Finite Model Theory – Homework 3

May 9, 2025

1 Ehrenfeucht-Fraisse Games

1. (0 points)

   (a) Let $\sigma = \{U_1, \ldots, U_m\}$ be a relational vocabulary with m unary predicate symbols. Prove that sentences over this vocabulary satisfy the *small model property*: if φ has any model (finite or infinite), then it also has a finite model of size $\leq f(|\varphi|)$, where f is some computable function f.

   (Hint: given any number k and a structure $\boldsymbol{A}$ over the vocabulary σ, if $\boldsymbol{A}$ is "large enough", then describe a "small" finite structure $\boldsymbol{B}$ s.t. $\boldsymbol{A} \sim_k \boldsymbol{B}$. Here "small" means that the size of $\boldsymbol{B}$ does not depend on $\boldsymbol{A}$, but it may depend on k and m.)

   (b) Prove that the satisfiability problem for a relational vocabulary consisting of only unary predicate symbols is decidable. (Hint: it should be easy once you solved the previous problem.)

   (c) Recall that $L_n = ([n], <)$: more precisely, the vocabulary is $\sigma = (<)$ and the structure L_n is a total order on n elements. Prove that, if $m < 2^k - 1$ then $L_m \sim_k L_n$ iff $m = n$. In other words, if $m < 2^k - 1$ then the duplicator has a winning strategy for the k-pebble game on L_m, L_n only if $m = n$. (Hint: we used induction on k to prove that, whenever $m, n \geq 2^k - 1$ then $L_m \sim_k L_n$, and induction served us well.)

   (d) Consider the vocabulary $\sigma = (E, B)$, where $E(x, y)$ represents an edge from x to y, and $B(x)$ means that x is black. Find a sentence $\varphi \in FO[3]$ that separates the two trees below. In other words, φ is true in one tree, and false in the other. Notice that you can only use up to 3 nested quantifiers.

   (Hint: review how we played the EF games on these two trees in the lecture)

2 Logic on Words

2. (0 points)
   (a) Consider the vocabulary $(<, P_a, P_b, P_c)$ of strings over the alphabet $\Sigma = \{a, b, c\}$.
      i. Write each of the regular expressions below in FO or in MSO. Use succ, $\leq$, $\min, \max$ when needed, since these are expressible using $<$.

$$E_1 = (a|b)^*.c^* \qquad E_2 = (a.b)^* \qquad E_3 = (a.a.a)^*$$

      ii. Write a regular expression describing the following language:

$$\forall S(\exists x(S(x) \land P_a(x)) \to \exists y(S(y) \land P_b(y)))$$