Modern Quantum Complexity Theory

Lecture 1

Open Questions

Apr 1

Lecture 1: Open Questions

Approach

goal of the course is to motivate research. This is not a finished body of knowledge. Lots of unknown and interesting questions and that is what I encourage you to explore.

There will be some problems assigned during the course, but the majority of the grade comes from participation. I won't assign a project or scribing, although I would appreciate scribing help if people want.

Take a North star approach to this course. Some of you have a fav. question already; if not find one or two today. As the course progresses, look at every lecture w.r.t. North star.

Hamiltonian complexity

$$H = \sum_{i=1}^{m} h_i \qquad \text{each } h_i \text{ acting on } k\text{-qudits and } \|h_i\| = 1.$$

$\lambda_{min}(H)$ = min eigenvalue of H

For $\alpha - \beta = 1/poly(n)$, deciding if $\lambda_{min}(H) \le \alpha$ or $\lambda_{min}(H) \ge \beta$ is QMA-complete.

Local Hamiltonian is q. analog of CSPs.

Most important result for CSPs is PCP-theorem.

Quantum PCP conjecture: $\exists \epsilon > 0$, s.t. $\alpha - \beta = \epsilon m$, deciding if $\lambda_{min}(H) \le \alpha$ or $\lambda_{min}(H) \ge \beta$ is also QMA-complete.

In this course, we will discuss why the classical PCP proofs elude quantization.

Are there subclasses of local Ham. that are easier to solve?

Introducing friends of QMA:

① Commuting local Hamiltonians

Consider $H = \sum h_i$ s.t. $[h_i, h_j] = 0 \quad \forall i, j$.

CSPs $\subseteq$ CLH $\subseteq$ LH so we know NP-hardness and PCP theorem says approximate version is also NP-hard.

What is the complexity of CLH? Is it NP, QMA or something in between?

Open question: Is CLH even MA-hard?

Even better: Is CLH even BPP-hard?

Since BPP $\subseteq$ QMA, $\exists$ a local Ham capturing (via reductions) any BPP problem but the only known construction rely on Kitaev's clock or perturbation theory and are non-commutative.

Why could it equal NP? There exists a basis under which each h_i is diagonal. Wlog, we can assume each h_i is a projector.

For any small collection of h_i, its easy to find the diagonalization. Hard to do in general.

For some families of LH, we know that the CLH problem is in NP

Two major families: 2-local Ham over qudits for any d.

2D local Ham over qubits.

Both rely on the structural lemma of Bravyi Vyali to decompose a CLH problem into a problem with an NP-witness.

At the same time, most QEC is a CLH so we know that they can have complex entanglement.

CLH PCP conjecture: How hard is it to decide if $\lambda_{min}(H) = 0$ or $\lambda_{min}(H) > \epsilon m$?

Interesting question is if it is strictly easier or as hard as deciding $\lambda_{min}(H) = 0$ or $\lambda_{min}(H) \geq 1$.

We hope that issues with cloning that plague QPCP in general can be circumvented by considering CLH.

② Stoquastic Hamiltonians.

H is stoquastic if each $h_i = \begin{bmatrix} \ddots & \leq 0 \\ \leq 0 & \ddots \end{bmatrix} \in \mathbb{R}^{k \times k}$

Hamiltonians with no "sign problem".

This is because we can prove that $|\psi\rangle$, the min eigenvector has only non-negative entries in the standard basis.

This will yield an AM-algorithm after some nice machinery.
Further every MA-promise problem reduces to stoquastic Ham.

Approximation of stoquastic Ham is known to be in NP.

Classification of 2-local Hamiltonians: Either P, NP, StoqMA, or QMA.

$$\begin{array}{c} \mathrm{AM} \\ \rotatebox{90}{$\subseteq$} \\ \mathrm{StoqMA} \\ \rotatebox{90}{$\subseteq$} \\ \mathrm{MA} \end{array}$$

where StoqMA is the complexity of Stoq. Ham.

Does the classification extend to other families of LH? Unclear due to the nature of CLH.

③ QCMA.

$$\text{Merlin} \xrightarrow{\text{classical}} \text{Arthur (BQP)}$$

instead of quantum.

$\mathrm{MA} \subseteq \mathrm{QCMA} \subseteq \mathrm{QMA}$

Known oracle separations (including classical).

How do QMA and QCMA compare?

Say we allow length q witness and time t verifier.

For what fns q, t is $QMA \subseteq QCMA[q, t]$?

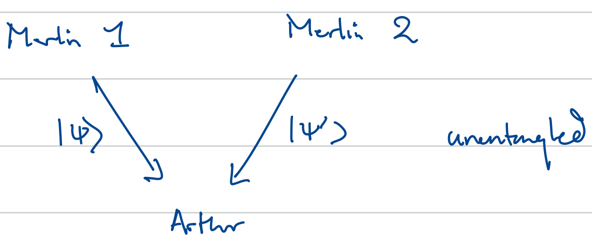
$$QMA \subseteq \underbrace{QCMA\left[2^{O(n)}, O(\log n)\right]}_{\substack{\text{PCP theorem} \\ \text{since } QMA \subseteq PP}} \cap \underbrace{QCMA\left[0, 2^{O(n)}\right]}_{\text{brute-force.}}$$

Open question: Let QMA_ϵ be the complexity of approximating to energy ϵm. So "QPCP" version.

What is the best QCMA upper bound on QMA_ϵ?

Interesting interplay with QPCP conjecture.

④ QMA(2).



What is the power of unentangled proofs? $QMA \subseteq QMA(2) \subseteq NEXP$.

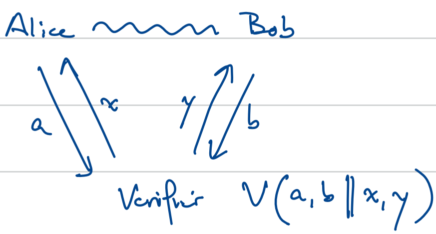
No known oracle separations in either direction.

Brier Tapp protocol. $NP \subseteq QMA(2)[\log n$ pfs, $1/poly(n)$ soundness$]$

If $NP \subseteq QMA[\log n, 1/poly(n)] \Rightarrow NP \subseteq BQP$.

2 proof system for 3-coloring.

5 Quantum games PCP.



given a game described by a fn V, decide the optimal quantum value

This problem is RE-complete.

$MIP^* = RE$.

So $QMA \subseteq MIP^*$. But is there a prover efficient solution?

For QMA problem with witness $|\psi\rangle$, a prover efficient protocol exists

Interactive Proofs

Merlin

$\downarrow\uparrow$ sequences of q. messages.

Arthur

QIP = QIP(3) = IP = PSPACE. Any protocol has a 3 round q. equivalent and any pspace problem has a 3 round protocol.

What is the power of two round interaction? What if the communication is classical but Arthur is quantum?

Personal conjecture: $P^{\#P} \subseteq$ QIP(2) with classical proofs.

BQP $\subseteq$ PSPACE = IP

Efficient interactive proofs for BQP (or oracle counterexample)

For yes instance, $\exists$ a communication strategy computable in q. poly-time but for no instance, all communication strategies are rejected whp

State complexity / Description complexity

Some states are important in q. computation. One collection is the groundstates of local Hamiltonians. What is the complexity of describing them?

$$\text{groundstate} = \lim_{\beta \to \infty} e^{-\beta H} \quad \text{if groundenergy} = 0.$$

So tautologically all the info. about the groundstate is encoded in H.

However, this description is not useful to a BQP device (unless BQP = QMA).

Useful description ex. circuit generating $|\psi\rangle$.

What is the minimum ckt size/depth generating $|\psi\rangle$?

QMA ≠ QCMA implies superpolynomial ckt depth l.bs.

QMA ≠ QCMA + QPCP implies superpolynomial ckt depth l.b.s. for all low-e. states.

We don't know how to prove robust lower bounds past log n.

Search-to-decision.

On input QMA problem H construct g.state $|\psi\rangle$ with access to an oracle for QMA questions.

Classically, use oracle for NP to "binary-search" for optimal proof.

Is there a search-to-decision alg or classical oracle counterexample?

Complexity of Cloning.

given H and one copy of g. state $|\psi\rangle$ how hard is it to create a state $|\varphi\rangle$ s.t. $|\varphi\rangle$ is g. state of $\mathbb{1} \otimes H + H \otimes \mathbb{1}$?

Note $|\varphi\rangle$ could be entangled if H is degenerate.

Afaik, no complexity theoretic bound is known. Since public key q. money is built on the assumption that cloning is hard, we have some crypto l.b.s. for this problem.

Low-depth q. circuits

Lots of models: QNC_0, QAC_0, few non-clifford etc, magic hierarchy

Are there provable state or unitary l.b.s. for these models?

Probably not going to cover much in this course.

Query complexity

"the one area where rigorous results can be proved"

Unitary synthesis problem Can any unitary U on n-qubits be implementable given access to general gates + access to fns $f: \{0,1\}^{poly(n)} \to \{0,1\}$ (in superposition)?

Restricted models are also interesting as only a 1 query l.b. is known.

Cryptography from truly q. assumptions

Can cryptography from quantum assumptions exist even with "bizarre" complexity collapses?

For example, hardness of cloning even if BQP = QMA.

Area: Construct oracle models proving such results could potentially be true.

Many of the results are known for quantum oracles. Conversion to classical oracle result is still open.

QMA Upper Bounds

QMA-hardness of Local Hamiltonians

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QMA :

Merlin

$\Big\downarrow$ poly(n) length quantum message $|\psi\rangle$

Arthur

Arthur runs verification alg $V(x, |\psi\rangle)$.

If $x \in L_{yes}$, $\exists\, |\psi\rangle$ s.t. Arthur accepts w pr $\geq 2/3$.

If $x \in L_{NO}$, $\forall\, |\psi\rangle$, Arthur accepts w pr $\leq 1/3$.

Complete problem for QMA:

given k-local Ham $H = \sum_{i=1}^{m} h_i$, decide if $\lambda_{min}(H) \leq 2^{-n}$

or $\lambda_{min}(H) \geq 1/n^c$.

What is an upper bound on QMA?

(1) QMA $\subseteq$ EXP.

Write out $2^n \times 2^n$ H and diagonalize.

(2) QMA $\subseteq$ PP.

PP = set of decision problems solvable by a ~~randomized~~ classical randomized computation s.t.

(1) yes instance if $\Pr[\text{comp accepts}] \geq 1/2$
(2) no instance if $\Pr[\text{comp accepts}] < 1/2$.

PP = "BPP without the promise gap".
PP = "decision version of #P".

Exercise. given polytime computable fn $f: \{0,1\}^n \to \{-2^n - 1, \ldots, 2^n - 1\}$ and threshold $T \in [-2^{2n}, 2^{2n}]$ deciding if

$$\sum_{x \in \{0,1\}^n} f(x) \leq T \quad \text{is in PP.}$$

We use this formulation to show QMA $\subseteq$ PP.

Start from the complete problem for QMA.

Decide if $\lambda_{min}(H) \le 2^{-n}$ (yes) or $\lambda_{min}(H) \ge n^{-c}$ (no).

Construct $H' = \mathbb{1} - H/m$ so $0 \le H' \le 1$.

Then $\lambda_{max}(H') \ge 1 - 2^{-n}/m$ for yes instances and
$\lambda_{max}(H') \le 1 - 1/n^c$ for no instances.

What is the top eigenvalue of $(H')^t$?

As long as $t \ll m \cdot 2^n$, for yes instances,

$$\lambda_{max}(H'^t) \ge 1 - t \cdot 2^{-n}/m$$

If $t \gg n^c$, then $\lambda_{max}(H'^t) \le e^{-t/n^c}$.

Recall trace of a matrix is the sum of eigenvalues. So,

if yes instance, $tr(H'^t) \ge 1 - t \cdot 2^{-n}/m$ since $H' \ge 0$.

for no instance, its possible that all 2^n eigenvalues are at e^{-t/n^c}.

So, $\operatorname{tr}(H'^t) \le 2^n \cdot e^{-t/n^c}$.

Pick $t = n^{c+1}$. Then $\operatorname{tr}(H'^t) \ge 1 - \frac{n^{c+1}}{m 2^n}$ vs $\left(\frac{2}{e}\right)^n$.

Remains to show that $\operatorname{tr}(H'^t)$ is calculable in PP.

$$H' = \sum_i \frac{1}{m}(\mathbb{1} - h_i) =: \sum_i \frac{1}{m} h_i'.$$

$$\operatorname{tr}(H'^t) = \frac{1}{m^t} \operatorname{tr}\left(\left(\sum_i h_i'\right)^t\right)$$

$$= m^{-t} \sum_x \langle x | \left(\sum_i h_i'\right)^t | x \rangle$$

$$= m^{-t} \sum_{x_1 \dots x_t} \langle x_1 | \left(\sum_i h_i'\right) | x_2 \rangle \langle x_2 | \left(\sum_i h_i'\right) | x_3 \rangle \dots \langle x_t | \left(\sum_i h_i'\right) | x_1 \rangle$$

$$= m^{-t} \sum_{\substack{x_1 \dots x_t \\ i_1 \dots i_t}} \underbrace{\langle x_1 | h_{i_1}' | x_2 \rangle \langle x_2 | h_{i_2}' | x_3 \rangle \dots \langle x_t | h_{i_t}' | x_1 \rangle}_{:= f(x_1 \dots x_t, i_1 \dots i_t)}$$

Sum of a fn (not integer but with a little work can be massaged).

So QMA $\subseteq$ PP.

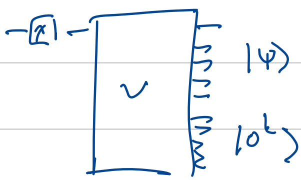
Key idea is that

$$\lim_{t\to\infty} \left(\frac{\mathbb{1} - H}{m}\right)^t \propto \text{groundstate projector.}$$

<u>QMA-hardness of the local Ham. problem</u>

Let's recall the Kitaev clock Ham transform.

given a verification ckt



where $V = g_T \cdots g_1$.

want Ham $H = H(V)$ s.t.

$\lambda_{min}(H)$ small if $\exists\ |\psi\rangle$ causing accept w pr $\geq 1 - 2^{O(n)}$

$\lambda_{min}(H)$ large if $\forall\ |\psi\rangle$, pr accept $\leq 2^{O(n)}$.

Let's create a $O(\log n)$-local Hamiltonian whose ground states are of the form $\frac{1}{\sqrt{T+1}} \sum_{t=0}^{T} |t\rangle \otimes |\psi_t\rangle \leftarrow$ state register

$\uparrow$ time register

t expressed in $\lceil \log(T+1) \rceil$ bits

and $|\psi_t\rangle = g_t \cdots g_1 |\psi_0\rangle$ (all related).

$$h_t = \frac{1}{2}\left(|t\rangle\langle t| \otimes \mathbb{1} - |t\rangle\langle t-1| \otimes g_t - |t-1\rangle\langle t| \otimes g_t^\dagger + |t-1\rangle\langle t-1| \otimes \mathbb{1} \right)$$

$$V = \sum_{t=0}^{T} |t\rangle\langle t| \otimes g_t \cdots g_1$$

$$V^\dagger = \sum_{t=0}^{T} |t\rangle\langle t| \otimes g_1^\dagger \cdots g_t^\dagger .$$

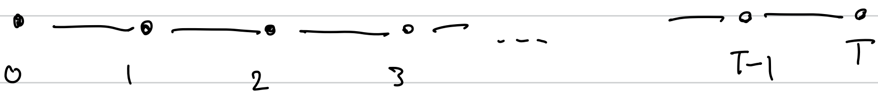
$$V^\dagger h_t V = \frac{1}{2}\left(|t\rangle\langle t| \otimes \mathbb{1} - |t-1\rangle\langle t| \otimes \mathbb{1} - |t\rangle\langle t-1| \otimes \mathbb{1} + |t-1\rangle\langle t-1| \otimes \mathbb{1} \right)$$

$$= \frac{1}{\sqrt{2}}\left(|t\rangle - |t-1\rangle \right) \frac{1}{\sqrt{2}}\left(\langle t| - \langle t-1| \right) \otimes \mathbb{1} .$$

So, $V^\dagger \left(\sum_{t=1}^{T} h_t \right) V =$

$$\frac{1}{2} \begin{pmatrix} 1 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & -1 & \\ & & & \ddots & 2 & -1 \\ & & & & -1 & 1 \end{pmatrix} \otimes \mathbb{1} .$$

This is a special matrix. It is the Laplacian of a line graph and also a circulant matrix. (also tri-diagonal)



for any $|\psi_0\rangle$

$$\lambda_0 = 0, \quad \text{eigenvector } |\Psi_0\rangle = \frac{1}{\sqrt{T+1}} \sum_{t=0}^{T} |t\rangle \otimes |\psi_0\rangle$$

$\lambda_1 \geq \frac{c}{T^2}$ $\leftarrow$ exercise / intuition from graph mixing time.

So, removing rotation by V:

$H_{prop} = \sum_{t=0}^{T} h_t$ is a Hamiltonian with

ground-energy $= 0$, groundstates of the form $\underbrace{\frac{1}{\sqrt{T+1}} \sum_{t=0}^{T} |t\rangle \otimes g_t \cdots g_1 |\psi_0\rangle}_{\text{known as a history state}}$

for any state $|\psi_0\rangle$

highly degenerate groundspace.

and first non-zero energy of $\geq c/T^2$.

Therefore, $H_{prop} \geq \frac{c}{T^2}\left(\mathbb{1} - \Pi_{prop}\right)$ Projector onto span of history states

History states just check evolution of the computation

To check output of computation, we want a term that checks that when the time register is set to T, the first qubit is in the $|1\rangle$ state.

$$H_{out} = |T\rangle\langle T|_{time} \otimes |0\rangle\langle 0|_1$$

Penalizes being time T and first qubit = 0.

To check input, we need to make sure that all the ancilla registers of the computation were set to $|0\rangle$.

$$H_{in} = \sum_{j=1}^{m} |0\rangle\langle 0|_{time} \otimes |1\rangle\langle 1|_{ancilla(j)}$$

$H_{out} + H_{in}$ is a commuting Hamiltonian meaning every pair of local terms commute.

Each local term is a projector. So $H_{out} + H_{in}$ has an integer spectrum.

projector onto passing all in & out checks.

$$H_{out} + H_{in} \geq \left(\mathbb{1} - \Pi_{inout}\right)$$

<u>Claim</u> $H = H_{prop} + H_{out} + H_{in}$ suffices as a Ham.

<u>Pf</u>. Consider a (yes) instance of QCIRCUIT SAT.

i.e. $\exists\, |\psi\rangle$ s.t. $C(|\psi\rangle)$ accepts w $pr \geq 1 - \epsilon$.

Then let $|\Psi\rangle = \frac{1}{\sqrt{T+1}} \sum_{t=0}^{T} |t\rangle \otimes g_t \cdots g_1 |\psi, 0^m\rangle$

the History state of $|\psi\rangle$. By construction,

$\langle\Psi| H_{prop} |\Psi\rangle = 0$

$\langle\Psi| H_{in} |\Psi\rangle = 0 \qquad \Rightarrow \qquad \langle\Psi|H|\Psi\rangle \leq \epsilon.$

$\langle\Psi| H_{out} |\Psi\rangle \leq \epsilon$

What about a (no) instance of QCircuit SAT?

Recall the max success prob. of a QCircuitSAT instance was $\cos^2\Theta$ where Θ was the angle between $\mathbb{1}_{2^n} \otimes |0\rangle\langle 0|^m$ and $C^\dagger (|1\rangle\langle 1| \otimes \mathbb{1}_{2^{n+m-1}}) C$ projectors.

If max success prob is small $(\leq \epsilon)$ then Θ is large (near $\frac{\pi}{2}$). as $\cos^2\Theta = \epsilon$.

We want to show that $\lambda_{min}(H)$ is not super small.

Since $H_{prop} \geq \frac{c}{T^2}(\mathbb{1} - \Pi_{prop})$

it suffices to show lower bound on:

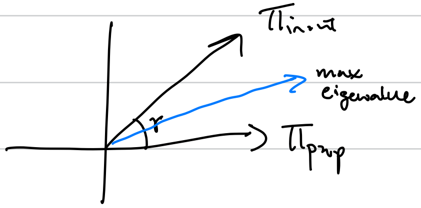
$$\lambda_{min}\left(\frac{c}{T^2}(\mathbb{1} - \Pi_{prop}) + (\mathbb{1} - \Pi_{inout})\right)$$

$$\geq \frac{c}{T^2}\lambda_{min}\left((\mathbb{1} - \Pi_{prop}) + (\mathbb{1} - \Pi_{inout})\right)$$

$$\geq \frac{c}{T^2}\left(2 - \lambda_{max}(\Pi_{prop} + \Pi_{inout})\right)$$

$$= \frac{c}{T^2}\left(2 - 2\cos^2\frac{\gamma}{2}\right) = \frac{2c}{T^2}\sin^2\frac{\gamma}{2}.$$

where γ = angle between Π_{prop} and Π_{inout}



$$\gamma = \text{angle}\left(\Pi_{prop}, \Pi_{inout}\right) = \text{angle}\left(V^{\dagger}\Pi_{prop}V, V^{\dagger}\Pi_{inout}V\right)$$

To calculate angle between spaces:

$$\cos^2\gamma = \max_{|\psi\rangle} \langle h_\psi | \Pi_{inout} | h_\psi \rangle$$

$$= \max_{|\psi\rangle} \frac{1}{T+1} \sum_{t=0}^{1} \langle t | \langle \psi | \Pi_{inout} | t \rangle | \psi \rangle$$

$$= \max_{|\psi\rangle} \frac{T-1}{T+1} + \frac{1}{T+1}\Big(\langle \psi | C^{\dagger} \left(|1\rangle\langle 1| \otimes \mathbb{1} \right) C | \psi \rangle + \langle \psi | \mathbb{1}_{2^m} \otimes |0^m\rangle\langle 0^m| \, | \psi \rangle \Big)$$

$$= \frac{T-1}{T+1} + \frac{1}{T+1} 2\cos^2\frac{\theta}{2}$$

as optimal $|\psi\rangle$ is midway between projectors $C^\dagger(|1\rangle\langle 1|_1 \otimes \mathbb{1})C$

and $\mathbb{1}_{2^m} \otimes |0^m\rangle\langle 0^m|$ which are θ apart.

Recall $\sqrt{\epsilon} = \cos\theta = 2\cos^2\frac{\theta}{2} - 1 \Rightarrow$

$\Rightarrow 2\cos^2\frac{\theta}{2} = 1-\sqrt{\epsilon} \Rightarrow$

$$\cos^2\gamma = \frac{T-1}{T+1} + \frac{1}{T+1}(1+\sqrt{\epsilon}) = 1 - \frac{(1-\sqrt{\epsilon})}{T+1}.$$

$$\sin^2\frac{\gamma}{2} = \frac{1-\cos^2\gamma}{2} = \frac{1-\sqrt{\epsilon}}{2(T+1)}.$$

Therefore, $\lambda_{min}(H) \geq \frac{2c}{T^2}\left(\frac{1-\sqrt{\epsilon}}{T+1}\right) = \Omega\left(\frac{1}{T^3}\right)$

for small ϵ.

So if yes instance, $\lambda_{min}(H) \leq \epsilon$

if no instance, $\lambda_{min}(H) \geq \frac{1}{T^3}$.

Once we add H_{clock} terms to enforce use of a unary clock instead of a $O(\log n)$ qubit clock, we can ensure every H_{prop} term becomes 5-local instead.

However, the local Hams we see are still "artificial" and "synthetic" and don't look like the real problems we see.

2-local, geometric locality, translational invariance

are all examples of properties we might want out of local Ham.

The hard family identified has none of these. Could relevant families not be QMA-hard?

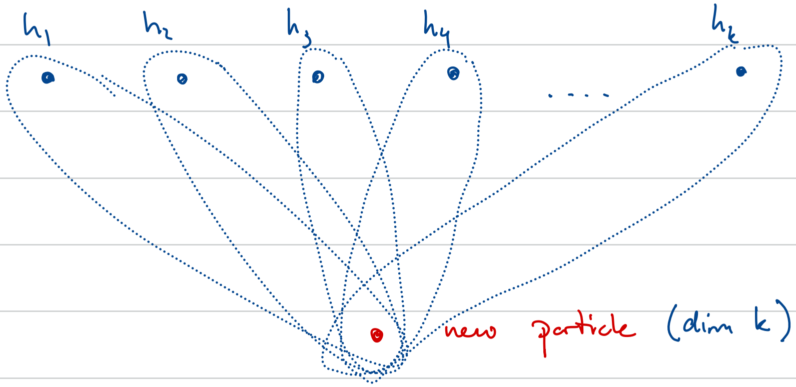
Reduce 5-local Ham to other families.

There is really only one trick to use:

Perturbation Theory.

Simple question: given $h = h_1 \otimes h_2 \otimes \dots \otimes h_k$

construct a 2-local Ham. where low-energy ground space looks like that of h.



Add new particle and terms connecting i^{th} qubit and ancilla.

But how?

We claim that for sufficiently large Δ, this should approach the original Ham.

Let $\mathcal{H} = S \oplus S^{\perp}$

<u>Lemma</u> Let $\tilde{H} = \Delta \Pi_{S^{\perp}} + V$, where $\Pi_{S^{\perp}}$ is a projector onto subspace $S^{\perp}$ and $-\mathbb{1} \leq V \leq \mathbb{1}$, Then

$$\lambda(V|_S) - \frac{1}{\Delta - 2} \leq \lambda(\tilde{H}) \leq \lambda(V|_S).$$

<u>Proof</u>. Write $|v\rangle \in \mathcal{H}$ as $\alpha_1 \underset{\underset{S}{\cap}}{|v_1\rangle} + \alpha_2 \underset{\underset{S^{\perp}}{\cap}}{|v_2\rangle}$ with unit vectors. $\alpha_1, \alpha_2 \geq 0$

Then
$$\begin{aligned}
\langle v|\tilde{H}|v\rangle &\geq \langle v|V|v\rangle + \Delta\alpha_2^2 \\
&= (1-\alpha_2^2)\langle v_1|V|v_1\rangle + 2\alpha_1\alpha_2 \operatorname{Re}\left(\langle v_1|V|v_2\rangle\right) \\
&\quad + \alpha_2^2\langle v_2|V|v_2\rangle + \Delta\alpha_2^2 \\
&\geq \langle v_1|V|v_1\rangle - \alpha_2^2 - 2\alpha_2 - \alpha_2^2 + \Delta\alpha_2^2
\end{aligned}$$

$$= \langle v_1 | V | v_1 \rangle + (\Delta - 2)\alpha_2^2 - 2\alpha_2$$

$$\gtrsim \lambda(V|_S) + \frac{1}{\Delta - 2}$$

$\lambda(\tilde{H}) \leq \lambda(V|_S)$ by def of min eigenvalue.

Cor If $\|V\| \leq K$, then error is $\frac{K^2}{\Delta - 2K}$

This analysis, known as the projection lemma, is useful for characterizing the ground-energy of Hamiltonian perturbations.

However, to conclude the analysis that we want, we will need to argue that the full spectrum of $V|_S$ is captured by $\tilde{H}$.

Want to show that low-energy spectrum of $\tilde{H}$ is well-approximated by spectrum of $V|_S$.

Assume $\|V\| \ll \Delta/2$ (perturbation is small).

$\tilde{H} = \Delta \Pi_{S^\perp} + V$. Write $V = \begin{pmatrix} A & B^\dagger \\ B & C \end{pmatrix}$

where $A = \Pi_S V \Pi_S$ so

$$\tilde{H} = \begin{pmatrix} A & B^\dagger \\ B & \Delta \mathbb{1} + C \end{pmatrix}$$

Consider a state $|\psi\rangle = |x\rangle + |y\rangle \in S \oplus S^\perp$ s.t. $\lambda = \langle \psi | \tilde{H} | \psi \rangle < \Delta/2$. Then

$$\tilde{H}|\psi\rangle = \begin{pmatrix} A & B^\dagger \\ B & \Delta \mathbb{1} + C \end{pmatrix} \begin{pmatrix} |x\rangle \\ |y\rangle \end{pmatrix} = \begin{pmatrix} A|x\rangle + B^\dagger |y\rangle \\ B|x\rangle + \Delta |y\rangle + C|y\rangle \end{pmatrix}$$

so
$$A|x\rangle + B^\dagger|y\rangle = \lambda |x\rangle$$
$$B|x\rangle + (\Delta \mathbb{1} + C)|y\rangle = \lambda |y\rangle$$

$$\Rightarrow (\Delta \mathbb{1} + C - \lambda \mathbb{1})|y\rangle = -B|x\rangle$$

Since λ is small, we have invertability, so

$$|y\rangle = -\left((\Delta-\lambda)\mathbb{1} + C\right)^{-1} B\,|x\rangle \quad (*)$$ and then

$$A|x\rangle - B^{+}\left((\Delta-\lambda)\mathbb{1} + C\right)^{-1} B\,|x\rangle = \lambda|x\rangle$$

$H_{eff}(\lambda) := A - B^{+}\left((\Delta-\lambda)\mathbb{1} + C\right)^{-1} B$. Then

$|x\rangle + |y\rangle$ is a λ-eigenvalue of $\tilde{H}$ if

$H_{eff}(\lambda)\,|x\rangle = \lambda|x\rangle$ and $|y\rangle$ relates to $|x\rangle$ by $(*)$.

Observe that by construction, $\||y\rangle\| \sim O\left(\frac{\|V\|}{\Delta}\right)\||x\rangle\|$

so eigenvectors of $\tilde{H}$ mostly live in the Hilbert space S.

However, the don't fully live in this space. The key is that the perturbation has some tail effects.

Solving $H_{eff}(\lambda)$:

$$\left((\Delta-\lambda)\mathbb{1}+C\right)^{-1}$$

$$= \frac{1}{\Delta}\mathbb{1} - \frac{(C-\lambda)}{\Delta^2} + \frac{(C-\lambda)^2}{\Delta^3} - \dots = \sum_k (-1)^k \frac{(C-\lambda)^k}{\Delta^{k+1}}$$

So $H_{eff}(\lambda) =$

$$\underbrace{A - \frac{1}{\Delta}B^\dagger B + \frac{1}{\Delta^2}B^\dagger(C-\lambda\mathbb{1})B}_{H_{eff}^{(2)}(\lambda)} + O\left(\frac{K^2}{\Delta^3}\right).$$

Cor Characterization of the eigenspaces of $\tilde{H}$ of energy $\ll \frac{\Delta}{2}$.

If $|\psi\rangle$ is an eigenvector of $\tilde{H}$ with eigenvalue $\ll \frac{\Delta}{2}$, then

$\Pi_S|\psi\rangle$ is an eigenvector of $H_{eff}(\lambda)$ which is approx. by $H_{eff}^{(k)}$.

Application: k-local to $\frac{k}{2}+1$-local Ham.

Consider a Ham term $h_2h_1 + h_1^\dagger h_2^\dagger$ where both h_i are $\frac{k}{2}$-local.

Add an ancilla system of 1 qubit.

Let $S = \mathrm{span}\{|0\rangle\}$ and $S^\perp = \mathrm{span}\{1\}$.

$$V = -|1\rangle\langle 0| \otimes h_1 - |0\rangle\langle 1| \otimes h_1^\dagger$$
$$- |1\rangle\langle 0| \otimes h_2^\dagger - |0\rangle\langle 1| \otimes h_2 .$$

Then, $V = \begin{pmatrix} 0 & -h_1 - h_2^\dagger \\ -h_1^\dagger - h_2 & 0 \end{pmatrix}$. Consider $\tilde{H} = \Delta |1\rangle\langle 1| + V$.

$$H_{\text{eff}}(\lambda) = -\left(\frac{1}{\Delta} + \frac{\lambda}{\Delta^2}\right) |0\rangle\langle 0| \otimes (h_1 + h_2^\dagger)^\dagger (h_1 + h_2^\dagger) + O\left(\frac{1}{\Delta^3}\right)$$

$$= -\left(\frac{1}{\Delta} + \frac{\lambda}{\Delta^2}\right) |0\rangle\langle 0| \otimes \left(h_1^\dagger h_1 + h_2 h_2^\dagger + h_2 h_1 + h_1^\dagger h_2^\dagger\right) + O\left(\frac{1}{\Delta^3}\right).$$

Observe that this term captures the k local term $h_2 h_1 + h_1^\dagger h_2^\dagger$ up to some $k/2$ local corrections.

In general we can add a qudit of deg d and construct a similar clock transformation.

However the perturbation will require a stronger choice of Δ and the Taylor expansion will require analyzing d^{th} order terms.

However, we can use this to locality reduce d local terms.

The Brandão-Harrow limitation

Apr 22

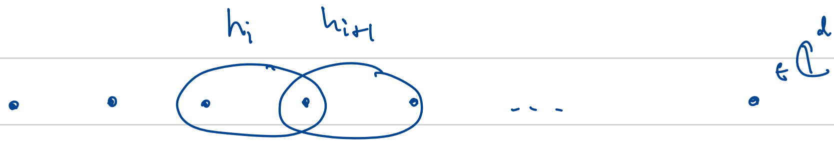
So far we have seen upper bounds on QMA as well as a technique (perturbation) theory for proving hardness of local Ham systems.

Using this technique, researchers have proven that many families of local Ham are QMA-complete.

1. 1D Hamiltonian with qudits of dim 12
2. Hamiltonians where $h_i = X \otimes X$ or $Z \otimes Z$ (with weights) (any geometry)
3. 2D Hamiltonians with qudits of dim 6.

Next, we will look at solving local Hamiltonian systems in specific settings towards our global goal of understanding when Hamiltonian systems are easy and when they are hard.

We start by investigating the complexity of 1D local Hamiltonians.



wlog by lumping together qubits into qudits we can assume interactions are 2-local with qudits of dimension d.

Our goal will be to show an upper bound on the efficiently verifiable description complexity of the groundstate of such Hamiltonian.

Our alg must depend on an additional parameter other than d and n since otherwise we can solve the QMA-complete problem.

Roadmap:

① Understand tensor network / MPS descriptions of quantum states

② Describe area laws and why area laws imply tractable ground state descriptions

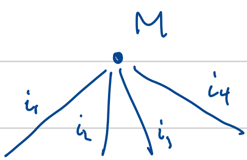
③ Sketch construction of 1D algorithm

Ex. Prove that for any classical CSP (k-local) in 1D, that $\exists$ a P algorithm for calculating the minimum energy solution.

Observe that our algorithm will have to incorporate this somehow.

Tensor networks

Tensor is a rep. of a linear operator.



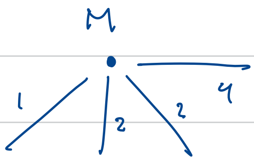
Think of a tensor as a vertex with protruding edges (named M)

- a set of labels for edges
$i_j \in [d]$

A tensor is a fn. $M : [d] \times [d] \times [d] \times [d] \longrightarrow \mathbb{C}$

Another way of thinking about M is as an array (4-dim) of numbers.

For ex. $\qquad d=10$



$M_{1,2,2,4} = 2.7$

Interpretations:

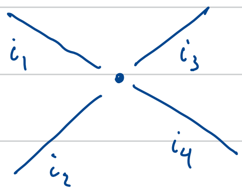
Let $\mathcal{H} = \mathbb{C}^d$ with a basis $|0\rangle, |1\rangle, \dots |d-1\rangle$.

<u>Vector interpretation</u>

Interpret M as a vector in $\mathcal{H}^{\otimes 4}$

$$|M\rangle := \sum_{i_1 \dots i_4} M_{i_1 \dots i_4} |i_1\rangle |i_2\rangle |i_3\rangle |i_4\rangle$$

<u>linear map</u> $\qquad$ Divide the edges into 2 groups.



Think of M as a linear map
$\mathcal{H} \otimes \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$.

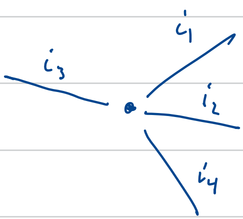
Since M will be a linear map, suffices to describe how to think of M acts on basis vectors.

$\{ |i_1\rangle \otimes |i_2\rangle \mid i_1, i_2 \in [d] \}$ is a basis for $\mathcal{H} \otimes \mathcal{H}$.

Then $$M(|i_1\rangle |i_2\rangle) = \sum_{i_3 i_4} M_{i_1 i_2 i_3 i_4} |i_3\rangle |i_4\rangle.$$

Another division:

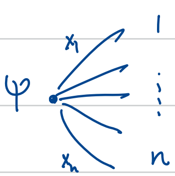
$$\mathcal{H} \longrightarrow \mathcal{H}^{\otimes 3}.$$



$$M(|i_3\rangle) = \sum_{i_1 i_2 i_4} M_{i_1 i_2 i_3 i_4} |i_1\rangle |i_2\rangle |i_4\rangle$$

$$M = \sum_{i_1 i_2 i_3 i_4} M_{i_1 i_2 i_3 i_4} |i_1\rangle |i_2\rangle |i_4\rangle \langle i_3|$$ (as a matrix of the linear transform)

__Quantum states__

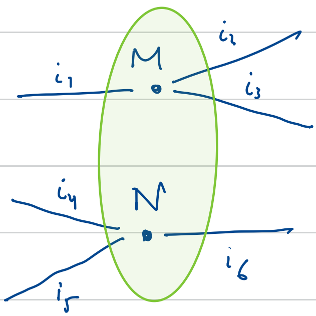


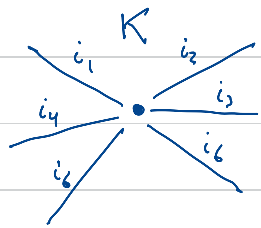
$$|\psi\rangle = \sum_{x_1 \dots x_n} \psi_{x_1 \dots x_n} |x_1\rangle |x_2\rangle \dots |x_n\rangle$$

is the tensor network representation of

a quantum state n - qudits.

Join / Tensor product

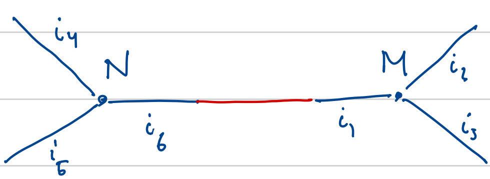


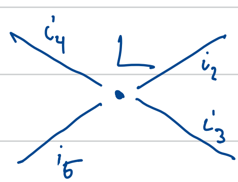


$$K\left(|i_1\rangle|i_4\rangle|i_6\rangle\right) = \sum_{i_2 i_3 i_6} M_{i_1 i_2 i_3} N_{i_4 i_5 i_6} |i_2\rangle|i_3\rangle|i_6\rangle$$

What is K? $\qquad K = M \otimes N$

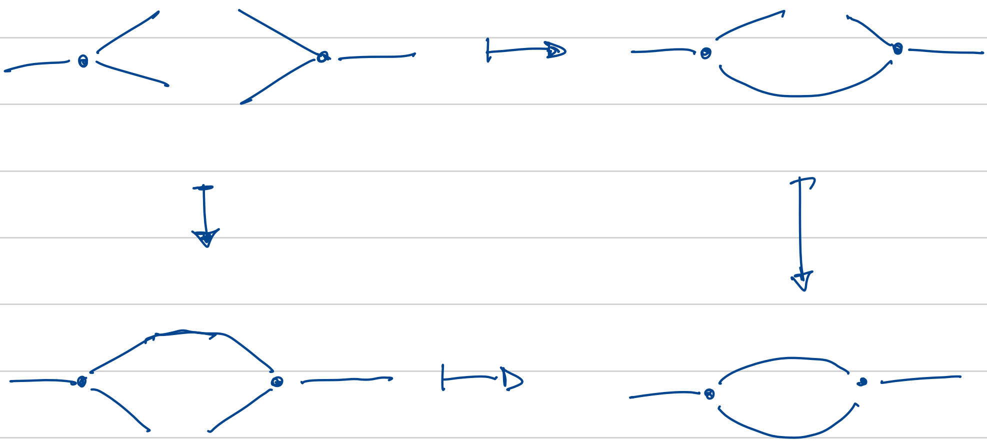
Contraction

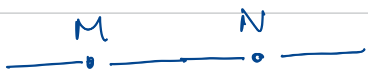




defined by $L(|i_4\rangle|i_6\rangle) = \sum_{i_6 = i_1} N_{i_4 i_5 i_6} M_{i_1 i_2 i_3} \; |i_2\rangle|i_3\rangle$

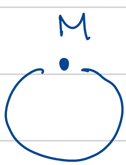
<u>Ex</u>. Contraction orders commute



<u>Questions</u> What is 

?

Answer: matrix multiplication.

What is 

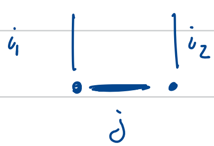
Answer $\mathrm{tr}(M)$

A bit more generality

Allow each edge to have a different number of valid labelings

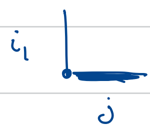
Allows us to study Hilbert spaces of the form $\mathbb{C}^{d_1} \otimes \dots \otimes \mathbb{C}^{d_n}$.

Ex.



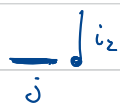
$i_1, i_2 \in [2]$

$j \in [5]$



$M_{i_1 j} = (i_1 + j)$

$|\psi\rangle$



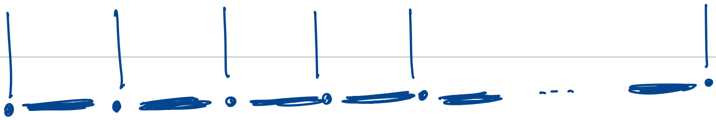
$M_{i_2 j} = (i_2 \cdot j)$

$$\psi_{i_1 i_2} = \sum_j (i_1 + j)(i_2 j) = \sum_{j=1}^{5} i_1 i_2 j + i_2 j^2$$

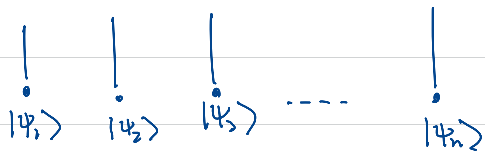
Matrix Product States

A state $|\psi\rangle \in (\mathbb{C}^d)^{\otimes n}$ is said to have a D-rank MPS

is $|\psi\rangle$ can be represented as a tensor network with middle bonds of dimension D.



Ex. Any state $|\psi\rangle = |\psi_1\rangle \otimes \dots \otimes |\psi_n\rangle$ is a 1-rank MPS



Ex. Show $\dfrac{|0\dots0\rangle + |1\dots1\rangle}{\sqrt{2}}$ is a 2-rank MPS.

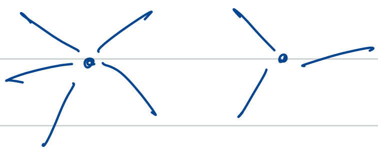
Ex. Show that $\dfrac{|0\dots01\rangle + |0\dots10\rangle + |0\dots100\rangle + \dots + |10\dots0\rangle}{\sqrt{n}}$

is a 2-rank MPS.

Ex. Show that any state is a d^n-rank MPS.

What is the comp complexity of join / contraction operations?

join:

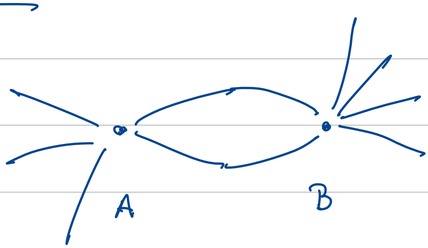


new tensor has 8 legs

d^8 time to write down.

in general, product of dimensions of legs.

contraction



think of $A \in \mathcal{H}^3 \to \mathcal{H}^2$

$B \in \mathcal{H}^2 \to \mathcal{H}^4$

so $B \cdot A$ as matrix multiplication in time

$$d^3 (d^2) d^4.$$

In general, $A \in \mathbb{C}^{d_1} \to \mathbb{C}^{d_2}$

$B \in \mathbb{C}^{d_2} \to \mathbb{C}^{d_3}$

then $B \cdot A \in \mathbb{C}^{d_1} \to \mathbb{C}^{d_3}$ computable in time

$$O(d_1 d_2 d_3)$$

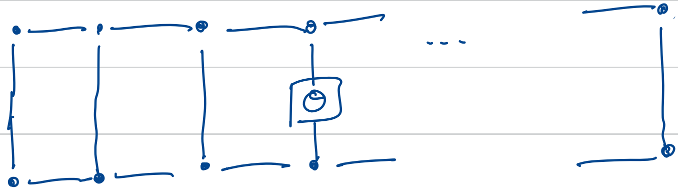
Aside: Fast Mat Mult gives $O\left((d_1 d_2 d_3)^{\omega/3}\right)$.

Ex. given $|\psi\rangle$ as a MPS of bond dimension D, compute

$\langle\psi|\psi\rangle$ in time $O(n d D^3)$.

Next show how to compute $\langle\psi|\mathcal{O}|\psi\rangle$ for any $O(1)$-local matrix $\mathcal{O}$.

Alg "Bubble over the MPS."

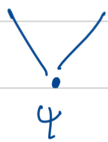


Evaluate the tensor from left to right so that at any time at most $O(1)$ legs of dim D and of dim d.

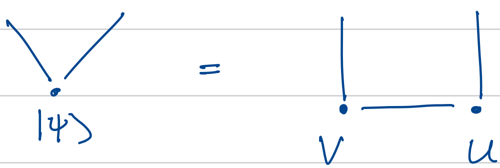
n such multiplications required.

Schmidt rank

Consider $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ s.t. d_A and d_B dim, resp.

Any state $|\psi\rangle$ 

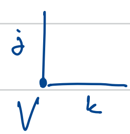
has a unique schmidt decomposition



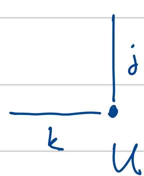
$$|\psi\rangle = \sum_{k=1}^{\min(d_A, d_B)} c_k \underbrace{|v_k\rangle \otimes |u_k\rangle}$$

$c_k \geq 0$.

unit vec.



$$V_{jk} = \langle j | v_k \rangle$$



$$U_{jk} = \langle j | u_k \rangle$$

What is Schmidt decomposition? It is the vectorization of the singular value decomposition.

Def. Schmidt rank $(|\psi\rangle)$ = number of non-zero c_j.

$$|EPR\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |i\rangle|i\rangle$$ has max Schmidt rank d.

$|\psi\rangle \otimes |\phi\rangle$ has Schmidt rk 1.

Observations
$$\operatorname{tr}_A(|\psi\rangle\langle\psi|) = \sum_k c_k^2 |u_k\rangle\langle u_k|$$

$$\operatorname{tr}_B(|\psi\rangle\langle\psi|) = \sum_k c_k^2 |v_k\rangle\langle v_k|$$

Entanglement entropy $S(\psi_A) = S(\psi_B) = -\sum_k c_k \log(c_k)$

Returning to solving 1D Hamiltonians.

Theorem (Landau, Vazirani, Vidick 2013)

Let H be a 2-local Ham. on a line with qudits. For simplicity,

- $H = \sum_i h_i$ h_i acts on $i, i+1$. h_i is a projector.

- $\lambda_{min}(H) = 0$ with unique groundstate $|\Gamma\rangle$. $\lambda_1(H) = \epsilon > 0$.

We construct a $n^{c(d,\epsilon)} \cdot poly\left(\frac{n}{\eta}\right)$ time alg for outputting a state $|\phi\rangle$ represented by MPS of bond dim $n^{c'(d,\epsilon)}$ s.t.

$|\langle\phi|\Gamma\rangle| \geq 1 - \eta$. $\qquad c(d,\epsilon) = 2^{O(\log^3 d/\epsilon)}$.

This says two things:

① We can construct a classical description of the groundstate (not just the groundenergy).

② Since such states can be described by MPS of poly(n) bond dim, they aren't very entangled.

Indeed this had been known since a result of Hastings called the area law (for 1D):

Area Law: Let H be a local Ham with interaction graph $G = ([n], E)$. Let $|\Gamma\rangle$ be g-state. For any set $S \subseteq [n]$,

$$S(\psi_S) = O_{\epsilon, d}\left(|E(S, \bar{S})|\right).$$

where ϵ = spectral gap of the Ham system.

<u>In 1D</u> Suffices to consider $S = \{1, \dots, i\}$ in which case

$$S(\psi_S) \leq O\left(\frac{\log^3 d}{\epsilon}\right) \quad \left[\text{originally} \leq \exp\left(\frac{\log d}{\epsilon}\right) \text{ by Hastings}\right].$$

<u>Step 1</u> Linear algebra over a more tractable subspace.

Brute-force, solving H is linear alg. over $(\mathbb{C}^d)^{\otimes n}$.

<u>Lemma</u> Let $|\phi_1\rangle \dots |\phi_J\rangle$ be a set of MPS of bond

dimension D. Let $S = \text{span}\{|\phi_1\rangle \ldots |\phi_J\rangle\}$. We can calculate the $|\phi\rangle \in S$ minimizing $\langle\phi|H|\phi\rangle$ in time $O(J^3 + J^2 n^2 d D^3)$ and moreover $|\phi\rangle$ is representable as a MPS of bond dim DJ.

<u>Pf of lemma</u>

Compute gram matrix G given by $\langle j|G|i\rangle = \langle\phi_j|\phi_i\rangle$.

Compute energy matrix E given by $\langle j|E|i\rangle = \langle\phi_j|H|\phi_i\rangle$.

Both computable in time $O(J^2 n^2 d D^3)$ using MPS representation & inner product algorithm.

Any vec $|\phi\rangle \in S$ is representable by $\sum \alpha_i |\phi_i\rangle$. Because $|\phi_i\rangle$ are not necessarily orthogonal,

$$\| |\phi\rangle \|^2 = \langle\phi|\phi\rangle = \sum_{i,j} \langle\phi_j| \alpha_j^* \alpha_i |\phi_i\rangle$$

$$= \sum_{i,j} \alpha_j^* \alpha_i \langle j|G|i\rangle$$

$$= \langle \alpha | G | \alpha \rangle$$

where $|\alpha\rangle = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_J \end{pmatrix}$.

Similarly, $\langle \phi | H | \phi \rangle = \langle \alpha | E | \alpha \rangle$

so $$\min_{\substack{|\phi\rangle \in S \\ \| |\phi\rangle \| = 1}} \langle \phi | H | \phi \rangle = \min_{|\alpha\rangle \neq 0} \frac{\langle \alpha | E | \alpha \rangle}{\langle \alpha | G | \alpha \rangle}.$$

Prune $|\phi_i\rangle$ so that linearly indep. Then $G > 0$ so let $G = B^\dagger B$ with B invertible.

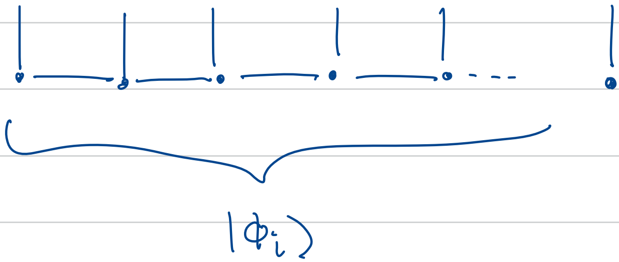
$$\min_{|\alpha\rangle \neq 0} \frac{\langle \alpha | E | \alpha \rangle}{\langle \alpha | B^\dagger B | \alpha \rangle} = \min_{|\beta\rangle \neq 0} \frac{\langle \beta | (B^{-1})^\dagger E (B^{-1}) | \beta \rangle}{\langle \beta | \beta \rangle}$$

$$= \lambda_{\min}\left((B^{-1})^\dagger E B^{-1} \right).$$

All solvable in time $O(J^3)$.

Lastly, given $|\alpha\rangle$, we know

$$|\phi\rangle = \sum_{i=1}^{J} \alpha_i$$



Create a new tensor of bond dim DJ that incorporates the sum along the comb of the tensor.

New goal: Let $V \subseteq (\mathbb{C}^d)^{\otimes n}$ be a δ- viable space if

$\exists\, |\phi\rangle \in V$ s.t. $|\langle \phi | \Gamma \rangle| \geq 1 - \delta$.

We want a viable space defined as the span of efficient MPS with the dimension of the viable space not being too large.

Let $R = \{a, a+1, \dots, b\} \subseteq [n]$ be a continous interval of points. We say that $V \subseteq (\mathbb{C}^d)^{\otimes |R|}$

is δ-viable for R if $\exists\ |\phi\rangle \in (\mathbb{C}^d)^{\otimes (a-1)} \otimes R \otimes (\mathbb{C}^d)^{\otimes (n-b)}$

s.t. $|\langle \phi | \Gamma \rangle| \geq 1 - \delta$.

We want to find a δ-viable space for $R = [n]$ spanned by some MPSs.

<u>Fact</u>

For $R = \{a\}$, $V = \mathbb{C}^d$ is a 0-viable set with basis $|0\rangle, |1\rangle, \dots |d-1\rangle$ (trivially an MPS).

<u>Fact</u>

Let R and R' be adjacent regions with δ-viable space V, V' spanned by m & m' MPS of bond dim D.

Then, for $R \cup R'$, $V \otimes V'$ is 2δ-viable and spanned by $m \cdot m'$ MPS of bond dim D.

If we were to just grow viable sets this way, we would end up with $J = d^n$. Need a way to join two viable spaces together and keep a handle on their dimension

Additionally, if the sets were only δ-viable the tensor product will not be δ-viable so we also need to improve the accuracy as we join the subspaces.

2013: Landau, Vazirani, and Vidick

Build the viable set $\{1, \dots i\} \mapsto \{1, \dots, i+1\}$.

Required a lemma: the area law about the entanglement entropy of $|\Gamma\rangle$ across cut $\{1, \dots i\}$ vs $\{i+1, \dots, n\}$.

2016: Arad + LVV

Build the viable set using a binary tree.

Proved the area law about the entanglement (on the go) but is a bit more challenging to prove.

We will follow the first approach roughly as understanding the area law

Roughly, the induction of the viable set growth is 4 steps:

Initial: start with viable space S_{i-1}.

$s := \mathrm{rk}(S_{i-1}) = p(n) \cdot p_1(n)$

B = bond dim of every basis MPS $= p(n) \cdot p_2(n)$

δ = level of viability $= c_\epsilon / n$

Extension: tensor each basis MPS with $|j\rangle$

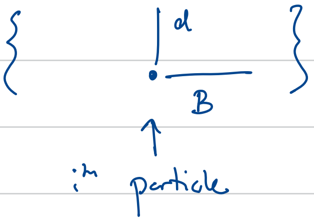
$s = d\, p(n)\, p_1(n)$

$B = p(n) \cdot p_2(n)$

$\delta = c_\epsilon / n$.

Size trimming:

Find a $\frac{c_\epsilon}{n}$ - approximation net over all tensors of the form



. As $B = \mathrm{poly}(n)$, such a net has only $\mathrm{poly}(n)$ size.

From area law (to be seen), we find that we can generate a

$s = p_1(n)$

$B = p'(n)\, p_2(n)$

$\delta = 1/12$ viable set.

Details to be seen shortly.

Bond trimming:

Remove all light Schmidt coefficients to decrease $B = p_2(n)$

$s = p_1(n), \; B = p_2(n), \; \delta = 1/2.$

Error reduction:

Apply AGSP to again increase Bond dimension and rank but fix error to $c\epsilon/n$.

This is only the high level view. We now will go into the details starting with the area law which will be a key step in the analysis of the above procedure.

Area law: Want to show that for $A = \{1, \dots i\}$, $B = \{i+1, \dots, n\}$

$$S(\Gamma_A) \leq O\left(\frac{\log^3 d}{\epsilon}\right).$$

Start by writing $|\Gamma\rangle = \sum_j c_j |\psi_A^{(j)}\rangle |\psi_B^{(j)}\rangle$

in its Schmidt

$|\Gamma\rangle = \sum_j c_j |\psi_j^{(A)}\rangle |\psi_j^{(B)}\rangle$ for

A | B

$\cdot \; \cdot \; \cdot \; \cdot \; \cdot \;|\; \cdot \; \cdot \; \cdot \; \cdot \; \cdot$

$1 \quad i \;|\; i+1 \quad n$

Not. $|\Gamma^{\perp}\rangle$ will be any state s.t. $\langle \Gamma^{\perp} | \Gamma \rangle = 0$.

Starting pt. Observe that if we want to prove that entanglement entropy is bounded by H, it follows that $\exists$ a product state which has at least $\geq 2^{-H}$ of the mass. ~~This is because~~

Pf. $p_1 \geq \dots p_{n+1} \geq 0$. ~~This is maximized~~ Entropy is maximized when

$\cancel{-p_1 \log(p_1)}$ $\quad p_1, \; p_2 = \dots = p_{n+1} = \frac{1-p_1}{n}$. Then

$$H = \cancel{p_1 \log\left(\frac{1}{p_1}\right) + (1-p_1)\log\left(\frac{n}{1-p_1}\right)} - \sum p_i \log(p_i) \geq -\sum p_i \log(p_i)$$

$$= \cancel{p_1 \log\left(\frac{1}{p_1}\right) + (1-p_1)\left(\log n - \log(1-p_1)\right)}$$

$$\geq \log\left(\frac{1}{p_1}\right). \quad \Longrightarrow \quad \frac{1}{p_1} \leq 2^{H} \quad \Longrightarrow \quad p_1 \geq 2^{-H}.$$

converse not necessarily true.

Gives us an idea for proving area law like entanglement.

Start with best product state approx. i.e, $|\psi_1^{(A)}\rangle \otimes |\psi_1^{(B)}\rangle$

and then evolve it (not too much) till it resembles $|\Gamma\rangle$.

We will do this by constructing an AGSP

approx. g-state proj.

Def. An ~~Hamn.~~ op. $K : (\mathbb{C}^d)^{\otimes n} \to (\mathbb{C}^d)^{\otimes n}$ is a (D, Δ)-agsp. if

1. $K|\Gamma\rangle = |\Gamma\rangle$
2. $\| K |\Gamma^\perp\rangle \|^2 \le \Delta$ for any $|\Gamma^\perp\rangle$ orthogonal and norm 1.
3. $K = \sum_{j=1}^{D} K_A^{(j)} \otimes K_B^{(j)}$, small Schmidt rk.

Before we construct a AGSP, lets see how it implies the entropy bounds we want.

For $D\Delta \le 1/2$,

Claim 1 If $\exists$ a (D, Δ)-AGSP, then $\exists$ product state $|\phi_A\rangle \otimes |\phi_B\rangle$ s.t.

$$\langle \Gamma | \left(|\phi_A\rangle \otimes |\phi_B\rangle \right) \ge \frac{1}{\sqrt{2D}}.$$

Pf. consider $|\phi_A\rangle|\phi_B\rangle$ of best value $=: a$. Then write

$$|\phi_A\rangle|\phi_B\rangle = a|\Gamma\rangle + \sqrt{1-a^2}\,|\Gamma^\perp\rangle \qquad \text{then}$$

$$K|\phi_A\rangle|\phi_B\rangle = a|\Gamma\rangle + b|\Gamma^\perp\rangle \quad \text{for } b \le \sqrt{\Delta}\sqrt{1-a^2}.$$ Normalizing

gives $$K|\phi_A\rangle|\phi_B\rangle \mapsto \frac{a}{\sqrt{a^2+b^2}}|\Gamma\rangle + \frac{b}{\sqrt{a^2+b^2}}|\Gamma^\perp\rangle$$

so this overlap is at least $$\frac{a}{\sqrt{a^2+\Delta}}$$

But this new state is not product and has Schmidt rk $\le D$.

As $$K|\phi_A\rangle|\phi_B\rangle = \sum_{j}^{D} \underbrace{K_A^{(j)}|\phi_A\rangle \otimes K_B^{(j)}|\phi_B\rangle}_{\text{these may not be orthogonal but prove rk} \le D}$$

Consider the Schmidt vector of best overlap; Since only D s.t. vectors

$\exists$ $|\phi_A'\rangle|\phi_B'\rangle$ s.t. overlap with $|\Gamma\rangle$ is $\geq \frac{1}{\sqrt{D}} \frac{a}{\sqrt{a^2+\Delta}}$.

As we assumed best $|\phi_A\rangle|\phi_B\rangle$, then

$$\frac{1}{\sqrt{D}} \frac{a}{\sqrt{a^2+\Delta}} \leq a \implies \frac{a^2}{D(a^2+\Delta)} \leq a^2$$

$$\implies 1 \leq Da^2 + \Delta D \leq Da^2 + \frac{1}{2}$$

$$\frac{1}{2} \leq Da^2 \implies a \geq \frac{1}{\sqrt{2D}}.$$

<u>Claim 2</u> Given (D,Δ)-AGSP with $D\Delta \leq \frac{1}{2}$,

the entanglement entropy in $|\Gamma\rangle$ across cut is $O(\log D)$.

~~Event~~ We will use this when we construct a AGSP with $D = \exp\left(\frac{\log^3 d}{\epsilon}\right)$.

Start from $|\phi_0\rangle = |\phi_A\rangle \otimes |\phi_B\rangle$

Define $|\phi_m\rangle = \dfrac{K^m|\phi_0\rangle}{\| K^m|\phi_0\rangle\|}$.

Fact 1. Schmidt rank of $|\phi_m\rangle \le D^m$.

Fact 2. Overlap $\langle \phi_m | \Gamma\rangle \ge \sqrt{\dfrac{2^{m-1}}{2^{m-1}+1}}$

Pf induction if $|\phi_m\rangle = a_m|\Gamma\rangle + \sqrt{1-a_m^2}\,|\Gamma^\perp\rangle$ then

$$K|\phi_m\rangle = a_m|\Gamma\rangle + c|\Gamma^\perp\rangle \quad \text{where} \quad c \le \sqrt{\Delta}\cdot\sqrt{1-a_m^2}$$

Then normalizing gives $|\phi_{m+1}\rangle$ which has overlap

$$a_{m+1} = \frac{a_m}{\sqrt{a_m^2 + c^2}} \ge \frac{a_m}{\sqrt{a_m^2 + (1-a_m^2)\Delta}}.$$

Eckert-Young thm If $|\Gamma\rangle$ has Schmidt decomp

$$\sum_j c_j |\psi_j^{(A)}\rangle|\psi_j^{(B)}\rangle$$

for $c_1 \ge c_2 \ge \dots$ then the best rank D approx is

$$\sum_{j=1}^{D} c_j |\psi_j^{(A)}\rangle|\psi_j^{(B)}\rangle, \text{ renormalized}.$$

Together these imply

$$\sum_{j=1}^{D^m} c_j^2 \geq \frac{2^{m-1}}{2^{m-1}+1} \qquad \forall \text{ values of } m.$$

By itself, no one equation (m) is enough. But together...

$$\underbrace{\overset{B_1}{\left[c_1^2 \cdots\cdots c_D^2\right]}}_{\geq \frac{1}{2}} \; \underbrace{\overset{B_2}{\left[c_{D+1}^2 \cdots\cdots c_{D^2}^2\right]}}_{\leq 2^{-1}} \; \underbrace{\overset{B_3}{\left[c_{D^2+1}^2 \cdots\cdots c_{D^3}^2\right]}}_{\leq 2^{-2}} \cdots \underbrace{\overset{B_m}{\left[c_{D^{m-1}+1}^2 \cdots c_{D^m}\right]}}_{2^{-m+1}}$$

Entropy per block: $S(\text{Block } m)$ maximized when 2^{-m+1} prob. mass spread out over all $D^m - D^{m-1} \leq D^m$ events. So

$$S(\text{Block } m) \leq 2^{-m+1} \cdot \left(-\log\left(\frac{2^{-m+1}}{D^m}\right)\right)$$

$$= 2^{-m+1}\left(m \log D + (m-1)\right)$$

$$= O\left(\frac{m}{2^{-m+1}} \log D\right).$$

$S(\text{Block } 1) \leq O(\log D)$.

$\Rightarrow \; S(\text{total}) \leq O(\log D)$.

<u>Cor</u> For $B_\delta = \exp\left(O(D \log 1/\delta)\right)$,

$$\sum_{j=1}^{B_\delta} c_j^2 \geq 1-\delta.$$

Recall from the previous lecture

$K : (\mathbb{C}^d)^{\otimes n} \to (\mathbb{C}^d)^{\otimes n}$ is a (Δ, D)-AGSP for groundstate $|\Gamma\rangle$ and partition of the qubits into registers A & B

1. $K|\Gamma\rangle = |\Gamma\rangle$
2. $\forall$ unit $|\Gamma^\perp\rangle \perp$ to $|\Gamma\rangle$, $\| K|\Gamma^\perp\rangle \|^2 \leq \Delta$
3. $K = \sum_{i=1}^{D} K_A^{(i)} \otimes K_B^{(i)}$.

Thm 1

We saw that for $D\Delta \leq 1/2$, the existence of a (Δ, D)-AGSP implies that

1. $S(\Gamma_A) \leq O(\log D)$
2. $\forall \eta > 0$, $\exists$ a state $|\phi\rangle$ s.t. $|\langle \phi | \Gamma \rangle| \geq 1 - \eta$

with Schmidt rk $2^{O(D \log(1/\eta))}$ across this cut.

Cor If we can construct such a AGSP for every cut $1 \ldots i \mid i+1 \ldots n$

then, $\exists$ a MPS of bond dim $2^{O(D \log(1/\eta))}$ forming an

$|\eta\rangle$ approx of the ground-state.

Let's defer AGSP construction till the end.

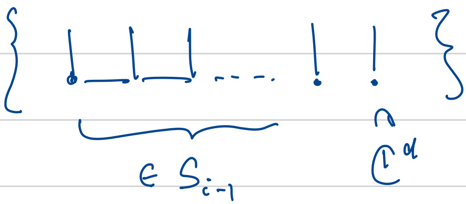
Armed with a (Δ, D)-AGSP for $D\Delta \leq 1/2$ and $D = \exp\left(\frac{\log^3 d}{\epsilon}\right)$

we go back to the 4 step process of constructing the inductive evolution of the viable sets.

size, bond dim, $c_\epsilon = c(d,\epsilon)$

Starting point: $(s_{i-1}, b_{i-1}, \delta_{i-1} = c/n)$ viable set on first $i-1$.

Extend:



$= S_i^{(1)}$

Problem: $S_i^{(1)}$ has too many elements.

$$(s_{i-1} \cdot d, \; b_{i-1}, \; \delta_{i-1} = \epsilon/n)$$

Fix: Let's trim the number of MPS.

$\exists\, |\phi\rangle \in S_i^{(1)} \otimes \mathcal{H}_{[i+1 \ldots n]}$ with $\langle \phi | H | \phi \rangle \leq c_\epsilon$ as $\|H\| \leq n$.

By Thm 1.2, we have that

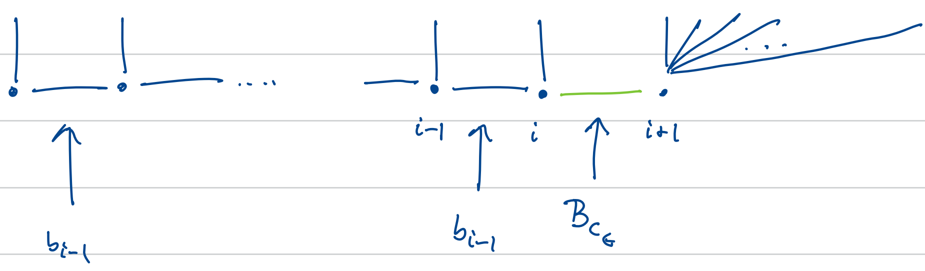
$$|\phi'\rangle \propto \text{trim}_{B_{c_\epsilon}}(|\phi\rangle) \quad \text{has energy} \leq 6\sqrt{c_\epsilon}$$

where $B_{c_\epsilon} = \exp\left(O\left(\frac{1}{\eta}\log^3 d \log 1/c_\epsilon\right)\right)$

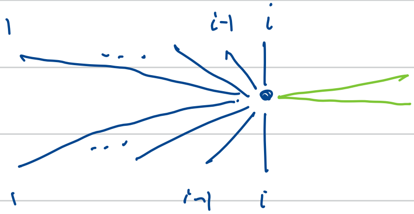
by the exponential drop-off of Schmidt coefficients and that we are only trimming one coefficient.

(Details not proven).

Picture of $|\phi'\rangle$:



Consider a density matrix ρ acting on $S_{i-1} \otimes \mathbb{C}^{B_{C_\epsilon}}$.

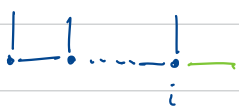


Tracing out registers $1 \ldots i-1$ is a tensor X 

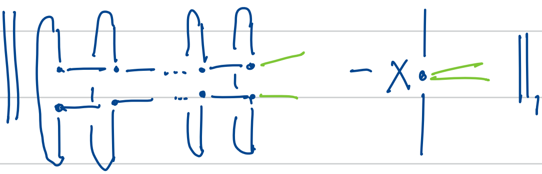
this is the boundary condition

Let's fix a tensor X  and solve

$$\min \quad \langle \phi'' | \sum_{j=1}^{i-1} | \phi'' \rangle$$



$= |\phi''\rangle$

s.t.  $\le c_\epsilon / n$

and $|\phi''\rangle \in S_i$

Solvable in poly time

For a fixed boundary condition X, we can solve the left side up till i to fit that boundary condition.

(This is exactly what the 1D classical dynamic programming algorithm does).

The result is an MPS $|\phi''\rangle$ of bond dim $d s_{i-1} b_{i-1}$.

How do we solve for all X? Classically we enumerate over all boundary conditions.

We solve for an ξ-net over $X \in \mathcal{H}_i^{\otimes 2} \otimes \mathbb{C}_{B_{C_\epsilon}}^{\otimes 2}$.

Thm The net size is $\left(\frac{3}{\xi}\right)^{O(d^2 B_{C_\epsilon}^2)}$

We will pick $\xi = C_\epsilon/n$ so the number of X considered is a constant.

Unfortunately, this will yield some error but now there are fewer MPS in the viable set.

To build $S_i^{(2)} = \bigcup_{X \in \xi\text{-net}} \left[\text{Left Schmidt vectors of } |\phi''(X)\rangle\right]$

Claim $S_i^{(2)}$ is a $\left(\text{size} = B_{C_\epsilon} d \; O(n)^{d^2 B_{C_\epsilon}^2},\right.$

$\left.\text{bond dim} = b_{i-1} s_{i-1} d, \quad \text{error} = 1/12\right)$

viable set.

Next: Bond-trimming

Trim each bond. Let $r(n)$ be a poly s.t. $\exists\, |\phi\rangle$ of Schmidt rank $\le r(n)$ for each cut $i\,|\,i+1$ s.t.

$$|\langle \phi | \Gamma \rangle| \ge 1 - 1/100.$$

Trim each bond of each MPS in $S_i^{(2)}$ to $100 n \cdot r(n)$. This gives a

$\left(O\left(n^{d^2 B_{C_\epsilon}^2}\right), \; O(n \cdot r(n)), \; \text{error} = 1/2 \right)$ viable-set

Error reduction:

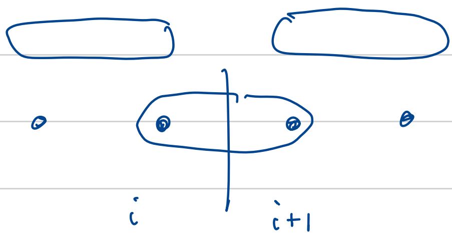
Let $\Pi_o = \prod (1 - h_{2i+1})$ and

$\Pi_e = \prod (1 - h_{2i})$ be odd, even projectors.

Let $K = \Pi_o \Pi_e$. Claim K is an AGSP for any cut

Easy to check $K|\Gamma\rangle = |\Gamma\rangle$.

Second, check $D = d^4$ as only one term crosses + light cone.



Claim: $\Delta = (1 + \epsilon/4)^{-1}$

This follows from what is called the Detectability Lemma and I'm going to defer the proof.

Let's use K to perform error-reduction.

For each $|\phi''\rangle \in S_i^{(3)}$, construct $\dfrac{K^m |\phi''\rangle}{\| K^m |\phi''\rangle \|}$ in $S_i^{(4)}$.

for m s.t. $\Delta^m \le \dfrac{c_\epsilon}{100n} \Rightarrow m \ge O\left(\frac{1}{\epsilon} \log n\right)$

If $|\lambda\rangle \in S_i^{(3)} \otimes \mathcal{H}_{[i+1 \dots n]}$ is a proof of $1/2$ viability,

then $\dfrac{K^m |\lambda\rangle}{\| K^m |\lambda\rangle \|} \in S_i^{(4)} \otimes \mathcal{H}_{[i+1 \dots n]}$

and the $|\Gamma\rangle$ orthogonal component will be dropped to $\leq c_\epsilon/100n$. So,

$$S_i^{(u)} \text{ is a } \left(O(n)^{d^2 B_{c_\epsilon}^2},\ n^{O\left(\frac{\log d}{\epsilon}\right)} \cdot r(n),\ c_\epsilon/n \right)$$

viable.

Some parameter cleanup and we are done.

Lastly, the AGSP for proving area law.

The previous K does not satisfy $D \cdot \Delta \leq 1/2$.

For that, we need some additional steps:

Now the hard part, argue $\exists$ a (D, Δ)-AGSP for our particular 1D Ham instance for $D = \exp\left(\frac{\log^3 d}{\epsilon}\right)$ and $D\Delta \le 1/2$.

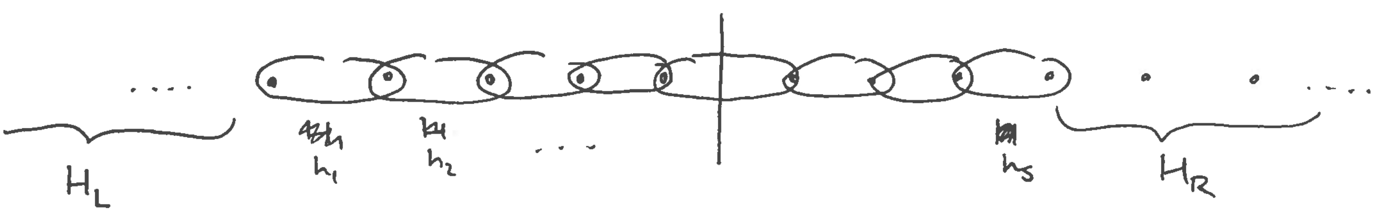
Naive first attempt: $H \ge 0$ with (by assumption) $\lambda_{min}(H) = 0$.

So try $K = \left(\mathbb{1} - \frac{H}{n}\right)$.

Then, $\Delta = 1 - 1/n$ but what is Schmidt rk of K? Ans: 3.

Step 1: Truncate the Hamiltonian

Find new Ham of smaller norm that resembles $|\Gamma\rangle$.



Let $H_L' = $ ~~projection onto the~~ $\text{trunc}_t(H_L)$ any eigenvalue $> t$ of H_L is replaced with 1.

$H_R' = \text{trunc}_t(H_R)$.

$H = H_L + h_1 + \ldots + h_s + H_R$

$H' = H_L' + h_1 + \ldots + h_s + H_R'$.

$|H'| \le s + 2t$

why 4 not 2?

Still $K = \mathbb{1} - \frac{H'}{s+2t}$ is not good enough for Δ.

Step 2: Consider a low-degree poly. $P(H')$

Considerations

~~→ the eigenv of H' of eigenvalue~~

– If $H'|\phi\rangle = \lambda|\phi\rangle$, then $P(H')|\phi\rangle = P(\lambda)|\phi\rangle$.

– H''s eigenvalues are $0, \geq \frac{\epsilon}{10}, \dots, s+2t$

for $t = \Omega(1/\epsilon)$ (not proven).

So we want ~~P(H')~~ $P(x)$ s.t. ~~P(x)~~

$$P(0) = 0$$
$$|P(x)| \leq \Delta \quad \text{if } x \in \left[\frac{\epsilon}{10}, s+2t\right]$$

~~If deg P~~ $P(H')$ has ~~degree~~ ~~Schmidt~~ Schmidt rank $D \leq O\left(d^{O(\deg P/s)}\right)$

~~End result choose s~~

Pick the correct Chebyshev polynomial of deg $P = s^2$ and $s = O(\log^2 d/\epsilon)$.

~~$\Rightarrow D = O(d^{O(\log^3 d)})$~~ Details ommitted.

Solvable local Hamiltonians

Apr 7

We saw that whenever a 1D Ham system is gapped, we have an alg for constructing a MPS representation of the ground state.

Next, we will see that whenever a Ham has "too many" terms, there exists a product state of low-energy.

This is a natural barrier for QPCP assuming QMA ≠ NP.
More on this later.

Setup: n qudits of dim d. $\mathcal{H} = \mathcal{H}_1 \otimes \ldots \otimes \mathcal{H}_n$.

We will mostly deal with density matrices in this result.

<u>Def</u>. A state σ is called (globally) seperable if

$$\sigma = \sum_{z} \underset{\substack{\uparrow \\ \text{prob. dist.}}}{P_z} \left(\sigma_1^{(z)} \otimes \sigma_2^{(z)} \otimes \ldots \otimes \sigma_n^{(z)} \right)$$

If $V_1 \sqcup V_2 \ldots \sqcup V_k = [n]$ is a partition, then

σ is sep. w.r.t. $\{V_i\}$ if

$$\sigma = \sum_z p_z \left(\sigma_{V_1}^{(z)} \otimes \ldots \otimes \sigma_{V_k}^{(z)} \right)$$

Not. For σ a d.m. on $\mathcal{H}_{[1\ldots n]}$, use

σ_S to refer to $\operatorname{tr}_{[n]\setminus S}(\sigma) = \operatorname{tr}_{-S}(\sigma)$

If $S = \{i\}$ or $\{i,j\}$, we often write σ_i or σ_{ij} for simplicity.

Fact If σ is sep, then

$$\sigma_{ij} = \sum_z p_z \, \sigma_i^{(z)} \otimes \sigma_j^{(z)}$$

$$\sigma_i = \sum_z p_z \, \sigma_i^{(z)} .$$

Thm (BH'13) Let $G = ([n], E)$ be a D-reg. graph.

For any $\rho \in \mathcal{H}_{[1\ldots n]}$, $\exists$ σ globally sep. s.t.

$$\mathbb{E}_{(i,j)\in E} \left\| \rho_{ij} - \sigma_{ij} \right\|_1 \le 12 \left(\frac{d^2 \ln d}{D} \right)^{1/3}.$$

$\|\cdot\|_1$ norm is sum of abs. value of eigenvalues.

<u>Cor</u> Let H be a 2-local Ham with interaction graph $([n], E)$.
D-regular

$$H = \mathbb{E}_{(i,j)\in E} h_{ij} \quad \text{where} \quad 0 \le h_{ij} \le \mathbb{1}. \text{ Then}$$

$\exists\ |\varphi\rangle = |\varphi_1\rangle \otimes \cdots \otimes |\varphi_n\rangle$ pure product state s.t.

$$\operatorname{tr}(H\varphi) \le \lambda_0(H) + 12 \left(\frac{d^2 \ln d}{D} \right)^{1/3}. \qquad (*)$$

<u>Pf.</u> Let ρ be g-state of H. Then $\exists$ globally sep σ

s.t. $\operatorname{tr}(H\sigma) \le (*)$.

Since σ globally sep $= \sum_z p_z\, \sigma_1^{(z)} \otimes \cdots \otimes \sigma_n^{(z)}$

$\exists$ a z_0 s.t. $\mathrm{tr}\left(H\, \sigma_1^{(z_0)} \otimes \dots \otimes \sigma_n^{(z_0)}\right) \leq (*)$.

$\sigma_1^{(z_0)} \otimes \dots \otimes \sigma_n^{(z_0)}$ is a lin comb. of product states.

So pick the product state minimizing the energy.

<u>Thm</u> For any 2-local D-reg Ham $H = \mathbb{E}_{(i,j)\in E} h_{ij}$ deciding

whether $\lambda_0(H) \leq \alpha$ or $\lambda_0(H) \geq \beta$ is $\in$ NP when

$$\beta - \alpha \geq 13\left(\frac{d^2 \ln d}{D}\right)^{1/3}$$

<u>Proof</u>. If $\lambda_0(H) \leq \alpha$, then $\exists\ |\psi\rangle$ product s.t.

$$\langle\psi|H|\psi\rangle \leq \alpha + 12\left(\frac{d^2 \ln d}{D}\right)^{1/3}.$$

Merlin describes $|\psi\rangle$, proving $\langle\psi|H|\psi\rangle < \beta$ and therefore assuming the promise, the problem is solved.

Observation 1 This statement using BH13 is only strong when the graph describing H is expanding.

Let G be a graph s.t. removal of ϵ-fraction of edges disconnects the graph into c.c's of size $\leq r$.

Then $\exists$ a classical time $O\left(\frac{n}{r}\right) \cdot 2^r$ time alg for calculating an ϵ-approx. of the groundenergy.

Pf. Assume E has m edges and removal set E' has size $\leq \epsilon m$.

If $H = \frac{1}{m} \sum_{(i,j) \in E} h_{ij}$ then let $H' = \frac{1}{m} \sum_{(i,j) \in E \setminus E'} h_{ij}$.

Observe $\langle \psi | H | \psi \rangle \leq \frac{\epsilon m}{m} + \langle \psi | H' | \psi \rangle$

so $\lambda_0(H) \leq \epsilon + \lambda_0(H')$.

Since H' consists on $\frac{n}{r}$ components non-interacting each can be solved individually in time 2^r. So $\lambda_0(H')$ calculable efficiently which is an ϵ-approx.

Rmk For a 1D Ham. $r = O(1/\epsilon)$ so we can get a ϵ-approx of groundenergy in time

$$\epsilon n \cdot 2^{O(1/\epsilon)}.$$

BH13 is demonstrating the existence of a product low-energy sol in the presence of expansion. In fact as $D \to \infty$, the product expansion has to get better.

BH13 is also intriguing because it provides a roadblock to proving the quantum PCP conjecture.

qPCP conjecture. $\exists$ a fam of local Ham $H = \mathbb{E}_{(i,j)\in E} h_{ij}$ where E is $D = O(1)$ regular and constants $\alpha < \beta$ s.t.

deciding if $\lambda_0(H) \leq \alpha$ or $\lambda_0(H) \geq \beta$ is QMA-complete.

A key step in the classical PCP theorem is the gap amplification step.

Given a 2-local CSP of alphabet d and graph of deg D, construct new 2-local CSP on n^2 vertices with alphabet d^2 and graph of deg D^2.

If $\lambda_0(C) = 0 \Rightarrow \lambda_0(C') = 0$

normalized CSPs

If $\lambda_0(C) \geq \alpha \Rightarrow \lambda_0(C') \geq \lambda_0(C) \geq \alpha$.

We say there exist a gap amplification procedure if for (H, α, β) given $H = \mathbb{E}_{(i,j)\in E} h_{ij}$, we can in classical polynomial time construct

a new Ham H' on poly(n) qudits of local dim d^2 and degree D^2 s.t.

1. if $\lambda_0(H) \leq \alpha \Rightarrow \lambda_0(H') \leq \alpha$
2. if $\lambda_0(H) \geq \beta \Rightarrow \lambda_0(H') \geq \beta$.

Thm If $\exists$ a gap amplification procedure, then for any constant $\epsilon > 0$, deciding (H, α, β) for $\beta = \alpha + \epsilon$ is in NP.

Proof.

Let (H, α, β) be the input problem. If H is k-local for $k = O(1)$, using perturbation gadgets, we can reduce to a 2-local Ham. H' and constants α', β' s.t.

$$\beta' - \alpha' \geq \Omega(\beta - \alpha) = \Omega(\epsilon).$$

Assume H' is over qudits of dim d, and is D-regular.

If $D \leq 2d^3$, add dummy checks so that $D \geq 2d^3$. This new Ham H'' yields the computational problem

$$(H'', \alpha'', \beta'') \text{ where } \beta'' - \alpha'' \geq \frac{\beta - \alpha}{2d^3} \geq \Omega\left(\frac{\epsilon}{d^3}\right).$$

Currently, the best product state approx. to gstate of H'' is at energy $\le \left(\frac{1}{2}\right)^{1/3} + \lambda_0(H'')$ by BH13.

Let's apply gap amplification. Let $H^{(1)}$ be the result.

$H^{(1)}$ is d^2 local and $D^2 \ge 4d^6$ regular. So $H^{(1)}$ has a product state approx at energy

$$\le \left(\frac{1}{4}\right)^{1/3} + \lambda_0(H'').$$

Repeat t times. $H^{(t)}$ has a product state approx at energy

$$\le 2^{-t/3} + \lambda_0(H'').$$

Pick t s.t. $2^{-t/3} \le O\left(\frac{\epsilon}{d^3}\right) \Rightarrow t = \Omega\left(\log \frac{d^3}{\epsilon}\right)$

If d, ϵ constants, $H^{(t)}$ is only on poly(n) qudits.

If $\lambda_0(H) \le \alpha \Rightarrow \lambda_0(H^{(t)}) \le \alpha^{(t)}$

and therefore $\exists$ a product state $|\psi\rangle$ of energy

$$\leq \alpha^{(t)} + 2^{-t/3} \leq \alpha^{(t)} + O\left(\frac{\epsilon}{d^3}\right) < \beta^{(t)}.$$

If $\lambda_0(H) \geq \beta \Rightarrow \lambda_0(H^{(t)}) \geq \beta^{(t)}$

and therefore no product state $|\psi\rangle$ of energy $\leq \beta^{(t)}$.

Merlin describes $|\psi\rangle$ the product witness for $H^{(t)}$. $\square$

$\in NP$.

So BHI3 restricts the types of transforms that are gap-preserving.
This suggests some trouble as to why we find QPCP so hard.

BHI3 is a statement about monogamy of entanglement.

We are going to get to this result over the next two lectures
so bear with me as we get there.

The result is information theoretic and non-constructive so we
will have to see how this plays out.

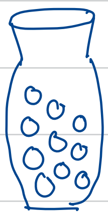
De Finetti theorems

These are information theoretic statements about how close a distribution is to product.

Classical version

multinomial dist $M_k^{(X)}$

$=$ dist. over k independent draws from urn X <u>with</u> replacement

X 

urn with n balls each color $\in [d]$

If $n = n_1 + \ldots + n_d$ (# of each color) then

$$M_k^{(X)}(s_1, \ldots, s_k) = \prod_{j=1}^{k} \left(\frac{n_{s_j}}{n} \right) \quad \text{for } s_1, \ldots, s_k \in [d].$$

Let $\nu_i = \left| \{ j \in [k] \mid s_j = i \} \right|$, then

$$M_k^{(X)}(s_1 \ldots s_k) = M_k^{(X)}(\nu_1, \ldots, \nu_d) = \prod_{i=1}^{d} \left(\frac{n_i}{n} \right)^{\nu_i}.$$

Hypergeometric dist. $H_k^{(x)}$ is indep. draws <u>without</u> replacement.

$$H_k^{(x)}(s_1,\dots,s_k) = H_k^{(x)}(v_1,\dots,v_d)$$

$$= \frac{\big(n_1\cdots(n_1-v_1+1)\big)\big(n_2\cdots(n_2-v_2+1)\big)\cdots\big(n_d\cdots(n_d-v_d+1)\big)}{n(n-1)\cdots(n-k+1)}$$

$$= \frac{\prod_{i=1}^{d} \frac{n_i!}{(n_i-v_i)!}}{\frac{n!}{(n-k)!}}$$

Intuition for $n \gg k$, these dist are close as repeat balls are rarely pulled.

<u>Lemma</u> ① $\| M_k^{(x)} - H_k^{(x)} \|_1 \leq \frac{2dk}{n}$

② For $d = 2$, $k \leq n$, this is asymptotically tight.

(3) If $d \geq n$, and $\forall i, n_i \leq 1$, $\| M_k^{(x)} - H_k^{(x)} \|_1 \leq \frac{k(k-1)}{n}$.

↑ Birthday paradox.

Def. A dist P over $[d]^n$ is n-exchangable (or permutation-invariant)

if $\forall \pi \in S_n$,

$$P(s_1, \ldots, s_n) = P(s_{\pi(1)}, \ldots, s_{\pi(n)}).$$

Examples: uniform, product, draw from urn

uniform dist over $[2]^n$ having exactly $n/2$ ones.

↑ very far from product.

For $X = (n_1, \ldots, n_d)$ s.t. $n_1 + \ldots + n_d = n$. (histogram)

Thm Any n-exchangable distribution $= \sum w_x H_n^{(x)}$
where w_x is a prob. dist.

Pf. $w_X = \Pr[\text{sample from } P \text{ matches histogram } X]$.

Thm (de Finetti)

For any P that is n-exchangable on $[d^n]$, let $tr_{n-k}(P)$ be the marginal on <u>any</u> k elements.

Then $\exists$ a measure μ on the distributions on $[d]$ s.t. $\forall k \le n$,

$$\left\| tr_{n-k}(P) - \int Q^{\otimes k} \, d\mu(Q) \right\| \le \min\left\{ \frac{2kd}{n}, \frac{k(k-1)}{n} \right\}.$$

Pf. $tr_{n-k}(P) = \sum_X w_X \, tr_{n-k}(P_X)$

$$= \sum_X w_X H_k^{(X)}$$

So this is close to $\sum_X w_X M_k^{(X)}$

and $M_k^{(x)}$ is a product distribution $= \left(M_1^{(x)}\right)^{\otimes k} = \left(H_1^{(x)}\right)^{\otimes k}$

Distance bound comes from previous lemma.

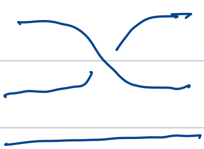
Quantum de Finetti theorems

Def. Let $\pi \in S_n$. We define operator $P_d(\pi)$ acting on $\left(\mathbb{C}^d\right)^{\otimes n}$

as $$P_d(\pi) = \sum_{i_1 \dots i_n \in [d]} |i_{\pi(1)}, \dots, i_{\pi(n)}\rangle\langle i_1, \dots, i_n|$$

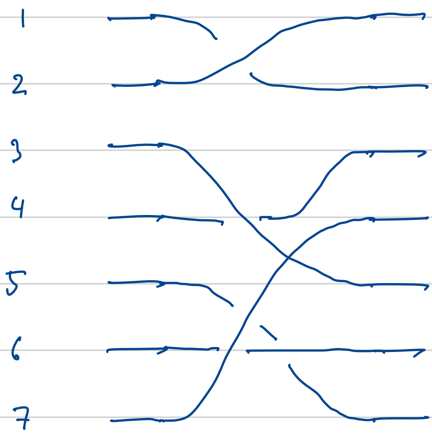
Ex. $\pi = (1\ 2)(3)$. Then

$$P_d(\pi) = \sum_{i_1 i_2 i_3} |i_2, i_1, i_3\rangle\langle i_1, i_2, i_3|$$



as a ckt / tensor network.

More generally, draw a permutation as tubes



$\pi = (1\ 2)(3\ 5\ 7\ 4)(6)$

Def. The symmetric subspace on n qudits is defined as

$$V^n(\mathbb{C}^d) := \operatorname{span}\{ |\psi\rangle : P_d(\pi)|\psi\rangle = |\psi\rangle \ \forall \pi \in S_n \}.$$

Ex. Any state $|\psi\rangle \in \mathbb{C}^d$, $|\psi\rangle^{\otimes n} \in V^n(\mathbb{C}^d)$.

For $\vec{t} = (t_0, \dots, t_{d-1})$ s.t. $\sum t_i = n$, call this a type.

For any type $\vec{t}$, let $S_{\vec{t}}$ be the set of $(i_1, \dots, i_n)$ matching $\vec{t}$.

meaning $\#\{i_j : i_j = 0\} = t_0, \dots, \#\{i_j : i_j = i\} = t_i$.

ex: for $n = 4$, and $\vec{t} = (3, 0, 1)$,

$$S_{\vec{t}} = \{(0,0,0,2), (0,0,2,0), (0,2,0,0), (2,0,0,0)\}.$$

Define $$|\vec{t}\rangle = \frac{1}{\sqrt{S_{\vec{t}}}} \sum_{(i_1 \dots i_n) \in S_{\vec{t}}} |i_1, \dots, i_n\rangle.$$

Observe that the sets $S_{\vec{t}}$ are disjoint. So $|\vec{t}\rangle$ are orthogonal.

Facts: Any $P_d(\pi)$ preserves span$\{S_{\vec{t}}\}$.

$P_d(\pi) =$

block-diagonal wrt this basis.

Within each block, the unique vector in $V^n(\mathbb{C}^d)$ is $|\vec{t}\rangle$.

Thm $\{|\vec{t}\rangle\}$ is a basis for $V^n(\mathbb{C}^d)$.

Then, $\dim V^n(\mathbb{C}^d) = \binom{n+d-1}{d-1}$

as this is the number of types (proof stars-and-bars counting).

For $n=2$, $\dim V^2(\mathbb{C}^d) = \frac{d(d+1)}{2}$.

For $n=2$, $d=2$, this is 3. Basis $|00\rangle, |11\rangle, \frac{1}{\sqrt{2}}|01\rangle + |10\rangle$.

Thm The projector onto $V^n(\mathbb{C}^d)$ has a nice decomp.

$$\Pi_{sym}^{d,n} = \frac{1}{n!}\sum_{\pi \in S_n} P_d(\pi).$$

Pf. First observe that $P_d(\pi)^\dagger = P_d(\pi^{-1})$, and

$$P_d(\pi) P_d(\pi') = P_d(\pi \cdot \pi').$$

Next,

$$\left(\Pi_{sym}^{d,n}\right)^2 = \frac{1}{(n!)^2} \sum_{\pi,\pi' \in S_n} P_d(\pi) P_d(\pi')$$

$$= \frac{1}{(n!)^2} \sum_{\pi \in S_n} n! \, P_d(\pi)$$

$$= \Pi_{sym}^{d,n} .$$

$\Pi_{sym}^{d,n}$ is a projector and

$$P_d(\pi)\, \Pi_{sym}^{d,n} = \frac{1}{n!} \sum_{\pi'} P_d(\pi) P_d(\pi')$$

$$= \frac{1}{n!} \sum_{\pi'} P_d(\pi') .$$

$$= \Pi_{sym}^{d,n} . \qquad \Longrightarrow \operatorname{im}\left(\Pi_{sym}^{n,d}\right) \subseteq V^{d}(\mathbb{C}^{n}) .$$

So for any $|\psi\rangle \in V^n(\mathbb{C}^d)$,

$$\Pi_{sym}^{d,n} |\psi\rangle = \frac{1}{n!} \sum_{\pi} P_d(\pi) |\psi\rangle = |\psi\rangle$$

$$\operatorname{tr}\left(\Pi_{sym}^{d,n}\right) = \dim V^n(\mathbb{C}^d) = \binom{n+d-1}{d-1}$$

<u>Claim</u>

$$\frac{\Pi_{sym}^{d,n}}{\operatorname{tr}\left(\Pi_{sym}^{d,n}\right)} = \underbrace{\mathop{\mathbb{E}}_{|\varphi\rangle \in \mathbb{C}^d} \left(|\varphi\rangle\langle\varphi|\right)^{\otimes n}}_{=: M_n} .$$

<u>Pf.</u> Observe that for any unitary g acting on $\mathbb{C}^d$,

$$g^{\otimes n} M_n = M_n \, g^{\otimes n} .$$

So, M_n commutes with the rep. $U(d) \longrightarrow V^d(\mathbb{C}^n) \times V^d(\mathbb{C}^n)$

given by $g \longmapsto g^{\otimes n}$.

Using rep. theory (pf omitted), the only irreducible representations must be multiples of identity on the subspace. So

M_n is a scalar multiple of $\Pi_{sym}^{d,n}$

Since $\mathrm{tr}(M_n) = 1$, the multiple must be $\mathrm{tr}\left(\Pi_{sym}^{d,n}\right)$.