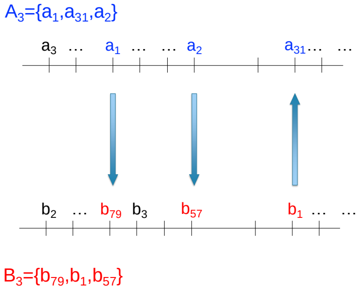

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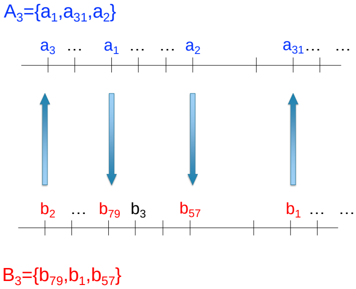

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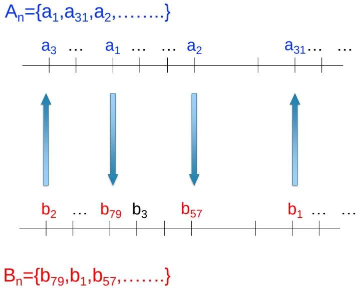

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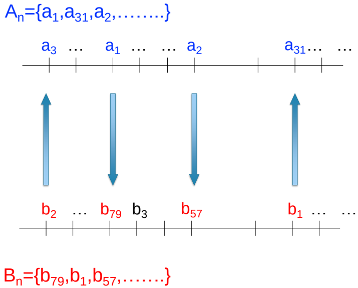

- Add b_i and any matching $a \in A$.

Then $A = \bigcup A_i$, $B = \bigcup B_i$ and $(A, <) \cong (B, <)$.

Discussion

The Los-Vaught test applies to any cardinal number, as follows:

- If Σ has no finite models and is categorical in some infinite cardinal κ (meaning: any two models of cardinality κ are isomorphic) then Σ is complete.

Useful for your homework.

The Classics

We discussed:

- Gödel's Completeness Theorem
- Church-Turing's Undecidability Theorem
- Löwenheim-Skolem(-Tarski) Used in the 0/1 law
- Los-Vaught Test. Used in the 0/1 law
- The Compactness Theorem (maybe next time?) Used in the 0/1 law

Next lecture: we will use these to prove the 0/1 law