CSE 599d Quantum Computing Problem Set 2

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Exercise 1: Four-in-one Grover

Let $f_\alpha : \{0,1\}^2 \to \{0,1\}$ be one of four functions from two bits to one bit defined as $f_\alpha(x_1, x_2) = \delta_{\alpha_1,x_1}\delta_{\alpha_2,x_2}$ where $\alpha \in \{0,1\}^2$ and $x \in \{0,1\}^2$ (The four different functions are label by the two bits of α.)

(a) Prove that in order to exactly (no probability of failure) distinguish between these four functions, you need to query this function three times in the worst case.

(b) Suppose that you have a unitary gate which enacts this function in the standard reversible manner:

$$U_\alpha = \sum_{x_1,x_2 \in \{0,1\}} |x_1, x_2\rangle\langle x_1, x_2| \otimes \sum_{y \in \{0,1\}} |y \oplus f_\alpha(x_1, x_2)\rangle\langle y| \tag{1}$$

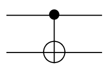
Explain how to use this unitary to create the state

$$|\alpha\rangle = \frac{1}{2} \sum_{x_1,x_2 \in \{0,1\}} (-1)^{f_\alpha(x_1,x_2)} |x_1, x_2\rangle \tag{2}$$

(c) Show that the $|\alpha\rangle$ states defined in the last problem are all orthonormal.

(d) Since the four states defined above are orthogonal, there is a measurement which distinguishes between the four states. Write down a two qubit unitary matrix which transforms the $|\alpha\rangle$ states into the four computational basis states $|00\rangle$, $|01\rangle$, $|10\rangle$, and $|11\rangle$. Express the elements of this unitary matrix in the computational basis.

(e) Construct a circuit which transforms $|\alpha\rangle$ to the computational basis elements using only controlled-NOT gates and Hadamard gates. Note that this need not be the identical matrix to that in part (d). Recall also that the controlled-NOT and Hadamard gates are

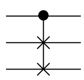


$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad -\boxed{H}- = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \tag{3}$$

What you've shown in this problem is that is possible to write a quantum algorithm which given a function one two bits which has one marked element (i.e. there is one input, $f(\alpha) = 1$ and the others, $x \neq \alpha$, $f(x) = 0$) which identifies this marked element using a single quantum query. This compares rather favorably with the worst case exact classical model where in the worst case we need four queries. The algorithm we have described is a version of Grover's search algorithm.

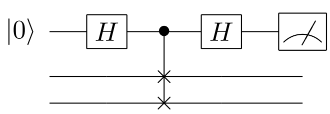
Exercise 2: The Swap Test

Recall that the three qubit gate the controlled-SWAP gate, also known as the Fredkin gate, is given in the computational basis as



$$= C_{SWAP} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Consider the following circuit using this gate:



(4)

(a) Suppose that we feed in two identical single qubit states $|\psi\rangle \otimes |\psi\rangle$ into the second and third qubits of this circuit. What are the probabilities of the two outcomes ($|0\rangle$ and $|1\rangle$) for the measurement meter in this circuit?
(b) Now suppose that instead of inputting identical qubits to the second and third qubits, we input two states which are orthogonal: $|\psi\rangle \otimes |\phi\rangle$, $\langle\psi|\phi\rangle = 0$. Show that the probabilities of the two outcomes for the measurement meter in the circuit are now both fifty percent.
(c) Find a state which can be inputed into the second and third qubits of this circuit and which will result in the measurement meter always resulting in the outcome $|1\rangle$. Show that this state is orthogonal to all two qubit states of the form $|\psi\rangle \otimes |\psi\rangle$.
(d) Now suppose that you are given two n qubit states which are either the same or orthogonal. Construct a circuit which will distinguish between these two possibilities with a failure probability of less than or equal to fifty percent.

The above circuit is called the SWAP test and is a very useful tool in algorithms and in quantum information theory.

Exercise 3: A Continuous Time Search Problem

The unitary evolutions we have been discussing in class are, in the real world, generated by the evolution of Schrodinger's equation. In particular a physical system has a Hamiltonian H and after a time t, the unitary evolution generated is given by the $U(t) = \exp(-iHt)$ (where we have used units where Planck's constant is one.) Here H is a hermitian operator and t is a real number. In this problem we will work on an algorithm which works with Hamiltonians instead of the traditional quantum gates. Throughout the problem we will work on a system of n qubits.

(a) Let $|s\rangle$ be a computational basis element and $|\psi\rangle = \frac{1}{\sqrt{2^n}} \sum_{x\in\{0,1\}^n} |x\rangle$ an equal superposition over all computational basis elements. These two vectors are not orthogonal but span a two dimensional subspace of the Hilbert space of the n qubits. Find a basis for this two dimensional subspace which of two vectors, one of which is $|s\rangle$. In other words, find a linear combination of $|s\rangle$ and $|\psi\rangle$ which is orthogonal to $|s\rangle$ and is properly normalized.
(b) Suppose that we have n qubits and the Hamiltonian $H = |s\rangle\langle s| + |\psi\rangle\langle\psi|$. Express this Hamiltonian in outer product form using the orthogonal basis you found in part (a).
(c) The Hamiltonian H preserves the subspace in part (a). Calculate the action of $U(t) = \exp(-iHt)$ on the subspace from part (a). Express it in the basis you found in part (a).
(d) Suppose that we start our system in the state $|\psi\rangle$ and then evolve the system by $U(t) = \exp(-iHt)$ for a time $t = T$. At time T we stop this evolution and perform a measurement in the computational basis. What is the probability that we will observe $|s\rangle$ at time T? For what time T is this probability maximized?

The above problem is a continuous time version of Grover's algorithm. Grover's algorithm can be viewed as the above problem made discrete.