CSE 599d Quantum Computing Problem Set 3

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For this problem set recall that the Pauli X, Y, and Z are

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \text{and} \quad Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{1}$$

Exercise 1: Tsirel'son's Inequality

Suppose that A, A', B, B' are operators on some Hilbert space $\mathcal{H}$ which satisfy $A^2 = A'^2 = B^2 = B'^2 = I$ and $[A, B] = [A, B'] = [A', B] = [A', B'] = 0$ (where the commutator is $[M, N] = MN - NM$.)
(a) Define $C = AB + AB' + A'B - A'B'$. Show that $C^2 = 4I - [A, A'][B, B']$.
(b) The *sup norm* of an operator M is defined as

$$||M||_{\text{sup}} = \sup_{|\psi\rangle \neq 0} \frac{||M|\psi\rangle||}{|||\psi\rangle||} \tag{2}$$

where $||\cdot||$ is the standard norm on our Hilbert space. Prove that

$$||M + N||_{\text{sup}} \leq ||M||_{\text{sup}} + ||N||_{\text{sup}} \tag{3}$$

and

$$||MN||_{sup} \leq ||M||_{\text{sup}} ||N||_{\text{sup}} \tag{4}$$

(c) Use these properties of the sup norm to show that

$$||C||_{\text{sup}} \leq 2\sqrt{2} \tag{5}$$

This is Tsirel'son's (or Cirel'son's) inequality. Suppose we are working on a Hilbert space of two qubits. If we take $A = A_1 \otimes I$, $A' = A_2 \otimes I$, $B = I \otimes B_1$, and $B' = I \otimes B_2$, then this expression is

$$||A_1 \otimes B_1 + A_1 \otimes B_1 + A_2 \otimes B_1 - A_2 \otimes B_2||_{\text{sup}} \leq 2\sqrt{2} \tag{6}$$

Recall that from class we saw that for local hidden variable theories satisfy the CHSH inequality: $|\langle C \rangle| \leq 2$. So Tsirel'son's inequality bounds the "amount" of violation that quantum states can have over the CHSH inequality. In fact quantum theory can saturate this bound.

Exercise 2: A Quantum Error Detecting Code

In this problem we will examine a quantum error detecting code on four qubits.
(a) Show that the three four-qubit Pauli group operators $S_1 = X \otimes X \otimes I \otimes I$, $S_2 = I \otimes I \otimes X \otimes X$, $S_3 = Z \otimes Z \otimes Z \otimes Z$ all commute with each other (two operators commute if $AB = BA$.)
(b) The subspace defined by the simultaneous equations $S_i|\psi\rangle = |\psi\rangle$ is two dimensional. Construct an operator made up of a sum of products of S_i operators which projects onto this subspace. Such an operator should satisfy $P|\psi\rangle = |\psi\rangle$ for $|\psi\rangle$ in the subspace and $P|\psi\rangle = 0$ for all $|\psi\rangle$ orthogonal to states in the subspace.
(c) Use the projector you constructed in the last problem to find a basis for the subspace defined by the simultaneous equations $S_i|\psi\rangle = |\psi\rangle$.
(d) Find a Pauli group operator (i.e. one that can be written as a product of Pauli matrices, see problem set 1) which commutes with each of the S_i but which is not a product of the S_is (and is not identity).
(e) Prove that $P \otimes I \otimes I \otimes I$ where P is a Pauli matrix anti-commutes (two operators anticommute if $AB = -BA$) with at least one of the elements S_i. Argue why this is true for $I \otimes P \otimes I \otimes I$, $I \otimes I \otimes P \otimes I$, and $I \otimes I \otimes I \otimes P$ where again P is a Pauli matrix.
(f) If $S_i|\psi\rangle = |\psi\rangle$ and $QS_i = -S_iQ$, prove that $S_i(Q|\psi\rangle) = -(Q|\psi\rangle)$.

(g) Suppose we encode a single qubit into the subspace defined by $S_i|\psi\rangle = |\psi\rangle$. Now suppose a malicious person comes along and applies a Pauli operator of the form $P \otimes I \otimes I \otimes I$, $I \otimes P \otimes I \otimes I$, $I \otimes I \otimes P \otimes I$, $I \otimes I \otimes I \otimes P$, or $I \otimes I \otimes I \otimes I$ producing the new state $|\psi'\rangle$. Explain how determining the value of $S_i|\psi'\rangle$ can tell you whether one of the nontrivial Pauli operators was applied to $|\psi\rangle$ or whether $I \otimes I \otimes I \otimes I$ was applied to $|\psi\rangle$.

The subspace you've considered above is an example of a four qubit error detecting code: we can use measurements of the eigenvalues of the S_i operators to detect when a single error has happened on our encoded qubit.

Exercise 3: Decoherence-Free Subspaces

(a) Consider the following two qubit operators $X_2 = X \otimes I + I \otimes X$, $Y_2 = Y \otimes I + I \otimes Y$ and $Z_2 = Z \otimes I + I \otimes Z$. Find the two qubit state $|\psi\rangle$ which is annihilated by these three operators: $X_2|\psi\rangle = Y_2|\psi\rangle = Z_2|\psi\rangle = 0$.

(b) Suppose that we evolve a two qubit quantum system according to the Hamiltonian

$$H = s_x X_2 + s_y Y_2 + s_z Z_2. \tag{7}$$

In other words the evolution after a time t is $U(t) = \exp(-iHt)$. Prove that $U(t)|\psi\rangle = |\psi\rangle$ where $|\psi\rangle$ is the state you found in part (a).

(c) Now consider two qubits which are attached to another quantum system whose Hilbert space is $\mathcal{H}$. Suppose that the two qubits and the bath interact via the Hamiltonian

$$H_{SB} = X_2 \otimes B_X + Y_2 \otimes B_Y + Z_2 \otimes B_Z \tag{8}$$

where the B_α operators act on $\mathcal{H}$. Show that if we start with the two qubits in the state from part (a) and the bath in an arbitrary state, then evolving using H_{SB} does change the state. In other words, defining $U_{SB}(t) = \exp(-iH_{SB}t)$, show that $U_{SB}(t)|\psi\rangle \otimes |\phi\rangle = |\psi\rangle \otimes |\phi\rangle$ where $|\psi\rangle$ is the state from part (a) and $|\phi\rangle$ is an arbitrary state in $\mathcal{H}$. What you've just shown is that for couplings between the system and bath of the above form, the state $|\psi\rangle$ is protected.

(d) Now consider the four qubit operators

$$\begin{aligned}
X_4 &= X \otimes I \otimes I \otimes I + I \otimes X \otimes I \otimes I + I \otimes I \otimes X \otimes I + I \otimes I \otimes I \otimes X \\
Y_4 &= Y \otimes I \otimes I \otimes I + I \otimes Y \otimes I \otimes I + I \otimes I \otimes Y \otimes I + I \otimes I \otimes I \otimes Y \\
Z_4 &= Z \otimes I \otimes I \otimes I + I \otimes Z \otimes I \otimes I + I \otimes I \otimes Z \otimes I + I \otimes I \otimes I \otimes Z
\end{aligned} \tag{9}$$

Show that each of these operators annihilates the states $|\psi\rangle_{12} \otimes |\psi\rangle_{34}$, $|\psi\rangle_{13} \otimes |\psi\rangle_{24}$ and $|\psi\rangle_{14} \otimes |\psi\rangle_{23}$ where $|\psi\rangle_{ij}$ is the state from part (a) shared between qubits i and j.

(e) Show that the states $|\psi\rangle_{12} \otimes |\psi\rangle_{34}$, $|\psi\rangle_{13} \otimes |\psi\rangle_{24}$ and $|\psi\rangle_{14} \otimes |\psi\rangle_{23}$ are not linearly independent.

(f) Construct a basis for the two dimensional space spanned by the states $|\psi\rangle_{12} \otimes |\psi\rangle_{34}$, $|\psi\rangle_{13} \otimes |\psi\rangle_{24}$ and $|\psi\rangle_{14} \otimes |\psi\rangle_{23}$.

(g) Suppose we encode a qubit of information into the subspace spanned by the two basis states in part (f). If these four qubits now interact with a bath via the Hamiltonian

$$H_4 = X_4 \otimes B_X + Y_4 \otimes B_Y + Z_4 \otimes B_Z \tag{10}$$

then show that the quantum information encoded into this subspace is unaffected by this evolution.

The two dimensional subspace described above is an example of a decoherence-free subspace. Such subspaces exist when the coupling between a system and its environment possess a symmetry: in this case the symmetry is that the qubits couple collectively to the bath. Such codes avoid symmetric decoherence without the need for quantum error correction.