Finite Model Theory – Homework 5

May 17, 2025

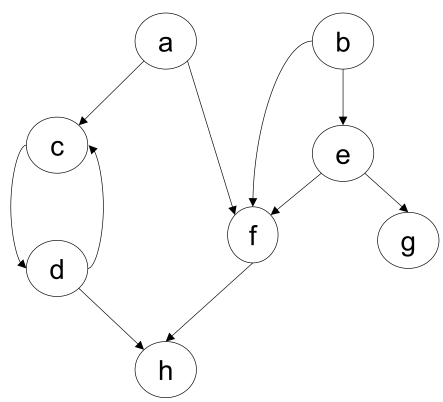
1. (0 points)

1 Least Fixpoint

(a) Let $G = (V, E)$ be a finite graph, and consider the following query:

$$q(x) = [\mathtt{lfp}_{S,x}(\forall y(E(x,y) \rightarrow S(y)))](x)$$

i. Which nodes x does the query return on the graph below?



ii. Write an FO sentence (without fixpoints!) that is equivalent to $\forall x \neg q(x)$.

iii. Consider these complexity classes: AC^0, $PTIME$, NP, $PSPACE$. Indicate the lowest complexity class to which q belongs. You can just indicate the lowest complexity class, no need to prove that it's not lower than that (but you are welcome to do so).

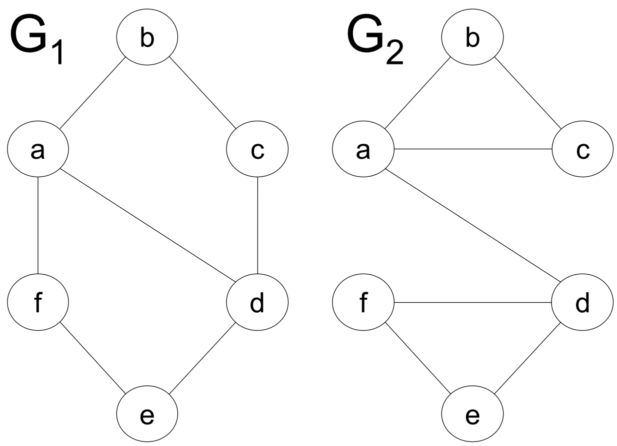
(b) Let $\varphi(x)$ be formula with a free variable x, and R be a unary relational symbol. We say that φ is monotone in a relational symbol R if for any two structures $\boldsymbol{A}, \boldsymbol{B}$ with the same domain and satisfying $R^A \subseteq R^B$, and $S^A = S^B$ for every other relational symbol S, we have $\{a \in A \mid \boldsymbol{A} \models \varphi(a)\} \subseteq \{b \in B \mid \boldsymbol{B} \models \varphi(b)\}$. (Note:

this is the semantic property needed for the least fixpoint, $[\mathtt{lfp}_{R,x}\varphi]$.) Prove that, if the vocabulary includes at least one binary relational symbol other than R, then the problem "given φ check if it is monotone in R over all finite structures" is undecidable.

2 Weisfeiler-Leman

2. (0 points)

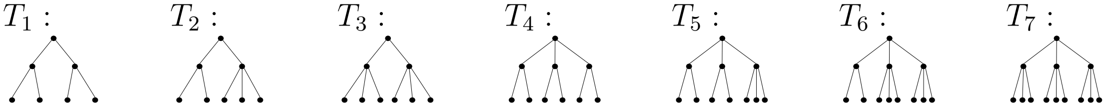
Consider the two graphs below:



(a) Compute a stable coloring for each of the two graphs. Does the stable coloring differentiate between them?

(b) Recall that, after t refinement steps, the color of a vertex can be identified with a tree of depth t. Considering the seven trees below, indicate the color of each vertex in G_1 and in G_2.

T_1 : T_2 : T_3 : T_4 : T_5 : T_6 : T_7 :



(c) Find a sentence $\varphi \in \mathrm{C}^k$ that differentiates between G_1 and G_2. Use the lowest possible value for k.