CSE 599d Quantum Computing Problem Set 2 Solutions

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Exercise 1: Four-in-one Grover

Let $f_\alpha : \{0,1\}^2 \to \{0,1\}$ be one of four functions from two bits to one bit defined as $f_\alpha(x_1,x_2) = \delta_{\alpha_1,x_1}\delta_{\alpha_2,x_2}$ where $\alpha \in \{0,1\}^2$ and $x \in \{0,1\}^2$ (The four different functions are label by the two bits of α.)

(a) Prove that in order to exactly (no probability of failure) distinguish between these four functions, you need to query this function three times in the worst case.

The function is 1 on only one possible query and 0 otherwise. Further each of the four functions is 1 on different possible queries. When we query the function at a particular input we obtain either 1 or 0. If we query and obtain 1 we can correctly identify the function. If we obtain 0 we only learn that the function is not the function that has 1 at the point where we query the function. Thus after two queries, it is possible in the worst case that we have obtained 0 both times. This will eliminate two possible functions, but the function could still be one of the remaining two. After three queries at different inputs, however, we will have obtained either a 1 output, which exactly identifies the function, or all 0s. In this latter case we have not queried the function at x_1, x_2. The function must output 1 there and thus the function must be the f_α with $\alpha_1 = x_1$ and $\alpha_2 = x_2$.

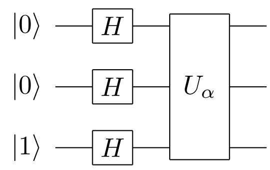
(b) Suppose that you have a unitary gate which enacts this function in the standard reversible manner:

$$U_\alpha = \sum_{x_1,x_2\in\{0,1\}} |x_1,x_2\rangle\langle x_1,x_2| \otimes \sum_{y\in\{0,1\}} |y \oplus f_\alpha(x_1,x_2)\rangle\langle y| \tag{1}$$

Explain how to use this unitary to create the state

$$|\alpha\rangle = \frac{1}{2} \sum_{x_1,x_2\in\{0,1\}} (-1)^{f_\alpha(x_1,x_2)} |x_1,x_2\rangle \tag{2}$$

We use the phase kickback trick. In particular, consider the following circuit



(3)

The state after the first Hadamard gates is

$$|\phi_0\rangle = \frac{1}{2\sqrt{2}}(|0\rangle + |1\rangle) \otimes (|0\rangle + |1\rangle) \otimes (|0\rangle - |1\rangle) \tag{4}$$

Now evaluate the unitary on this state,

$$U_\alpha|\phi_0\rangle = \left[\sum_{x_1,x_2\in\{0,1\}} |x_1,x_2\rangle\langle x_1,x_2| \otimes \sum_{y\in\{0,1\}} |y \oplus f_\alpha(x_1,x_2)\rangle\langle y| \right] |\phi_0\rangle \tag{5}$$

Explicitly acting with the unitary, we obtain

$$\frac{1}{2\sqrt{2}} \Big(|0\rangle \otimes |0\rangle \otimes (|f_\alpha(0,0)\rangle - |\bar{f}_\alpha(0,0)\rangle) + |0\rangle \otimes |1\rangle \otimes (|f_\alpha(0,1)\rangle - |\bar{f}_\alpha(0,1)\rangle)$$
$$+ |1\rangle \otimes |0\rangle \otimes (|f_\alpha(1,0)\rangle - |\bar{f}_\alpha(1,0)\rangle) + |1\rangle \otimes |1\rangle \otimes (|f_\alpha(1,1)\rangle - |\bar{f}_\alpha(1,1)\rangle) \Big) \tag{6}$$

Now notice that (phase kickback)

$$\frac{1}{\sqrt{2}}(|f_\alpha(x_1,x_2)\rangle - |\bar{f}_\alpha(x_1,x_2)\rangle) = \frac{(-1)^{f_\alpha(x_1,x_2)}}{\sqrt{2}}(|0\rangle - |1\rangle) \tag{7}$$

Thus we find that the state output from this circuit is

$$\frac{1}{2\sqrt{2}}\Big(|0\rangle \otimes |0\rangle \otimes (-1)^{f_\alpha(0,0)}(|0\rangle - |1\rangle) + |0\rangle \otimes |1\rangle \otimes (-1)^{f_\alpha(0,1)}(|0\rangle - |1\rangle)$$
$$+|1\rangle \otimes |0\rangle \otimes (-1)^{f_\alpha(0,1)}(|0\rangle - |1\rangle) + |1\rangle \otimes |1\rangle \otimes (-1)^{f_\alpha(0,1)}(|0\rangle - |1\rangle)\Big) \tag{8}$$

or, factoring out the last qubit, which is separable with the first two,

$$\frac{1}{2}\Big((-1)^{f_\alpha(0,0)}|00\rangle + (-1)^{f_\alpha(0,1)}|01\rangle + (-1)^{f_\alpha(1,0)}|10\rangle + (-1)^{f_\alpha(1,1)}|11\rangle\Big)\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \tag{9}$$

which is just $|\alpha\rangle \otimes \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$. Thus ignoring the final qubit, we have created $|\alpha\rangle$.

(c) Show that the $|\alpha\rangle$ states defined in the last problem are all orthonormal.

Consider two states call them $|\alpha\rangle$ and $|\alpha'\rangle$. Then

$$\langle\alpha|\alpha'\rangle = \frac{1}{2}\sum_{x_1,x_2\in\{0,1\}}(-1)^{f_\alpha(x_1,x_2)}\langle x_1,x_2| \sum_{x_1',x_2'\in\{0,1\}}(-1)^{f_{\alpha'}(x_1',x_2')}|x_1',x_2'\rangle \tag{10}$$

But using the orthogonality of the computational basis states, this is just

$$\langle\alpha|\alpha'\rangle = \frac{1}{4}\sum_{x_1,x_2\in\{0,1\}}(-1)^{f_\alpha(x_1,x_2)+f_{\alpha'}(x_1,x)} \tag{11}$$

Now if $\alpha = \alpha'$, then this is

$$\langle\alpha|\alpha'\rangle = \frac{1}{4}\sum_{x_1,x_2\in\{0,1\}}(-1)^{2f_\alpha(x_1,x_2)} = \frac{1}{4}\sum_{x_1,x_2\in\{0,1\}} 1 = 1 \tag{12}$$

If $\alpha \neq \alpha'$, then

$$\langle\alpha|\alpha'\rangle = \frac{1}{4}\sum_{x_1,x_2\in\{0,1\}}(-1)^{\delta_{x_1,\alpha_1}\delta_{x_2,\alpha_2}+\delta_{x_1,\alpha_1'}\delta_{x_2,\alpha_2'}} \tag{13}$$

The term in the exponent of -1, $\delta_{x_1,\alpha_1}\delta_{x_2,\alpha_2}+\delta_{x_1,\alpha_1'}\delta_{x_2,\alpha_2'}$, is 0 if $x_1 \neq \alpha_1$, $x_2 \neq \alpha_2$ and $x_1 \neq \alpha_1'$, $x_2 \neq \alpha_2'$, but is 1 otherwise. Since there are four possible x_1, x_2 values, and two of these fall into the former and two into the latter case, then this sum must be zero. Thus we see that the states are indeed orthonormal: $\langle\alpha|\alpha'\rangle = \delta_{\alpha,\alpha'}$.

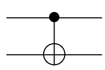
(d) Since the four states defined above are orthogonal, there is a measurement which distinguishes between the four states. Write down a two qubit unitary matrix which transforms the $|\alpha\rangle$ states into the four computational basis states $|00\rangle$, $|01\rangle$, $|10\rangle$, and $|11\rangle$. Express the elements of this unitary matrix in the computational basis.

The unitary which does this transform will have it's rows equal to the state $|\alpha\rangle$. This unitary matrix, expressed in the computational basis is

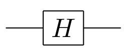
$$U = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \tag{14}$$

Using the orthogonality of the $|\alpha\rangle$s it follows that this matrix is unitary. Further it will transform $|\alpha\rangle$ into a computational basis state with $|\alpha_1,\alpha_2\rangle$.

(e) Construct a circuit which transforms $|\alpha\rangle$ to the computational basis elements using only controlled-NOT gates and Hadamard gates. Note that this need not be the identical matrix to that in part (d). Recall also that the controlled-NOT and Hadamard gates are

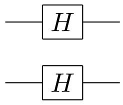


$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$



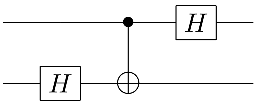
$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \tag{15}$$

Begin by noticing that



$$= \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \quad (16)$$

Which at least has the proper normalization for the states, but doesn't seem to be correct, since the rows are not the $|\alpha\rangle$ states. Perhaps using a controlled-NOT between the Hadamards?



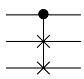
$$= \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix} \quad (17)$$

This circuit has rows that a either the $|\alpha\rangle$ states or such states multiplied by a -1 (like the last row.) Thus it will take the $|\alpha\rangle$ states to computational basis states. Unlike the U I gave in the solution above, this one will not have the values of α_1 and α_2 in the same locations, indeed we see that $|00\rangle$ output corresponds to $\alpha_1 = 1, \alpha_2 = 1$, $|01\rangle$ output corresponds to $\alpha_1 = 0, \alpha_2 = 1$, $|10\rangle$ output corresponds to $\alpha_1 = 1, \alpha_2 = 0$, and $|11\rangle$ output corresponds to $\alpha_1 = 0, \alpha_2 = 0$.

What you've shown in this problem is that is possible to write a quantum algorithm which given a function one two bits which has one marked element (i.e. there is one input, $f(\alpha) = 1$ and the others, $x \neq \alpha$, $f(x) = 0$) which identifies this marked element using a single quantum query. This compares rather favorably with the worst case exact classical model where in the worst case we need four queries. The algorithm we have described is a version of Grover's search algorithm.

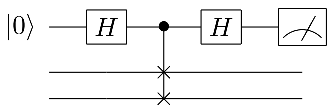
Exercise 2: The Swap Test

Recall that the three qubit gate the controlled-SWAP gate, also known as the Fredkin gate, is given in the computational basis as



$$= C_{SWAP} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Consider the following circuit using this gate:



(18)

(a) Suppose that we feed in two identical single qubit states $|\psi\rangle \otimes |\psi\rangle$ into the second and third qubits of this circuit. What are the probabilities of the two outcomes ($|0\rangle$ and $|1\rangle$) for the measurement meter in this circuit?

The state after the first Hadamard is

$$\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |\psi\rangle \otimes |\psi\rangle \quad (19)$$

The Fredkin gate acts as $C_{SWAP}|0\rangle \otimes |\psi\rangle \otimes |\phi\rangle = |0\rangle \otimes |\psi\rangle \otimes |\phi\rangle$ and $C_{SWAP}|1\rangle \otimes |\psi\rangle \otimes |\phi\rangle = |0\rangle \otimes |\phi\rangle \otimes |\psi\rangle$. Thus

$$C_{SWAP}\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |\psi\rangle \otimes |\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |\psi\rangle \otimes |\psi\rangle \quad (20)$$

Now applying the second Hadamard, we obtain the state

$$|0\rangle \otimes |\psi\rangle \otimes |\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |\psi\rangle \otimes |\psi\rangle \tag{21}$$

Thus with probability 1 we obtain outcome $|0\rangle$ and with probability 0 we obtain $|1\rangle$.

(b) Now suppose that instead of inputting identical qubits to the second and third qubits, we input two states which are orthogonal: $|\psi\rangle \otimes |\phi\rangle$, $\langle\psi|\phi\rangle = 0$. Show that the probabilities of the two outcomes for the measurement meter in the circuit are now both fifty percent.

The state after the first Hadamard is

$$\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |\psi\rangle \otimes |\phi\rangle \tag{22}$$

Applying the Fredkin gate now produces

$$\frac{1}{\sqrt{2}}\left(|0\rangle \otimes |\psi\rangle \otimes |\phi\rangle + |1\rangle \otimes |\phi\rangle \otimes |\psi\rangle\right) \tag{23}$$

Now when we apply the second Hadamard operator we obtain

$$|v\rangle = \frac{1}{2}\left((|0\rangle + |1\rangle) \otimes |\psi\rangle \otimes |\phi\rangle + (|0\rangle - |1\rangle) \otimes |\phi\rangle \otimes |\psi\rangle\right) \tag{24}$$

The projection operators for the measurement are $P_0 = |0\rangle\langle 0|$ and $P_1 = |1\rangle\langle 1|$. Thus the probability of outcomes 0 is

$$Pr(0) = \langle v|P_0 \otimes I \otimes I|v\rangle \tag{25}$$

Now

$$\begin{aligned} P_0 \otimes I \otimes I|v\rangle &= (|0\rangle\langle 0| \otimes I \otimes I)\frac{1}{2}\left((|0\rangle + |1\rangle) \otimes |\psi\rangle \otimes |\phi\rangle + (|0\rangle - |1\rangle) \otimes |\phi\rangle \otimes |\psi\rangle\right) \\ &= \frac{1}{2}|0\rangle \otimes |\psi\rangle \otimes |\phi\rangle + \frac{1}{2}|0\rangle \otimes |\phi\rangle \otimes |\psi\rangle \end{aligned} \tag{26}$$

Taking the inner product of this state with $\langle v|$ yields

$$\langle v|P_0 \otimes I \otimes I|v\rangle = \frac{1}{2}\left(((\langle 0| + \langle 1|) \otimes \langle\psi| \otimes \langle\phi| + ((\langle 0| - \langle 1|) \otimes \langle\phi| \otimes \langle\psi|\right)\frac{1}{2}\left[|0\rangle \otimes |\psi\rangle \otimes |\phi\rangle + |0\rangle \otimes |\phi\rangle \otimes |\psi\rangle\right] \tag{27}$$

which is, using $\langle\psi|\phi\rangle =$, $\frac{1}{4} + \frac{1}{4} = \frac{1}{2}$. Thus the probability of obtaining outcome 0 is one half. The probability of obtaining 1 is therefore also one half.

(c) Find a state which can be inputed into the second and third qubits of this circuit and which will result in the measurement meter always resulting in the outcome $|1\rangle$. Show that this state is orthogonal to all two qubit states of the form $|\psi\rangle \otimes |\psi\rangle$.

The state which is orthogonal to all $|\psi\rangle \otimes |\psi\rangle$ must be orthogonal to $|00\rangle$ and $|11\rangle$. Thus it must be a superposition of $|01\rangle$ and $|10\rangle$. Further, if, under swapping, this state has eignevalue -1, then phase kickback will result in the outcome $|1\rangle$. Thus the state is

$$|\psi_-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) \tag{28}$$

Notice that applying swap to this state results in $-|\psi_-\rangle$. We will check that this state indeed produces the outcome $|1\rangle$. After the first Hadamard, the state is

$$\frac{1}{\sqrt{2}}(|0\rangle + 1\rangle) \otimes |\psi_-\rangle \tag{29}$$

After the Fredkin gate, this becomes

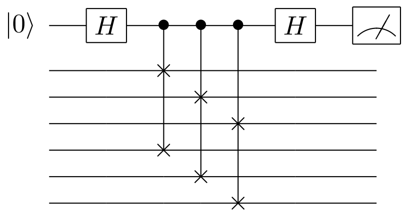
$$\frac{1}{\sqrt{2}}(|0\rangle \otimes |\psi_-\rangle - |1\rangle \otimes |\psi_-\rangle) = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \otimes |\psi_-\rangle \tag{30}$$

Applying the final Hadamard to the first qubit then produces the state $|1\rangle \otimes |\psi_-\rangle$. Measuring this first qubit thus always results in the measurement outcome $|1\rangle$. Now lets check that $|\psi_-\rangle$ is orthogonal to all states of the form $|\psi\rangle \otimes |\psi\rangle$. Express $|\psi_-\rangle = \alpha|0\rangle + \beta|1\rangle$. Then

$$\begin{aligned} \langle\psi_-|(|\psi\rangle \otimes |\psi\rangle) &= \frac{1}{\sqrt{2}}(\langle 01| - \langle 10|)[(\alpha|0\rangle + \beta|1\rangle) \otimes (\alpha|0\rangle + \beta|1\rangle)] \\ &= \alpha\beta - \beta\alpha = 0 \end{aligned} \tag{31}$$

(d) Now suppose that you are given two n qubit states which are either the same or orthogonal. Construct a circuit which will distinguish between these two possibilities with a failure probability of less than or equal to fifty percent.

Notice that nothing in our above constructions relied on the fact that the second and third systems were qubits. Indeed as long as we perform a controlled-SWAP on the two systems, the arguments are the same. Thus to obtain a circuit we simply subsistute the Fredkin gate with n Fredkin gates. For example, for $n = 3$, the circuit will be



(32)

Where we feed the two 3 qubit states into the first three and second three qubits respectively. The analysis for this circuit is the same as for the single qubit circuit and thus achieves the task of distinguishing with at least 50% probability.

The above circuit is called the SWAP test and is a very useful tool in algorithms and in quantum information theory.

Exercise 3: A Continuous Time Search Problem

The unitary evolutions we have been discussing in class are, in the real world, generated by the evolution of Schrodinger's equation. In particular a physical system has a Hamiltonian H and after a time t, the unitary evolution generated is given by the $U(t) = \exp(-iHt)$ (where we have used units where Planck's constant is one.) Here H is a hermitian operator and t is a real number. In this problem we will work on an algorithm which works with Hamiltonians instead of the traditional quantum gates. Throughout the problem we will work on a system of n qubits.

(a) Let $|s\rangle$ be a computational basis element and $|\psi\rangle = \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} |x\rangle$ an equal superposition over all computational basis elements. These two vectors are not orthogonal but span a two dimensional subspace of the Hilbert space of the n qubits. Find a basis for this two dimensional subspace which of two vectors, one of which is $|s\rangle$. In other words, find a linear combination of $|s\rangle$ and $|\psi\rangle$ which is orthogonal to $|s\rangle$ and is properly normalized.

We want to find a state which is a superposition of $|s\rangle$ and $|\psi\rangle$ but which is orthogonal to $|s\rangle$. Lets call this state $|\psi'\rangle = \alpha|s\rangle + \beta|\psi\rangle$. The condition that this state is properly normalized is

$$\langle\psi'|\psi'\rangle = |\alpha|^2 + |\beta|^2 + (\alpha^*\beta\langle s|\psi\rangle + \alpha\beta^*\langle\psi|s\rangle) = 1 \tag{33}$$

That it is orthogonal to $|s\rangle$ yields the condition

$$\langle s|\psi'\rangle = \alpha + \beta\langle s|\psi\rangle = 0 \tag{34}$$

Now $\langle s|\psi\rangle = \frac{1}{\sqrt{2^n}}$, so the ration of α to β must be $-\frac{1}{\sqrt{2^n}}$. Pick α real (by adjusting the global phase freedom), this implies that β is real, and

$$|\psi' = -\frac{\beta}{\sqrt{2^n}}|s\rangle + \beta|\psi\rangle \tag{35}$$

Proper normalization then requires that

$$\frac{1}{2^n}\beta^2 + \beta^2 - \frac{2}{2^n}\beta^2 = 1 \tag{36}$$

Or,

$$\beta = \sqrt{\frac{2^n}{2^n - 1}} \tag{37}$$

and hence

$$\alpha = -\frac{1}{\sqrt{2^n - 1}} \tag{38}$$

Thus an orthogonal vector is

$$|\psi'\rangle = -\frac{1}{\sqrt{2^n - 1}}|s\rangle + \sqrt{\frac{2^n}{2^n - 1}}|\psi\rangle \tag{39}$$

(b) Suppose that we have n qubits and the Hamiltonian $H = |s\rangle\langle s| + |\psi\rangle\langle\psi|$. Express this Hamiltonian in outer product form using the orthogonal basis you found in part (a).

We can express $|\psi\rangle$ in terms of $|s\rangle$ and $|\psi'$,

$$|\psi\rangle = \frac{1}{\sqrt{2^n}}|s\rangle + \sqrt{\frac{2^n - 1}{2^n}}|\psi'\rangle \tag{40}$$

Then we calculate that

$$\begin{aligned} |\psi\rangle\langle\psi| &= \left[\frac{1}{\sqrt{2^n}}|s\rangle + \sqrt{\frac{2^n - 1}{2^n}}|\psi'\rangle\right]\left[\frac{1}{\sqrt{2^n}}\langle s| + \sqrt{\frac{2^n - 1}{2^n}}\langle\psi'|\right] \\ &= \frac{1}{2^n}|s\rangle\langle s| + \frac{\sqrt{2^n - 1}}{2^n}|s\rangle\langle\psi'| + \frac{\sqrt{2^n - 1}}{2^n}|\psi'\rangle\langle s| + \frac{2^n - 1}{2^n}|\psi'\rangle\langle\psi'| \end{aligned} \tag{41}$$

Addining in $|s\rangle\langle s|$, yields

$$|\psi\rangle\langle\psi| + |s\rangle\langle s| = \left(1 + \frac{1}{2^n}\right)|s\rangle\langle s| + \frac{\sqrt{2^n - 1}}{2^n}|s\rangle\langle\psi'| + \frac{\sqrt{2^n - 1}}{2^n}|\psi'\rangle\langle s| + \frac{2^n - 1}{2^n}|\psi'\rangle\langle\psi'| \tag{42}$$

(c) The Hamiltonian H preserves the subspace in part (a). Calculate the action of $U(t) = \exp(-iHt)$ on the subspace from part (a). Express it in the basis you found in part (a).

The subspace is two dimensional, and the Hamiltonian is given by

$$H = \begin{bmatrix} 1 + \frac{1}{2^n} & \frac{\sqrt{2^n-1}}{2^n} \\ \frac{\sqrt{2^n-1}}{2^n} & 1 - \frac{1}{2^n} \end{bmatrix} \tag{43}$$

which we can expand in terms of Pauli operators as

$$H = I + \frac{1}{2^n}Z + \frac{\sqrt{2^n - 1}}{2^n}X \tag{44}$$

Recall that

$$\exp(-i\vec{n}\cdot\vec{\sigma}) = \cos|\vec{n}|I - i\sin|\vec{n}|\hat{n}\cdot\vec{\sigma} \tag{45}$$

The identity component of H just produces a global phase e^{-it}. Then

$$\vec{n} = \left(\frac{\sqrt{2^n - 1}}{2^n}, 0, \frac{1}{2^n}\right) \tag{46}$$

so

$$|\vec{n}| = \sqrt{\frac{2^n - 1}{(2^n)^2} + \frac{1}{(2^n)^2}} = \frac{1}{\sqrt{2^n}} \tag{47}$$

Thus we calculate that

$$U(t) = e^{-it}\left(\cos\frac{t}{\sqrt{2^n}} I - i\sin\frac{t}{\sqrt{2^n}}\hat{n}\cdot\vec{\sigma}\right) \tag{48}$$

The unit vector $\hat{n}$ is given by $\vec{n}/|\vec{n}|$, or

$$\hat{n} = \left(\sqrt{\frac{2^n-1}{2^n}}, 0, \frac{1}{\sqrt{2^n}}\right) \tag{49}$$

So

$$U(t) = e^{-it}\left(\cos\left(\frac{t}{\sqrt{2^n}}\right) I - i\sin\left(\frac{t}{\sqrt{2^n}}\right)\left(\sqrt{\frac{2^n-1}{2^n}}X + \frac{1}{\sqrt{2^n}}Z\right)\right) \tag{50}$$

or, as a two by two matrix

$$U(t) = e^{-it}\begin{bmatrix} \cos\left(\frac{t}{\sqrt{2^n}}\right) - \frac{1}{\sqrt{2^n}} i\sin\left(\frac{t}{\sqrt{2^n}}\right) & -\sqrt{\frac{2^n-1}{2^n}} i\sin\left(\frac{t}{\sqrt{2^n}}\right) \\ -\sqrt{\frac{2^n-1}{2^n}} i\sin\left(\frac{t}{\sqrt{2^n}}\right) & \cos\left(\frac{t}{\sqrt{2^n}}\right) + \frac{1}{\sqrt{2^n}} i\sin\left(\frac{t}{\sqrt{2^n}}\right) \end{bmatrix} \tag{51}$$

(d) Suppose that we start our system in the state $|\psi\rangle$ and then evolve the system by $U(t) = \exp(-iHt)$ for a time $t = T$. At time T we stop this evolution and perform a measurement in the computational basis. What is the probability that we will observe $|s\rangle$ at time T? For what time T is this probability maximized?

We need to calculate $\langle s|U(t)|\psi\rangle$ (and take its magnitude squared to get the probability.) This is just

$$\langle s|U(t)\left(\frac{1}{\sqrt{2^n}}|s\rangle + \sqrt{\frac{2^n-1}{2^n}}|\psi'\rangle\right) \tag{52}$$

Which is

$$\langle s|U(t)|\psi\rangle = \frac{1}{\sqrt{2^n}}e^{-it}\left[\cos\left(\frac{t}{\sqrt{2^n}}\right) - \frac{1}{\sqrt{2^n}} i\sin\left(\frac{t}{\sqrt{2^n}}\right)\right] + \sqrt{\frac{2^n-1}{2^n}}e^{-it}\left[-\sqrt{\frac{2^n-1}{2^n}} i\sin\left(\frac{t}{\sqrt{2^n}}\right)\right] \tag{53}$$

or

$$\langle s|U(t)|\psi\rangle = e^{-it}\left[\frac{1}{\sqrt{2^n}}\cos\left(\frac{t}{\sqrt{2^n}}\right) - i\sin\left(\frac{t}{\sqrt{2^n}}\right)\right] \tag{54}$$

From which we calculate that

$$Pr(|s\rangle) = |\langle s|U(T)|\psi\rangle|^2 = \frac{1}{2^n}\cos^2\left(\frac{T}{\sqrt{2^n}}\right) + \sin^2\left(\frac{T}{\sqrt{2^n}}\right) \tag{55}$$

To find where this is maximized, we need to calculate the derivative of $p(T) = Pr(|s\rangle)$ with respect to T:

$$\frac{dp}{dT} = -\frac{1}{2^n}\sin\left(\frac{2T}{\sqrt{2^n}}\right) + \sin\left(\frac{2T}{\sqrt{2^n}}\right) \tag{56}$$

The maximum (and minimums) thus occur at

$$\sin\left(\frac{2T}{\sqrt{2^n}}\right) = 0 \tag{57}$$

or

$$T = \sqrt{2^n}\frac{\pi}{2}(2k+1) \tag{58}$$

where $k \in \mathbb{Z}$.

The above problem is a continuous time version of Grover's algorithm. Grover's algorithm can be viewed as the above problem made discrete.