Finite Model Theory
Lecture 1: Introduction, Classical Model Theory

Spring 2025

Course Organization

Lectures:

- Time: MW 10 - 11:20
- Room: CSE2 371
- Canceled: May 26, May 28
- Makeup: TBD

Homework assignment:

- Short problems
- Submit on Canvas
- Ignore points
- Collaborations strongly encouraged
- Deadlines are flexible

Grading: Credit / No Credit
If prefer a numerical grade, email me.

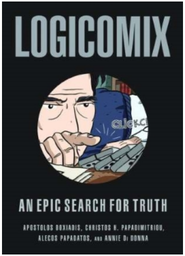
Course Outline

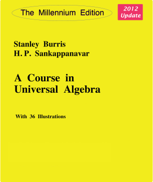
Week of 3/31 Intro, Classical Model Theory, 0/1 Laws
Week of 4/7 Conjunctive Queries, Homomorphism Order
Week of 4/14 EF Games, FO Types
Week of 4/21 Recursion, Datalog, Infinitary Logics, and Pebble Games
Week of 4/28 FO2, C2, Bisimulation, Stable Coloring, GNNs
Week of 5/5 SO: Fagin's Theorem, MSO and Regular expressions
Week of 5/12 Descriptive Complexity
Week of 5/19 FO over Semirings
Week of 5/26 CANCELED
Week of 6/2 TBD

VERY tentative! We may go slower.

Resources

- Doxiadis, Papadimitriou, *Logicomix*.
- Libkin *Finite Model Theory*.
- Enderton *A Mathematical Introduction to Logic*.
- Course on Friendly Logics from UPenn
  Val Tannen and Scott Weinstein
  `http://www.cis.upenn.edu/~val/CIS682/`
- Burris, Sankappanavar, *A Course in Universal Algebra*
- Abiteboul, Hull, Vianu, *Database Theory*
- Some lecture may refer to additional material.





Plan

Today: basic definitions, classical theorems in model theory

Wednesday: $0/1$ law for finite models and its suprising proof using classical theorems

Basic Definitions

Structures

A vocabulary σ is a set of relation symbols $R_1, \ldots, R_k$,
and function symbols $f_1, \ldots, f_m$, each with a fixed arity.

A structure (a.k.a. *model*) is $\boldsymbol{A} = (A, R_1^A, \ldots, R_k^A, f_1^A, \ldots, f_m^A)$,
where $R_i^A \subseteq (A)^{\text{arity}(R_i)}$ and $f_j^A : (A)^{\text{arity}(f_j)} \to A$.

The domain (a.k.a. *universe*), $\text{Dom}(\boldsymbol{A}) \stackrel{\text{def}}{=} A$ is is assumed $\neq \emptyset$.

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Discussion

- We don't really need functions, since $f : A^k \to A$ is represented by its graph $\subseteq A^{k+1}$, but we keep them when convenient.
- A constant, c, is just a 0-ary function $c : A^0 \to A$.
- The structure $\boldsymbol{A}$ may be finite or infinite.
- Sometimes, even the vocabulary σ is infinite!

First Order Logic (FO or FOL)

Fix a vocabulary σ and a set of variables $x_1, x_2, \dots$

Terms:

- Every constant c and every variable x is a term.
- If $t_1, \dots, t_k$ are terms and $f \in \sigma$, then $f(t_1, \dots, t_k)$ is a term.

Formulas:

- $\boldsymbol{F}$ is a formula (means *false*).
- If t_1, t_2 are terms then $t_1 = t_2$ is a formula.
- If $t_1, \dots, t_k$ are terms and $R \in \sigma$ then $R(t_1, \dots, t_k)$ is a formula.
- If φ, ψ are formulas, then so are $\varphi \to \psi$ and $\forall x(\varphi)$.

Discussion

We were very frugal! We used only $\boldsymbol{F}, \rightarrow, \forall$.

Sometimes it's good to be frugal. Sometimes we want more operations:

- $\neg\varphi$ is a shorthand for $\varphi \rightarrow \boldsymbol{F}$.
- $\varphi \vee \psi$ is a shorthand for $(\neg\varphi) \rightarrow \psi$.
- $\varphi \wedge \psi$ is a shorthand for $\neg(\neg\varphi \vee \neg\psi)$.
- $\exists x(\varphi)$ is a shorthand for $\neg(\forall x(\neg\varphi))$.

$\boldsymbol{F}$ often denoted: `false` or $\perp$ or 0.

Sometimes equality (=) is not included in the language.

Formulas, Sentences, Substitution

We say that $\forall x(\varphi)$ *binds* x in φ.
A variable occurrence can be *bound* or *free*.

A sentence is a formula φ without free variables.

Examples:

- Formula: $\varphi(x, z) = \exists y(E(x, y) \wedge E(y, z))$. Free variables: x, z.
- Sentence: $\varphi = \exists x \forall z \exists y(E(x, y) \wedge E(y, z))$.

$\varphi[t/x]$ is formula obtained by substituting free occurrences of x with t

Truth

Let φ be a sentence, and $\boldsymbol{A}$ a structure
Every ground term (i.e. has no variables), t, evaluates to a constant $t^A \in A$

Definition

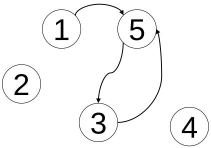
We say that the sentence φ is true in $\boldsymbol{A}$, written $\boxed{\boldsymbol{A} \models \varphi}$, if:

- φ is $t_1 = t_2$ and t_1^A, t_2^A are the same value.
- φ is $R(t_1, \ldots, t_n)$ and $(t_1^A, \ldots, t_n^A) \in R^A$.
- φ is $\psi_1 \to \psi_2$ and $\boldsymbol{A} \not\models \psi_1$, or $\boldsymbol{A} \models \psi_1$ and $\boldsymbol{A} \models \psi_2$.
- φ is $\forall y(\psi)$, and, forall $b \in A$, $\boldsymbol{A} \models \psi[b/y]$.

If Σ is a set of sentences then $\boxed{\boldsymbol{A} \models \Sigma}$ means: for every $\varphi \in \Sigma$, $\boldsymbol{A} \models \varphi$.

This definition is boring but important!

Example 1: Graphs



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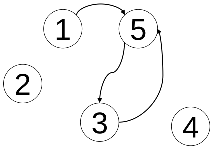

What do these sentences say about a graph?

- $\forall x \exists y E(x, y)$
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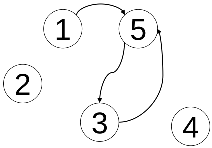

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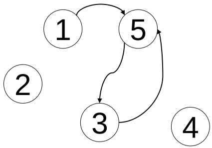

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- $\forall x \exists y E(x, y)$ — every node has an outgoing edge
- $\exists x \forall y E(x, y)$ — some node has outgoing edges to everyone
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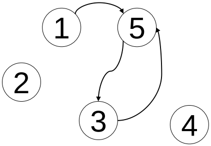

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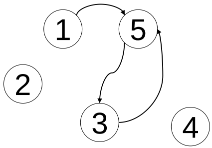

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How do you express the following?

- G is symmetric.
- Every two nodes are connected by path of length 2
- Every two nodes are connected by a path.

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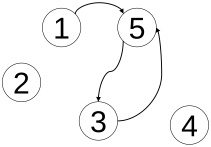

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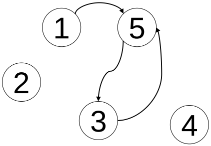

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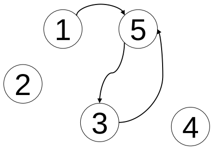

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- Every two nodes are connected by a path. — NOT POSSIBLE!

Example 2: Relational Databases

Vocabulary: $\sigma = (\texttt{User}, \texttt{Order}, \texttt{Product})$

A structure is a database instance: $\boldsymbol{D} = (D, \texttt{User}^D, \texttt{Order}^D, \texttt{Product}^D)$

$\texttt{User}^D$

uid	name
u001	Alice
u002	Bob
u003	Alice

$\texttt{Order}^D$

uid	pid
u001	p555
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What do these sentences say? Which are true in $\boldsymbol{D}$?

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- $\forall u \forall p (\texttt{Order}(u, p) \Rightarrow \exists n (\texttt{User}(u, n)))$
- $\forall u \forall n (\texttt{User}(u, n) \rightarrow$
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- Bob ordered everything that Alice ordered.
- Bob ordered fewer products than Alice.

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Special Case: Propositional Logic

A nullary relation, $P()$, is the same as a propositional variable p:

- In any structure $\boldsymbol{A}$, P^A can be either $\emptyset$ or $\{()\}$.
- If $P^A = \{()\}$ then we say that $p = \texttt{true}$.
- If $P^A = \emptyset$ then we say that $p = \texttt{false}$.

Sentences over nullary relations are the same as propositional formulas:

$$P() \wedge (Q() \vee \neg R())$$

$$p \wedge (q \vee \neg r)$$

Take Away from the Basics

Make sure you have a good understanding of the basic operators:[1]

$$\vee, \wedge, \neg, \rightarrow, \forall, \exists$$

Know simple tricks of the trade:

- Propositional calculus is a very simple special case of FO.
- $\neg\varphi$ is the same as $\varphi \rightarrow \boldsymbol{F}$.
- $\neg(\varphi \rightarrow \psi) = \varphi \wedge \neg\psi$.
- Common idioms for asserting [Property of x]:

$$\begin{array}{lcl} \exists x(R(x,\ldots) & \wedge & [\text{Property of } x]) \\ \forall x(R(x,\ldots) & \Rightarrow & [\text{Property of } x]) \end{array}$$

[1] $\rightarrow$ and $\Rightarrow$ are the same thing.

Problems in Classical, and in Finite Model Theory

Problems in Classical Model Theory

Fix a sentence φ, and a set of sentences Σ (may be infinite).

- Satisfiability: Σ is satisfiable if there exists $\boldsymbol{A}$ such that $\boldsymbol{A} \vDash \Sigma$.
- Implication: Σ implies φ, written $\Sigma \vDash \varphi$, if for every structure $\boldsymbol{A}$:

$$\text{if } \boldsymbol{A} \vDash \Sigma \text{ then } \boldsymbol{A} \vDash \varphi$$

Validity: φ is valid, written $\vDash \varphi$, if for every structure $\boldsymbol{A}$, $\boldsymbol{A} \vDash \varphi$.

- We write $\texttt{SAT}(\varphi)$, or $\texttt{VAL}(\varphi)$ when φ is satisfiable, or valid.

$$\boxed{\neg\texttt{SAT}(\varphi) \text{ iff } \texttt{VAL}(\neg\varphi)}$$

Problems in Finite Model Theory

All previous problems, where the models are restricted to be finite:

- Finite satisfiability, $\mathtt{SAT}_{\mathsf{fin}}(\Sigma)$.
- Finite implication, finite validity: we write $\Sigma \models_{\mathsf{fin}} \varphi$, or $\mathtt{VAL}_{\mathsf{fin}}(\varphi)$.

New problems that make sense only in the finite:

- Model checking: Given φ, $\boldsymbol{A}$, determine whether $\boldsymbol{A} \models \varphi$.
- Query evaluation: Given $\varphi(\boldsymbol{x})$, $\boldsymbol{A}$, compute $\{\boldsymbol{a} \mid \boldsymbol{A} \models \varphi[\boldsymbol{a}/\boldsymbol{x}]\}$.
- Expressibility: can we express a given property in FO?

An Example

$\Sigma = \{\varphi_2, \varphi_3, \varphi_4, \ldots\}$ where:

$$\begin{aligned}
\varphi_2 =& \exists x_1 \exists x_2 (x_1 \neq x_2) \\
\varphi_3 =& \exists x_1 \exists x_2 \exists x_3 (x_1 \neq x_2) \wedge (x_1 \neq x_3) \wedge (x_2 \neq x_3) \\
\varphi_4 =& \exists x_1 \exists x_2 \exists x_3 \exists x_4 (x_1 \neq x_2) \wedge (x_1 \neq x_3) \wedge \cdots \wedge (x_3 \neq x_4) \\
& \ldots
\end{aligned}$$

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$\mathtt{SAT}(\Sigma)$?

$\mathtt{SAT}_{\mathsf{fin}}(\Sigma)$?

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$\mathtt{SAT}(\Sigma)$?

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What does φ_n say?

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What does φ_n say? "There exists at least n elements"

An Example

$\Sigma = \{\varphi_2, \varphi_3, \varphi_4, \dots\}$ where:

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\varphi_2 =& \exists x_1 \exists x_2 (x_1 \neq x_2) \\
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What does φ_n say? "There exists at least *n* elements"

In general $\boxed{\mathtt{SAT}_{\mathtt{fin}}(\Sigma) \Rightarrow \mathtt{SAT}(\Sigma)}$, but not conversely

The Implication Problem

The Implication Problem

Given a set of sentences Σ and a sentence φ, check whether $\Sigma \vDash \varphi$.

When $\Sigma = \varnothing$, then we ask whether $\vDash \varphi$, i.e. $\mathtt{VAL}(\varphi)$.

The Satisfiability Problem is a special case:

$$\boxed{\mathtt{SAT}(\Sigma) \text{ iff not}(\Sigma \vDash \boldsymbol{F})}$$

Example: End of the Line

$\Sigma = \{\varphi_1, \varphi_2, \varphi_3\}$, where:

$$\begin{aligned}
\varphi_1 =& \forall x \forall y_1 \forall y_2 (E(x, y_1) \wedge E(x, y_2) \Rightarrow y_1 = y_2) \\
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Will prove: $\boxed{\Sigma \not\models \varphi}$ but $\boxed{\Sigma \models_{\text{fin}} \varphi}$.

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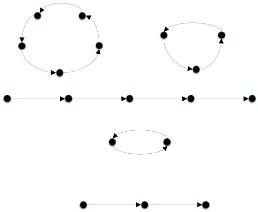

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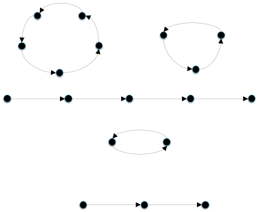

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How do you define a finite set? Or an infinite set?

Review: Cardinal Numbers

Two sets A, B are *equipotent*, or *equipollent*, or *equinumerous*, If there exists a bijection $f : A \to B$. We write $A \cong B$.

Definition

The cardinal number of A is the equivalence class $|A|$ under $\cong$.
We write $|A| \leq |B|$ if there exists an injective function $A \to B$;
Equivalently, if there exists a surjective function $B \to A$.

Cantor-Schröder-Bernstein Theorem: if $|A| \leq |B|$ and $|B| \leq |A|$ then $|A| = |B|$

Consequence: $\leq$ is a total order on cardinal numbers.

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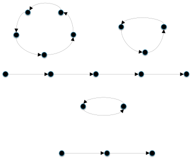
- Is there a cardinal number κ s.t. $\aleph_0 < \kappa < \mathfrak{c}$? Independent of ZFC!!!

Example: Proof for End of the Line

Claim: if $G = (V, E)$ is finite, all nodes have indegree, outdegree ≤ 1, and some node has indegree $= 0$, then some node has outdegree $= 0$.



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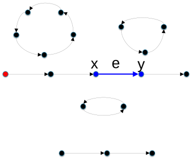
Proof If every node x has outdegree = 1, then we define $f : V \to V$:

$$f(x) \stackrel{\text{def}}{=} \text{"unique successor } y \text{ of } x\text{"}$$



Then f is injective, not surjective. Contradiction.

Discussion

- The implication problem $\Sigma \models \varphi$ is: check if for any structure $\boldsymbol{A}$,
  if $\boldsymbol{A} \models \Sigma$ then $\boldsymbol{A} \models \varphi$
- Different for arbitrary structures $\boldsymbol{A}$ and finite structures:
  $\boxed{\Sigma \models \varphi \text{ implies } \Sigma \models_{\text{fin}} \varphi}$ but not vice versa
- Classical model theory: theorems about arbitrary structures.
- They do not hold in the finite, but are useful nevertheless.