Finite Model Theory – Homework 1

May 9, 2025

1 Satisfiability

1. (0 points)

   (a) For each of the sentences below indicate whether it is satisfiable or not. If it is satisfiable, given an example of a structure that satisfies that sentence.

$$\varphi_1 = \exists x \forall y \, (U(y) \Rightarrow x = y)$$

$$\varphi_2 = \exists x_1 \exists x_2 \, ((x_1 \neq x_2) \land V(x_1) \land V(x_2))$$

$$\varphi_3 = \varphi_1 \land \varphi_2 \land \forall z \, (U(z) \Rightarrow V(z))$$

$$\varphi_4 = \varphi_1 \land \varphi_2 \land \forall z \, (V(z) \Rightarrow U(z))$$

$$\begin{aligned} \varphi_5 = & \forall x \, (U(x) \to (V(x) \lor W(x))) \\ & \land \forall y \, (V(y) \to (U(y) \lor W(y))) \\ & \land \forall z \, (W(z) \to (U(z) \lor V(z))) \\ & \land \neg \exists x (U(x) \land V(x) \land W(x)) \end{aligned}$$

2 Zero-One Law

2. (0 points)

(a) The Zero-One Law for First Order Logic *fails* if the vocabulary contains constants. For example, if we have one binary relation E (describing a graph) and a constant a then:

$$\mu_n(E(a,a)) = \frac{1}{2}$$

Where did we use the assumption that there are no constants in the proof of the zero-one law?

(b) Consider the following two sentences:

$$\begin{aligned}\varphi =& \forall x \exists y (E(x,y) \wedge \forall z (E(y,z) \rightarrow \exists u E(z,u)))\\ \psi =& \exists x \forall y (E(x,y) \rightarrow \exists z (E(y,z) \wedge \forall u (E(z,u) \rightarrow \boldsymbol{F})))\end{aligned}$$

i. Compute $\mu_n(\varphi) + \mu_n(\psi)$. (Hint: answer is 2-3 lines max)

ii. Compute $\lim_{n\to\infty} \mu_n(\varphi)$ for the following sentences φ:

$$\begin{aligned}&\exists x \exists y E(x,y)\\ &\exists x \exists y \exists z (E(x,y) \wedge E(y,z) \wedge E(z,x))\\ &\forall x \forall y \exists z (E(x,z) \wedge E(z,y))\\ &\forall x \forall y (\text{there exists a path from } x \text{ to } y)\end{aligned}$$

(Hint: the last sentece is not in FO, hence it is unclear a priori if $\lim \mu_n(\varphi)$ is 0 or 1, but the answer should be very simple once you solve the others.)

3 Completeness and Decidability

3. (0 points)

   Consider the vocabulary consisting of the constant 0 and the function succ. Let Σ be the following set of axioms:

$\forall x \forall y(\texttt{succ}(x) = \texttt{succ}(y) \to x = y)$	injective
$\forall x(\texttt{succ}(x) \neq 0)$	
$\forall x(x \neq 0 \to \exists y(\texttt{succ}(y) = x)$	almost surjective
$\forall x(\texttt{succ}(x) \neq x)$	no cycle of length n, forall $n \geq 1$
$\forall x(\texttt{succ}(\texttt{succ}(x)) \neq x)$	
$\forall x(\texttt{succ}(\texttt{succ}(\texttt{succ}(x))) \neq x)$	
$\ldots$	

   (a) Prove that Σ is complete, i.e. for any sentence φ, either $\Sigma \models \varphi$ or $\Sigma \models \neg\varphi$.
   (Hint: $(\mathbb{N}, 0, \texttt{succ})$ where $\texttt{succ}(x) = x + 1$ is indeed a model of Σ, but it is not the only countable model! So you need to try a bit harder.)

   (b) Let $\mathbb{N}_S = (\mathbb{N}, 0, \texttt{succ})$ be the structure of *natural numbers with 0 and successor*. Prove that $\text{Th}(\mathbb{N}_S)$ is decidable. (Hint: should be very easy after you solve the previous item.)