Finite Model Theory
Unit 1

Dan Suciu

Spring 2018
Welcome to 599c: Finite Model Theory

- Logic is the foundation of Mathematics (see Logicomix).
- Logic is the foundation of computing (see Turing Machines).
- Finite Model Theory is Logic restricted to finite models.
- Applications of FMT: Verification, Databases, Complexity
- This course is about:
  - Classic results in Mathematical Logic
  - Classic results in Finite Model Theory
  - New results in Finite Model Theory
  - Most results are negative, but some positive results too.

This course is not about: systems, implementation, writing programs.
Course Organization

Lectures:

- Regular time: MW 10 - 11:20, CSE 303
- Canceled: April 9, 11; May 14, 16.
- Makeup (all in CSE 303):

Homework assignment:

- 6 Homework assignments
- Short problems, but some require thinking.
- Email them to me by the due date.
- Ignore points: I will grade all 6 together as Credit/No-credit.
- Discussion on the bboard encouraged!
- Goal: no stress, encourage to participate and think.
Resources

- Required (fun) reading: Logicomix.
- Libkin *Finite Model Theory*.
- Enderton *A Mathematical Introduction to Logic*.
- Barnes and Mack *An Algebraic Introduction to Logic*.
- Abiteboul, Hull, Vianu, *Database Theory*
- Several papers, talks, etc.
- Course on Friendly Logics from UPenn (by Val Tannen and Scott Weinstein) (older version: http://www.cis.upenn.edu/~val/CIS682/)
Course Outline

Unit 1 Classical Model Theory and Applications to FMT.
Unit 2 Games and expressibility.
Unit 3 Descriptive Complexity.
Unit 4 Query Containment.
Unit 5 Algorithmic FMT.
Unit 6 Tree Decomposition. Guest lecturer: Hung Ngo.
Unit 7 Provenance semirings. Guest lecturer: Val Tannen.
Unit 8 Semantics of datalog programs.
## Structures

A **vocabulary** $\sigma$ is a set of relation symbols $R_1, \ldots, R_k$ and function symbols $f_1, \ldots, f_m$, each with a fixed arity.

A **structure** is $\mathcal{D} = (\mathcal{D}, R_1^D, \ldots, R_k^D, f_1^D, \ldots, f_m^D)$, where $R_i^D \subseteq (\mathcal{D})^{\text{arity}(R_i)}$ and $f_j^D : (\mathcal{D})^{\text{arity}(f_j)} \to \mathcal{D}$.

$\mathcal{D}$ = the **domain** or the **universe**.

$v \in \mathcal{D}$ is called a **value** or a **point**.

$\mathcal{D}$ called a **structure** or a **model** or **database**.
Structures

A vocabulary $\sigma$ is a set of relation symbols $R_1, \ldots, R_k$ and function symbols $f_1, \ldots, f_m$, each with a fixed arity.

A structure is $D = (D, R_1^D, \ldots, R_k^D, f_1^D, \ldots, f_m^D)$, where $R_i^D \subseteq (D)^{\text{arity}(R_i)}$ and $f_j^D : (D)^{\text{arity}(f_j)} \rightarrow D$.

$D = \text{the domain or the universe}$.  
$v \in D$ is called a value or a point.  
$D$ called a structure or a model or database.
### Structures

A **vocabulary** $\sigma$ is a set of relation symbols $R_1, \ldots, R_k$ and function symbols $f_1, \ldots, f_m$, each with a fixed arity.

A **structure** is $D = (D, R_1^D, \ldots, R_k^D, f_1^D, \ldots, f_m^D)$, where $R_i^D \subseteq (D)^{\text{arity}(R_i)}$ and $f_j^D : (D)^{\text{arity}(f_j)} \rightarrow D$.

$D = \text{the domain or the universe}.$

$v \in D$ is called a **value** or a **point**.

$D$ called a **structure** or a **model** or **database**.
Examples

A graph is $G = (V, E), E \subseteq V \times V$.

A field is $F = (F, 0, 1, +, \cdot)$ where

- $F$ is a set.
- 0 and 1 are constants (i.e. functions $F^0 \to F$).
- $+$ and $\cdot$ are functions $F^2 \to F$.

An ordered set is $S = (S, \leq)$ where $\leq \subseteq S \times S$.

A database is $D = (\text{Domain}, \text{Customer}, \text{Order}, \text{Product})$. 
Discussion

- We don’t really need functions, since $f: D^k \rightarrow D$ is represented by its graph $\subseteq D^{k+1}$, but we keep them when convenient.

- If $f$ is a 0-ary function $D^0 \rightarrow D$, then it is a constant $D$, and we denote it $c$ rather than $f$.

- $D$ can be a finite or an infinite structure.
First Order Logic

Fix a vocabulary $\sigma$ and a set of variables $x_1, x_2, \ldots$

Terms:

- Every constant $c$ and every variable $x$ is a term.
- If $t_1, \ldots, t_k$ are terms then $f(t_1, \ldots, t_k)$ is a term.

Formulas:

- $F$ is a formula (means $false$).
- If $t_1, \ldots, t_k$ are terms, then $t_1 = t_2$ and $R(t_1, \ldots, t_k)$ are formulas.
- If $\varphi, \psi$ are formulas, then so are $\varphi \rightarrow \psi$ and $\forall x(\varphi)$. 
Discussion

\[ F \] often denoted: false or \( \bot \) or 0.

= is not always part of the language

Derived operations:

- \( \neg \varphi \) is a shorthand for \( \varphi \rightarrow F \).
- \( \varphi \lor \psi \) is a shorthand for \( \neg (\neg \varphi) \rightarrow \psi \).
- \( \varphi \land \psi \) is a shorthand for \( \neg (\varphi \lor \psi) \).
- \( \exists x(\varphi) \) is a shorthand for \( \neg (\forall x(\neg \varphi)) \).
Formulas and Sentences

We say that $\forall x(\varphi)$ binds $x$ in $\varphi$. Every occurrence of $x$ in $\varphi$ is bound. Otherwise it is free.

A sentence is a formula $\varphi$ without free variables.

E.g. formula $\exists y(E(x, y) \land E(y, z))$.

E.g. sentence $\exists x \forall z \exists y(E(x, y) \land E(y, z))$. 
Truth

Let $\varphi$ be a formula with free variables $\mathbf{x} = (x_1, \ldots, x_k)$. Let $D$ be a structure, and $\mathbf{a} = (a_1, \ldots, a_k) \in D^k$. We say that $\varphi$ is true in $D$, written:

$$D \models \varphi[\mathbf{a}/\mathbf{x}]$$

if:

- $\varphi$ is $x_i = x_j$ and $a_i, a_j$ are the same value.
- $\varphi$ is $R(x_{i_1}, \ldots, x_{i_n})$ and $(a_{i_1}, \ldots, a_{i_n}) \in R^D$.
- $\varphi$ is $\psi_1 \rightarrow \psi_2$ and $D \not\models \psi_1[\mathbf{a}/\mathbf{x}]$, or $D \models \psi_1[\mathbf{a}/\mathbf{x}]$ and $D \models \psi_2[\mathbf{a}/\mathbf{x}]$.
- $\varphi$ is $\forall y(\psi)$, and, for all $b \in D$, $D \models \psi[(a_1, \ldots, a_k, b)/(x_1, \ldots, x_k, y)]$. 
Problems

- Classical model theory:
  
  - Satisfiability Is \( \varphi \) true in some structure \( D \)?
  
  - Validity Is \( \varphi \) true in all structures \( D \)?

- Finite model theory, databases, verification:
  
  - Finite satisfiability/validity Is \( \varphi \) true in some/every finite structure \( D \)?
  
  - Model checking Given \( \varphi, D \), determine whether \( D \models \varphi \).
  
  - Query evaluation Given \( \varphi(x), D \), compute \( \{ a \mid D \models \varphi[a/x] \} \).
What do these sentences say about \( D \)?

\[
\exists x \exists y \exists z (x \neq y) \land (x \neq z) \land (y \neq z)
\]

\[
\exists x \exists y \forall z (z = x) \lor (z = y)
\]
What do these sentences say about $D$?

$$\exists x \exists y \exists z (x \neq y) \land (x \neq z) \land (y \neq z)$$

“There are at least three elements”, i.e. $|D| \geq 3$

$$\exists x \exists y \forall z (z = x) \lor (z = y)$$
What do these sentences say about $D$?

$$\exists x \exists y \exists z (x \neq y) \land (x \neq z) \land (y \neq z)$$

“There are at least three elements”, i.e. $|D| \geq 3$

$$\exists x \exists y \forall z (z = x) \lor (z = y)$$

“There are at most two elements”, i.e. $|D| \leq 2$
What do these sentences say about $D$?

$$\forall x \exists y E(x, y) \lor E(y, x)$$

$$\forall x \forall y \exists z E(x, z) \land E(z, y)$$

$$\exists x \exists y \exists z (\forall u (u = x) \lor (u = y) \lor (u = z))$$

$$\land \neg E(x, x) \land E(x, y) \land \neg E(x, z)$$

$$\land \neg E(y, z) \land \neg E(y, y) \land E(y, z)$$

$$\land E(z, x) \land \neg E(z, y) \land \neg E(z, z)$$
What do these sentences say about $D$?

\[ \forall x \exists y E(x, y) \lor E(y, x) \]

“There are no isolated nodes”

\[ \forall x \forall y \exists z E(x, z) \land E(z, y) \]

\[ \exists x \exists y \exists z (\forall u (u = x) \lor (u = y) \lor (u = z)) \]
\[ \land \neg E(x, x) \land E(x, y) \land \neg E(x, z) \]
\[ \land \neg E(y, z) \land \neg E(y, y) \land E(y, z) \]
\[ \land E(z, x) \land \neg E(z, y) \land \neg E(z, z) \]
What do these sentences say about $D$?

$\forall x \exists y E(x, y) \lor E(y, x)$

“There are no isolated nodes”

$\forall x \forall y \exists z E(x, z) \land E(z, y)$

“Every two nodes are connected by a path of length 2”

$\exists x \exists y \exists z (\forall u (u = x) \lor (u = y) \lor (u = z))$

$\land \neg E(x, x) \land E(x, y) \land \neg E(x, z)$

$\land \neg E(y, z) \land \neg E(y, y) \land E(y, z)$

$\land E(z, x) \land \neg E(z, y) \land \neg E(z, z)$
What do these sentences say about $D$?

$$\forall x \exists y E(x, y) \lor E(y, x)$$

"There are no isolated nodes"

$$\forall x \forall y \exists z E(x, z) \land E(z, y)$$

"Every two nodes are connected by a path of length 2"

$$\exists x \exists y \exists z \left( \forall u \left( u = x \right) \lor (u = y) \lor (u = z) \right)$$
$$\land \neg E(x, x) \land E(x, y) \land \neg E(x, z)$$
$$\land \neg E(y, z) \land \neg E(y, y) \land E(y, z)$$
$$\land E(z, x) \land \neg E(z, y) \land \neg E(z, z)$$

It completely determines the graph: $D = \{a, b, c\}$ and $a \rightarrow b \rightarrow c \rightarrow a$. 
Logical Implication

Fix a set of sentences $\Sigma$ (may be infinite).

$\Sigma$ implies $\varphi$, $\Sigma \models \varphi$, if every model of $\Sigma$ is also a model of $\varphi$:

$D \models \Sigma$ implies $D \models \varphi$.

$\text{Con}(\Sigma) \overset{\text{def}}{=} \{ \varphi \mid \Sigma \models \varphi \}$. Sometimes called the theory of $\Sigma$, $\text{Th}(\Sigma)$.

$\Sigma$ finitely implies $\varphi$, $\Sigma \models_{\text{fin}} \varphi$ if every finite model of $\Sigma$ is also a model of $\varphi$. 
Discussion

- $F \models \varphi$ for any sentence $\varphi$. Why?
Discussion

- $F \models \varphi$ for any sentence $\varphi$. Why?

- $\Sigma \models F$ iff $\Sigma$ is unsatisfiable. Why?
Discussion

- $F \models \varphi$ for any sentence $\varphi$ why?

- $\Sigma \models F$ iff $\Sigma$ is unsatisfiable why?

- If $\Sigma \models \varphi$ and $\Sigma, \varphi \models \psi$ then $\Sigma \models \psi$ why?
Discussion

- $F \models \varphi$ for any sentence $\varphi$ why?

- $\Sigma \models F$ iff $\Sigma$ is unsatisfiable why?

- If $\Sigma \models \varphi$ and $\Sigma, \varphi \models \psi$ then $\Sigma \models \psi$ why?

- If $\Sigma \models \varphi$ then $\Sigma \models_{\text{fin}} \varphi$, but the converse fails in general why?
Discussion

- $F \models \varphi$ for any sentence $\varphi$ why?.

- $\Sigma \models F$ iff $\Sigma$ is unsatisfiable why?.

- If $\Sigma \models \varphi$ and $\Sigma, \varphi \models \psi$ then $\Sigma \models \psi$ why?.

- If $\Sigma \models \varphi$ then $\Sigma \models_{\text{fin}} \varphi$, but the converse fails in general why?. Let $\lambda_n$ say “there are at least $n$ elements, and $\Sigma = \{ \lambda_n \mid n \geq 1 \}$. Then $\Sigma \models_{\text{fin}} F$ but $\Sigma \not\models F$ why?. 
Discussion

- $F \models \varphi$ for any sentence $\varphi$ why?

- $\Sigma \models F$ iff $\Sigma$ is unsatisfiable why?

- If $\Sigma \models \varphi$ and $\Sigma, \varphi \models \psi$ then $\Sigma \models \psi$ why?

- If $\Sigma \models \varphi$ then $\Sigma \models_{\text{fin}} \varphi$, but the converse fails in general why?
  Let $\lambda_n$ say “there are at least $n$ elements, and $\Sigma = \{\lambda_n \mid n \geq 1\}$.
  Then $\Sigma \models_{\text{fin}} F$ but $\Sigma \not\models F$ why?

- If $\models \varphi$ then we call $\varphi$ a tautology.
A **theory** is a set of sentences $\Sigma$ closed under implication, i.e. $\Sigma = \text{Con}(\Sigma)$.

A theory $\Sigma$ is **complete** if, for every sentence $\varphi$, either $\varphi \in \Sigma$ or $\neg \varphi \in \Sigma$.

The theory of a set of structures $\mathcal{D}$ is

$$\text{Th}(\mathcal{D}) \overset{\text{def}}{=} \{ \varphi \mid \varphi \text{ is true in every } D \in \mathcal{D} \}$$

closed under implication?

For a single structure $D$, $\text{Th}(D)$ is complete why?
A **theory** is a set of sentences $\Sigma$ closed under implication, i.e. $\Sigma = \text{Con}(\Sigma)$.

A theory $\Sigma$ is **complete** if, for every sentence $\varphi$, either $\varphi \in \Sigma$ or $\neg \varphi \in \Sigma$.

The theory of a set of structures $\mathcal{D}$ is

$$\text{Th}(\mathcal{D}) \overset{\text{def}}{=} \{ \varphi \mid \varphi \text{ is true in every } D \in \mathcal{D} \}$$

closed under implication?

For a single structure $D$, Th($D$) is complete why?
A theory is a set of sentences $\Sigma$ closed under implication, i.e. $\Sigma = \text{Con}(\Sigma)$.

A theory $\Sigma$ is complete if, for every sentence $\varphi$, either $\varphi \in \Sigma$ or $\neg \varphi \in \Sigma$.

The theory of a set of structures $\mathcal{D}$ is

$$\text{Th}(\mathcal{D}) \overset{\text{def}}{=} \{ \varphi \mid \varphi \text{ is true in every } D \in \mathcal{D} \} \quad \text{closed under implication?}$$

For a single structure $D$, $\text{Th}(D)$ is complete why?
Theory

A **theory** is a set of sentences $\Sigma$ closed under implication, i.e. $\Sigma = \text{Con}(\Sigma)$.

A theory $\Sigma$ is **complete** if, for every sentence $\varphi$, either $\varphi \in \Sigma$ or $\neg \varphi \in \Sigma$.

The theory of a set of structures $\mathcal{D}$ is

$$\text{Th}(\mathcal{D}) \overset{\text{def}}{=} \{ \varphi \mid \varphi \text{ is true in every } D \in \mathcal{D} \} \quad \text{closed under implication?}$$

For a single structure $D$, $\text{Th}(D)$ is complete **why?**
Discussion

Which of the following theories are complete?

- The theory of fields $\mathbb{F} = (F, 0, 1, +, \cdot)$.
- The theory $\text{Th}(\mathbb{R})$ (vocabulary $0, 1, +, \cdot$).
- The theory of total orders:
  \[
  \forall x \forall y \neg ((x < y) \land (y < x))
  \]
  \[
  \forall x \forall y ((x < y) \lor (x = y) \lor (y < x))
  \]
  \[
  \forall x \forall y \forall z ((x < y) \land (y < z) \rightarrow (x < z))
  \]
- The theory of dense total orders without endpoints:
  axioms above plus
  \[
  \text{Dense: } \forall x \forall y (x < y \rightarrow \exists v (x < v < y))
  \]
  \[
  \text{W/o Endpoints: } \forall x \exists u \exists w (u < x < w)
  \]
Discussion

Which of the following theories are complete?

- The theory of fields $\mathbb{F} = (F, 0, 1, +, \cdot)$. No: $\exists x(x^2 + 1 = 0)$
- The theory $\text{Th(}\mathbb{R}\text{)}$ (vocabulary $0, 1, +, \cdot$).
- The theory of total orders:
  \[
  \forall x \forall y \neg((x < y) \land (y < x))
  \]
  \[
  \forall x \forall y ((x < y) \lor (x = y) \lor (y < x))
  \]
  \[
  \forall x \forall y \forall z ((x < y) \land (y < z) \rightarrow (x < z))
  \]

- The theory of dense total orders without endpoints:
  axioms above plus
  \[
  \text{Dense: } \forall x \forall y (x < y \rightarrow \exists v (x < v < y))
  \]
  \[
  \text{W/o Endpoints: } \forall x \exists u \exists w (u < x < w)
  \]
Discussion

Which of the following theories are complete?

- The theory of fields \( \mathbb{F} = (F, 0, 1, +, \cdot) \). No: \( \exists x (x^2 + 1 = 0) \)
- The theory \( \text{Th}(\mathbb{R}) \) (vocabulary 0, 1, +, \cdot). Yes
- The theory of total orders:

  \[
  \forall x \forall y \neg((x < y) \land (y < x))
  
  \forall x \forall y ((x < y) \lor (x = y) \lor (y < x))
  
  \forall x \forall y \forall z ((x < y) \land (y < z) \rightarrow (x < z))
  \]

- The theory of dense total orders without endpoints:
  axioms above plus

  Dense: \( \forall x \forall y (x < y \rightarrow \exists v (x < v < y)) \)

  W/o Endpoints: \( \forall x \exists u \exists w (u < x < w) \)
Discussion

Which of the following theories are complete?

- The theory of fields $\mathbb{F} = (F, 0, 1, +, \cdot)$. No: $\exists x (x^2 + 1 = 0)$
- The theory $\text{Th}(\mathbb{R})$ (vocabulary $0, 1, +, \cdot$). Yes
- The theory of total orders:

  $$\forall x \forall y \neg((x < y) \land (y < x))$$
  $$\forall x \forall y ((x < y) \lor (x = y) \lor (y < x))$$
  $$\forall x \forall y \forall z ((x < y) \land (y < z) \rightarrow (x < z))$$

  No: $\forall x \exists y (x < y)$.
- The theory of dense total orders without endpoints:
  axioms above plus

  **Dense:** $\forall x \forall y (x < y \rightarrow \exists v (x < v < y))$
  **W/o Endpoints:** $\forall x \exists u \exists w (u < x < w)$
Discussion

Which of the following theories are complete?

- The theory of fields $\mathbb{F} = (F, 0, 1, +, \cdot)$. **No:** $\exists x(x^2 + 1 = 0)$
- The theory $\text{Th}(\mathbb{R})$ (vocabulary $0, 1, +, \cdot$). **yes**
- The theory of total orders:
  
  \[
  \forall x \forall y \neg((x < y) \land (y < x)) \\
  \forall x \forall y ((x < y) \lor (x = y) \lor (y < x)) \\
  \forall x \forall y \forall z ((x < y) \land (y < z) \rightarrow (x < z))
  \]
  **No:** $\forall x \exists y (x < y)$.
- The theory of dense total orders without endpoints: axioms above plus
  
  Dense: $\forall x \forall y (x < y \rightarrow \exists v (x < v < y))$
  
  W/o Endpoints: $\forall x \exists u \exists w (u < x < w)$

  **Yes! Will prove later**
The Sentence Map

Give examples for each of the five classes
The Sentence Map

FO sentences

Unsat  Finitely unsat  Finitely valid  Valid

Give examples for each of the five classes

\( \exists x (\neg (x = x)) \)
The Sentence Map

FO sentences

Unsat

Finitely unsat

Valid

Finitely valid

Give examples for each of the five classes

\(\exists x (\neg (x = x))\)  

“\(<\) is a dense total order”
Give examples for each of the five classes

\( \exists x (\neg (x = x)) \)  \hspace{1cm} \text{“< is a dense total order”}  \hspace{1cm} \exists x \exists y (E(x, y))
The Sentence Map

Give examples for each of the five classes

$\exists x (\neg (x = x))$  “$<$ is a dense total order”

$\forall x (x = x)$  “if $<$ is a total order, then it has a maximal element”

$\exists x \exists y (E(x, y))$
The Sentence Map

Give examples for each of the five classes

\( \exists x (\neg (x = x)) \)  
“\(<\) is a dense total order”

“if \(<\) is a total order, then it has a maximal element”

\( \exists x \exists y (E(x, y)) \)

\( \forall x (x = x) \)
The Zero-One Law for FO

- Some sentences are neither true (in all structures) nor false.

- The Zero-One Law says this: over finite structures, every sentence is true or false with high probability.

- Proven by Fagin in 1976 (part of his PhD thesis).

- Although the statement is about finite structures, the proof uses theorems on finite and infinite structures.
The Zero-One Law for FO

Consider a relational vocabulary (i.e. no functions, no constants).
Let $\varphi$ be a sentence. Forall $n \in \mathbb{N}$ denote:

$$\#_n \varphi \overset{\text{def}}{=} \left\{ D \mid D = [n], D \models \varphi \right\}$$

$$\#_n T \overset{\text{def}}{=} \text{number of models with universe } [n]$$

$$\mu_n(\varphi) \overset{\text{def}}{=} \frac{\#_n \varphi}{\#_n T}$$

**Theorem (Fagin’1976)**

For every sentence $\varphi$, either $\lim_{n \to \infty} \mu_n(\varphi) = 0$ or $\lim_{n \to \infty} \mu_n(\varphi) = 1$.

Informally: for every $\varphi$, its probability goes to either 0 or 1, when $n \to \infty$; it is either almost certainly true, or almost certainly false.
The Zero-One Law for FO

Consider a relational vocabulary (i.e. no functions, no constants). Let \( \varphi \) be a sentence. For all \( n \in \mathbb{N} \) denote:

\[
\#_n \varphi \overset{\text{def}}{=} \left| \{ D \mid D = [n], D \models \varphi \} \right|
\]

\[
\#_n T \overset{\text{def}}{=} \text{number of models with universe } [n]
\]

\[
\mu_n(\varphi) \overset{\text{def}}{=} \frac{\#_n \varphi}{\#_n T}
\]

**Theorem (Fagin’1976)**

For every sentence \( \varphi \), either \( \lim_{n \to \infty} \mu_n(\varphi) = 0 \) or \( \lim_{n \to \infty} \mu_n(\varphi) = 1 \).

Informally: for every \( \varphi \), its probability goes to either 0 or 1, when \( n \to \infty \); it is either almost certainly true, or almost certainly false.
The Zero-One Law for FO

Consider a relational vocabulary (i.e. no functions, no constants). Let $\varphi$ be a sentence. For all $n \in \mathbb{N}$ denote:

\[
\#_n \varphi \overset{\text{def}}{=} \left| \{ D \mid D = [n], D \models \varphi \} \right|
\]

\[
\#_n T \overset{\text{def}}{=} \text{number of models with universe } [n]
\]

\[
\mu_n(\varphi) \overset{\text{def}}{=} \frac{\#_n \varphi}{\#_n T}
\]

**Theorem (Fagin’1976)**

*For every sentence $\varphi$, either $\lim_{n \to \infty} \mu_n(\varphi) = 0$ or $\lim_{n \to \infty} \mu_n(\varphi) = 1$.*

Informally: for every $\varphi$, its probability goes to either 0 or 1, when $n \to \infty$; it is either almost certainly true, or almost certainly false.
The Zero-One Law for FO

Consider a relational vocabulary (i.e. no functions, no constants). Let \( \varphi \) be a sentence. For all \( n \in \mathbb{N} \) denote:

\[
\#_n \varphi \overset{\text{def}}{=} \{ D \mid D = [n], D \models \varphi \}
\]

\[
\#_n T \overset{\text{def}}{=} \text{number of models with universe } [n]
\]

\[
\mu_n(\varphi) \overset{\text{def}}{=} \frac{\#_n \varphi}{\#_n T}
\]

**Theorem (Fagin’1976)**

*For every sentence \( \varphi \), either \( \lim_{n \to \infty} \mu_n(\varphi) = 0 \) or \( \lim_{n \to \infty} \mu_n(\varphi) = 1 \).*

Informally: for every \( \varphi \), its probability goes to either 0 or 1, when \( n \to \infty \); it is either almost certainly true, or almost certainly false.
Examples

Vocabulary of graphs: $\sigma = \{E\}$. Compute these probabilities:

$$\phi = \forall x \forall y E(x, y)$$

$$\phi = \exists x \exists y E(x, y)$$

$$\phi = \forall x \exists y E(x, y)$$
Examples

Vocabulary of graphs: $\sigma = \{E\}$. Compute these probabilities:

- $\varphi = \forall x \forall y E(x, y)$ \quad $\#_n(\varphi) = 1$ \quad $\mu_n = \frac{1}{2n^2} \to 0$

- $\varphi = \exists x \exists y E(x, y)$

- $\varphi = \forall x \exists y E(x, y)$
Examples

Vocabulary of graphs: $\sigma = \{ E \}$. Compute these probabilities:

$$\varphi = \forall x \forall y E(x, y) \quad \#_n(\varphi) = 1$$

$$\mu_n = \frac{1}{2n^2} \to 0$$

$$\varphi = \exists x \exists y E(x, y) \quad \#_n(\varphi) = 2^n - 1$$

$$\mu_n = \frac{2^n - 1}{2n^2} \to 1$$

$$\varphi = \forall x \exists y E(x, y)$$
Examples

Vocabulary of graphs: $\sigma = \{E\}$. Compute these probabilities:

$$\varphi = \forall x \forall y E(x, y) \quad \#_n(\varphi) = 1 \quad \mu_n = \frac{1}{2n^2} \to 0$$

$$\varphi = \exists x \exists y E(x, y) \quad \#_n(\varphi) = 2^{n^2} - 1 \quad \mu_n = \frac{2^{n^2} - 1}{2n^2} \to 1$$

$$\varphi = \forall x \exists y E(x, y) \quad \mu_n = \frac{(2^n - 1)^n}{2n^2} \to 1$$
The Sentence Map Revised

FO sentences

Unsat w.h.p.

Unsat
Finitely unsat

Finitely valid
Valid

Valid w.h.p.
Discussion

**Attempted proof:** Derive the general formula $\#_n \varphi$, then compute $\lim \frac{\#_n \varphi}{2^n}$ and observe it is 0 or 1.

**Problem:** we don’t know how to compute $\#_n \varphi$ in general: there is evidence this is “hard”

Instead, we will prove the 0/1 law using three results from classical model theory.
Three Classical Results in Model Theory

We will discuss and prove:

- Compactness Theorem.
- Löwenheim-Skolem Theorem.
- Los-Vaught Test.

Then will use them to prove Fagin’s 0/1 Law for First Order Logic.

Later we will discuss:

- Gödel’s completeness theorem.
- Decidability of theories.
- Gödel’s incompleteness theorem.
Compactness Theorem

Recall: $\Sigma$ is **satisfiable** if it has a model, i.e. there exists $D$ s.t. $D \models \varphi$, for all $\varphi \in \Sigma$.

**Theorem (Compactness Theorem)**

*If every finite subset of $\Sigma$ is satisfiable, then $\Sigma$ is satisfiable.*

Short: if $\Sigma$ is finitely satisfiable\(^1\), then it is satisfiable.

Considered to be the most important theorem in Mathematical Logic.

\(^1\)Don’t confuse with saying “$\Sigma$ has a finite model”!
Compactness Theorem - Alternative Formulation

The following is equivalent to the Compactness Theorem:

**Theorem**

If $\Sigma \models \varphi$ then there exists a finite subset $\Sigma_{\text{fin}} \subseteq \Sigma$ s.t. $\Sigma_{\text{fin}} \models \varphi$.

Proof: assume Compactness holds, and assume $\Sigma \models \varphi$. If $\Sigma_{\text{fin}} \not\models \varphi$ for any finite subset, then the set $\Sigma \cup \{\neg \varphi\}$ is finitely satisfiable, hence it is satisfiable, contradiction.

In the other direction, let $\Sigma$ be finitely satisfiable. If $\Sigma$ is not satisfiable, then $\Sigma \models F$, hence there is a finite subset s.t. $\Sigma_{\text{fin}} \models F$, contradicting the fact that $\Sigma_{\text{fin}}$ has a model.
Compactness Theorem - Alternative Formulation

The following is equivalent to the Compactness Theorem:

**Theorem**

If $\Sigma \models \varphi$ then there exists a finite subset $\Sigma_{\text{fin}} \subseteq \Sigma$ s.t. $\Sigma_{\text{fin}} \models \varphi$.

**Proof:** assume Compactness holds, and assume $\Sigma \models \varphi$. If $\Sigma_{\text{fin}} \not\models \varphi$ for any finite subset, then the set $\Sigma \cup \{\neg \varphi\}$ is finitely satisfiable, hence it is satisfiable, contradiction.

In the other direction, let $\Sigma$ be finitely satisfiable. If $\Sigma$ is not satisfiable, then $\Sigma \models F$, hence there is a finite subset s.t. $\Sigma_{\text{fin}} \models F$, contradicting the fact that $\Sigma_{\text{fin}}$ has a model.
Compactness Theorem - Alternative Formulation

The following is equivalent to the Compactness Theorem:

**Theorem**

*If* \( \Sigma \models \varphi *then there exists a finite subset* \( \Sigma_{\text{fin}} \subseteq \Sigma *s.t.* \( \Sigma_{\text{fin}} \models \varphi *.*

Proof: assume Compactness holds, and assume \( \Sigma \models \varphi *.* If \( \Sigma_{\text{fin}} \not\models \varphi *for any finite subset, then the set \( \Sigma \cup \{\neg \varphi\} *is finitely satisfiable, hence it is satisfiable, contradiction.*

In the other direction, let \( \Sigma *be finitely satisfiable. If \( \Sigma *is not satisfiable, then \( \Sigma \models F *,** hence there is a finite subset s.t. \( \Sigma_{\text{fin}} \models F *,** contradicting the fact that \( \Sigma_{\text{fin}} *has a model.*
Warmup: The Propositional Case

Let $\Sigma$ be a set of Boolean formulas, a.k.a. Propositional formulas.

**Theorem (Compactness for Propositional Logic)**

*If every finite subset of $\Sigma$ is satisfiable, then $\Sigma$ is satisfiable.*

Application: $G = (V, E)$ is an infinite graph s.t. every finite subgraph is 3-colorable. Prove: $G$ is 3-colorable.

Boolean Variables: $\{R_i, G_i, B_i \mid i \in V\}$ (“$i$ is colored Red/Green/Blue”).

$$\Sigma = \{R_i \lor G_i \lor B_i \mid i \in V\}$$

$\cup \{\neg R_i \lor \neg R_j \mid (i, j) \in E\}$ adjacent nodes get different colors

$\cup \{\neg G_i \lor \neg G_j \mid (i, j) \in E\}$

$\cup \{\neg B_i \lor \neg B_j \mid (i, j) \in E\}$

Every finite subset of $\Sigma$ is satisfiable, hence so is $\Sigma$. 
Warmup: The Propositional Case

Let $\Sigma$ be a set of Boolean formulas, a.k.a. Propositional formulas.

**Theorem (Compactness for Propositional Logic)**

*If every finite subset of $\Sigma$ is satisfiable, then $\Sigma$ is satisfiable.*

Application: $G = (V, E)$ is an infinite graph s.t. every finite subgraph is 3-colorable. Prove: $G$ is 3-colorable.

Boolean Variables: $\{R_i, G_i, B_i \mid i \in V\}$ (“$i$ is colored Red/Green/Blue”).

$$
\Sigma = \{R_i \lor G_i \lor B_i \mid i \in V\} \quad \text{every node gets some color}
\cup \{\neg R_i \lor \neg R_j \mid (i, j) \in E\} \quad \text{adjacent nodes get different colors}
\cup \{\neg G_i \lor \neg G_j \mid (i, j) \in E\}
\cup \{\neg B_i \lor \neg B_j \mid (i, j) \in E\}
$$

Every finite subset of $\Sigma$ is satisfiable, hence so is $\Sigma$. 
Warmup: The Propositional Case

Let $\Sigma$ be a set of Boolean formulas, a.k.a. Propositional formulas.

**Theorem (Compactness for Propositional Logic)**

*If every finite subset of $\Sigma$ is satisfiable, then $\Sigma$ is satisfiable.*

Application: $G = (V, E)$ is an infinite graph s.t. every finite subgraph is 3-colorable. Prove: $G$ is 3-colorable.

Boolean Variables: $\{R_i, G_i, B_i \mid i \in V\}$ ("$i$ is colored Red/Green/Blue").

$$\Sigma = \{R_i \lor G_i \lor B_i \mid i \in V\}$$  
$$\cup \{\neg R_i \lor \neg R_j \mid (i, j) \in E\}$$  
$$\cup \{\neg G_i \lor \neg G_j \mid (i, j) \in E\}$$  
$$\cup \{\neg B_i \lor \neg B_j \mid (i, j) \in E\}$$

Every finite subset of $\Sigma$ is satisfiable, hence so is $\Sigma$. 
Warmup: The Propositional Case

Let $\Sigma$ be a set of Boolean formulas, a.k.a. Propositional formulas.

**Theorem (Compactness for Propositional Logic)**

*If every finite subset of $\Sigma$ is satisfiable, then $\Sigma$ is satisfiable.*

Application: $G = (V, E)$ is an infinite graph s.t. every finite subgraph is 3-colorable. Prove: $G$ is 3-colorable.

Boolean Variables: $\{R_i, G_i, B_i \mid i \in V\}$ (“$i$ is colored Red/Green/Blue”).

$$
\Sigma = \{R_i \lor G_i \lor B_i \mid i \in V\}
\cup \{\neg R_i \lor \neg R_j \mid (i, j) \in E\}
\cup \{\neg G_i \lor \neg G_j \mid (i, j) \in E\}
\cup \{\neg B_i \lor \neg B_j \mid (i, j) \in E\}
$$

Every finite subset of $\Sigma$ is satisfiable, hence so is $\Sigma$. 
Warmup: The Propositional Case

Let $\Sigma$ be a set of Boolean formulas, a.k.a. Propositional formulas.

**Theorem (Compactness for Propositional Logic)**

*If every finite subset of $\Sigma$ is satisfiable, then $\Sigma$ is satisfiable.*

Application: $G = (V, E)$ is an infinite graph s.t. every finite subgraph is 3-colorable. Prove: $G$ is 3-colorable.

Boolean Variables: $\{R_i, G_i, B_i \mid i \in V\}$ ("$i$ is colored Red/Green/Blue").

\[
\Sigma = \{R_i \lor G_i \lor B_i \mid i \in V\} \\
\lor \{\neg R_i \lor \neg R_j \mid (i, j) \in E\} \quad \text{every node gets some color} \\
\lor \{\neg G_i \lor \neg G_j \mid (i, j) \in E\} \quad \text{adjacent nodes get different colors} \\
\lor \{\neg B_i \lor \neg B_j \mid (i, j) \in E\}
\]

Every finite subset of $\Sigma$ is satisfiable, hence so is $\Sigma$. 
Warmup: The Propositional Case

Two steps:

- Extend $\Sigma$ to $\bar{\Sigma}$ that is both complete and finitely satisfiable.
- Use the Inductive Structure of a complete and finite satisfiable set.
Step 1: Extend $\Sigma$ to a complete $\bar{\Sigma}$

Enumerate all formulas $\varphi_1, \varphi_2, \ldots$, and define:

$$
\Sigma_0 = \Sigma \quad \Sigma_{i+1} = \begin{cases} 
\Sigma_i \cup \{\varphi_i\} & \text{if } \Sigma_i \cup \{\varphi_i\} \text{ is finitely satisfiable} \\
\Sigma_i \cup \{\neg \varphi_i\} & \text{if } \Sigma_i \cup \{\neg \varphi_i\} \text{ is finitely satisfiable}
\end{cases}
$$

One of the two cases above must hold, because, otherwise both $\Sigma_i \cup \{\varphi_i\}$ and $\Sigma_i \cup \{\neg \varphi_i\}$ are finitely UNSAT, then $\Sigma_{\text{fin}} \cup \{\varphi_i\}$ and $\Sigma_{\text{fin}}' \cup \{\neg \varphi_i\}$ are UNSAT for $\Sigma_{\text{fin}}$, $\Sigma_{\text{fin}}' \subseteq \Sigma_i$, hence $\Sigma_{\text{fin}} \cup \Sigma_{\text{fin}}'$ is UNSAT, contradiction.

Then $\bar{\Sigma} \overset{\text{def}}{=} \bigcup_i \Sigma_i$ is complete and finitely satisfiable
Step 2: Inductive Structure of a Complete Set

Lemma

If $\bar{\Sigma}$ is a complete, and finitely satisfiable set, then:

1. $\varphi \land \psi \in \bar{\Sigma}$ iff $\varphi, \psi \in \bar{\Sigma}$.
2. $\varphi \lor \psi \in \bar{\Sigma}$ iff $\varphi \in \bar{\Sigma}$ or $\psi \in \bar{\Sigma}$.
3. $\neg \varphi \in \bar{\Sigma}$ iff $\varphi \notin \bar{\Sigma}$.

Proof in class

To prove Compactness Theorem for Propositional Logic, define this model:

$$\theta(X) \overset{\text{def}}{=} 1 \text{ if } X \in \bar{\Sigma}$$

$$\theta(X) \overset{\text{def}}{=} 0 \text{ if } X \notin \bar{\Sigma}$$

Then $\theta(\varphi) = 1$ iff $\varphi \in \bar{\Sigma}$ (proof by induction on $\varphi$).
Hence $\theta$ is a model for $\bar{\Sigma}$, and thus for $\Sigma$. 
Step 2: Inductive Structure of a Complete Set

Lemma

If $\bar{\Sigma}$ is a complete, and finitely satisfiable set, then:

- $\varphi \land \psi \in \bar{\Sigma}$ iff $\varphi, \psi \in \bar{\Sigma}$.
- $\varphi \lor \psi \in \bar{\Sigma}$ iff $\varphi \in \bar{\Sigma}$ or $\psi \in \bar{\Sigma}$.
- $\neg \varphi \in \bar{\Sigma}$ iff $\varphi \not\in \bar{\Sigma}$.

Proof in class

To prove Compactness Theorem for Propositional Logic, define this model:

$$\theta(X) \overset{\text{def}}{=} 1 \text{ if } X \in \bar{\Sigma}$$

$$\theta(X) \overset{\text{def}}{=} 0 \text{ if } X \notin \bar{\Sigma}$$

Then $\theta(\varphi) = 1$ iff $\varphi \in \bar{\Sigma}$ (proof by induction on $\varphi$).
Hence $\theta$ is a model for $\bar{\Sigma}$, and thus for $\Sigma$. 

Dan Suciu

Finite Model Theory – Unit 1

Spring 2018 32 / 80
Proof of the Compactness Theorem for FO

In addition to the propositional case, we need to handle $\exists$

$\Sigma$ is *witness-complete* if, for all $\exists x(\varphi) \in \Sigma$, there is some $c$ s.t. $\varphi[c/x] \in \Sigma$.

Extend $\Sigma$ to a complete and witness-complete set $\bar{\Sigma}$, by adding countably many new constants $c_1, c_2, \ldots$ proof in class.

Define a model $D$ for $\bar{\Sigma}$ as follows:

- Its domain $D$ consists of all terms$^2$.
- For each relation $R$, $R^D \overset{\text{def}}{=} \{ (t_1, \ldots, t_k) \mid R(t_1, \ldots, t_k) \in \bar{\Sigma} \}$.
- Similarly for a function $f$.

Check this is a model of $\bar{\Sigma}$ (by showing $D \models \varphi$ iff $\varphi \in \bar{\Sigma}$), hence of $\Sigma$.

---

$^2$ Up to the equivalence defined by $t_1 = t_2 \in \bar{\Sigma}$. 
Proof of the Compactness Theorem for FO

In addition to the propositional case, we need to handle $\exists$

$\Sigma$ is *witness-complete* if, for all $\exists x (\varphi) \in \Sigma$, there is some $c$ s.t. $\varphi[c/x] \in \Sigma$.

Extend $\Sigma$ to a complete *and* witness-complete set $\tilde{\Sigma}$, by adding countably many new constants $c_1, c_2, \ldots$ *proof in class*

Define a model $D$ for $\tilde{\Sigma}$ as follows:

- Its domain $D$ consists of all terms$^2$.
- For each relation $R$, $R^D \overset{\text{def}}{=} \{(t_1, \ldots, t_k) \mid R(t_1, \ldots, t_k) \in \tilde{\Sigma}\}$.
- Similarly for a function $f$.

Check this is a model of $\tilde{\Sigma}$ (by showing $D \models \varphi$ iff $\varphi \in \tilde{\Sigma}$), hence of $\Sigma$.

---

$^2$Up to the equivalence defined by $t_1 = t_2 \in \tilde{\Sigma}$. 
Proof of the Compactness Theorem for FO

In addition to the propositional case, we need to handle $\exists$

$\Sigma$ is *witness-complete* if, for all $\exists x(\varphi) \in \Sigma$, there is some $c$ s.t. $\varphi[c/x] \in \Sigma$.

Extend $\Sigma$ to a complete *and* witness-complete set $\bar{\Sigma}$, by adding countably many new constants $c_1, c_2, \ldots$ proof in class.

Define a model $D$ for $\bar{\Sigma}$ as follows:

- Its domain $D$ consists of all terms$^2$.
- For each relation $R$, $R^D \overset{\text{def}}{=} \{(t_1, \ldots, t_k) \mid R(t_1, \ldots, t_k) \in \bar{\Sigma}\}$.
- Similarly for a function $f$.

Check this is a model of $\bar{\Sigma}$ (by showing $D \models \varphi$ iff $\varphi \in \bar{\Sigma}$), hence of $\Sigma$.

---

$^2$Up to the equivalence defined by $t_1 = t_2 \in \bar{\Sigma}$.
Discussion

- Compactness Theorem is considered the most important theorem in Mathematical Logic.

- Our discussion was restricted to a finite vocabulary $\sigma$, but compactness holds for any vocabulary; e.g. think of having infinitely many constants $c$.

- Gödel proved compactness as a simple consequence of his completeness theorem.

- We will later prove Gödel’s completeness following a similar proof as for compactness.
Application of the Compactness Theorem

Can we say in FO “the world is infinite”? Or “the world is finite”? 

- Find a set of sentences $\Lambda$ whose models are precisely the infinite structures.

- Find a set of sentences $\Sigma$ whose models are precisely the finite structures.
Application of the Compactness Theorem

Can we say in FO “the world is infinite”? Or “the world is finite”?

- Find a set of sentences $\Lambda$ whose models are precisely the infinite structures.
  \[ \Lambda = \{ \lambda_1, \lambda_2, \ldots \} \]  where $\lambda_n$ says “there are $\geq n$ elements”:
  \[ \lambda_n = \exists x_1 \cdots \exists x_n \bigwedge_{i<j} (x_i \neq x_j) \]

- Find a set of sentences $\Sigma$ whose models are precisely the finite structures.
Application of the Compactness Theorem

Can we say in FO “the world is infinite”? Or “the world is finite”?

- Find a set of sentences $\Lambda$ whose models are precisely the infinite structures.
  $\Lambda = \{\lambda_1, \lambda_2, \ldots\}$ where $\lambda_n$ says “there are $\geq n$ elements”:
  \[
  \lambda_n = \exists x_1 \ldots \exists x_n \bigwedge_{i<j} (x_i \neq x_j)
  \]

- Find a set of sentences $\Sigma$ whose models are precisely the finite structures.
  Impossible! If we could, then $\Sigma \cup \Lambda$ were finitely satisfiable, hence satisfiable, contradiction.
Löwenheim-Skolem Theorem

Suppose the vocabulary $\sigma$ is finite.

**Theorem (Löwenheim-Skolem)**

*If $\Sigma$ admits an infinite model, then it admits a countable model.*

In other words, we can say “the world is infinite”, but we can’t say how big it is.
If there is a bijection $f : A \to B$ then we say that $A, B$ are \textit{equipotent}, or \textit{equipollent}, or \textit{equinumerous}, and write $A \simeq B$.

We write $|A|$ for the equivalence class of $A$ under $\simeq$.

**Definition**

A \textit{cardinal number} is an equivalence class $|A|$.
We write $|A| \leq |B|$ if there exists an injective function $A \to B$; equivalently, if there exists a surjective function $B \to A$. 
Background: Cardinal Numbers

- 4 is a cardinal number, why?
Background: Cardinal Numbers

• 4 is a cardinal number, why? The equivalence class of \{a, b, c, d\}.
Background: Cardinal Numbers

- 4 is a cardinal number, *why?* The equivalence class of \{a, b, c, d\}.
- 4 < 7, *why?*
Background: Cardinal Numbers

- **4** is a cardinal number, **why?** The equivalence class of \{a, b, c, d\}.

- **4 < 7, why?** \{a, b, c, d\} → \{x, y, z, u, v, w, m\}: \(a \mapsto x, \ b \mapsto y\) etc.

\(\aleph_0\) is the infinite countable cardinal; equivalence class of \(N\).

\(c\) is the cardinality of the continuum; equivalence class of \(R\).

What is the cardinality of the even numbers \{0, 2, 4, 6, ...\}? \(\aleph_0\).

What is the cardinality of \([0, 1]\)? \(c\).

What is the cardinality of \(Q\)? \(\aleph_0\).

Is there a cardinal number between \(\aleph_0\) and \(c\)? Either yes or no! (Recall Logicomix!)

What is the cardinality of the set of sentences over a finite vocabulary? \(\aleph_0\).
Background: Cardinal Numbers

- **4** is a cardinal number, **why?** The equivalence class of \( \{a, b, c, d\} \).
- **4 < 7, why?** \( \{a, b, c, d\} \rightarrow \{x, y, z, u, v, w, m\} : a \mapsto x, \ b \mapsto y \) etc.
- \( \aleph_0 \) is the **infinite countable cardinal**; equivalence class of \( \mathbb{N} \).
Background: Cardinal Numbers

- **4** is a cardinal number, **why?** The equivalence class of \{a, b, c, d\}.
- **4 < 7, why?** \{a, b, c, d\} → \{x, y, z, u, v, w, m\}: a → x, b → y etc.
- \(\aleph_0\) is the **infinite countable cardinal**; equivalence class of \(\mathbb{N}\).
- \(\mathfrak{c}\) is the **cardinality of the continuum**; equivalence class of \(\mathbb{R}\).
Background: Cardinal Numbers

- **4** is a cardinal number, *why?* The equivalence class of \(\{a, b, c, d\}\).
- **4 < 7, why?** \(\{a, b, c, d\} \rightarrow \{x, y, z, u, v, w, m\}: a \mapsto x, b \mapsto y\) etc.
- \(\aleph_0\) is the *infinite countable cardinal*; equivalence class of \(\mathbb{N}\).
- \(\mathfrak{c}\) is the *cardinality of the continuum*; equivalence class of \(\mathbb{R}\).
- What is the cardinality of the even numbers \(\{0, 2, 4, 6, \ldots\}\)?
Background: Cardinal Numbers

- 4 is a cardinal number, why? The equivalence class of \( \{a, b, c, d\} \).
- \( 4 < 7 \), why? \( \{a, b, c, d\} \rightarrow \{x, y, z, u, v, w, m\} \): \( a \mapsto x \), \( b \mapsto y \) etc.
- \( \aleph_0 \) is the *infinite countable cardinal*; equivalence class of \( \mathbb{N} \).
- \( \mathfrak{c} \) is the *cardinality of the continuum*; equivalence class of \( \mathbb{R} \).
- What is the cardinality of the even numbers \( \{0, 2, 4, 6, \ldots\} \)?
- \( \aleph_0 \).
Background: Cardinal Numbers

- 4 is a cardinal number, why? The equivalence class of \( \{a, b, c, d\} \).
- 4 < 7, why? \( \{a, b, c, d\} \rightarrow \{x, y, z, u, v, w, m\}: a \mapsto x, b \mapsto y \) etc.
- \( \aleph_0 \) is the infinite countable cardinal; equivalence class of \( \mathbb{N} \).
- \( c \) is the cardinality of the continuum; equivalence class of \( \mathbb{R} \).
- What is the cardinality of the even numbers \( \{0, 2, 4, 6, \ldots\} \)? \( \aleph_0 \).
- What is the cardinality of \([0, 1]\)?
Background: Cardinal Numbers

- 4 is a cardinal number, why? The equivalence class of \( \{a, b, c, d\} \).
- 4 < 7, why? \( \{a, b, c, d\} \rightarrow \{x, y, z, u, v, w, m\}: a \mapsto x, b \mapsto y \) etc.
- \( \aleph_0 \) is the infinite countable cardinal; equivalence class of \( \mathbb{N} \).
- \( c \) is the cardinality of the continuum; equivalence class of \( \mathbb{R} \).
- What is the cardinality of the even numbers \( \{0, 2, 4, 6, \ldots\} \)? \( \aleph_0 \).
- What is the cardinality of \([0, 1]\)? \( c \).
Background: Cardinal Numbers

- 4 is a cardinal number, why? The equivalence class of \( \{a, b, c, d\} \).
- 4 < 7, why? \( \{a, b, c, d\} \rightarrow \{x, y, z, u, v, w, m\} \): \( a \mapsto x, \ b \mapsto y \) etc.
- \( \aleph_0 \) is the *infinite countable cardinal*; equivalence class of \( \mathbb{N} \).
- \( c \) is the *cardinality of the continuum*; equivalence class of \( \mathbb{R} \).
- What is the cardinality of the even numbers \( \{0, 2, 4, 6, \ldots\} \)? \( \aleph_0 \).
- What is the cardinality of \([0, 1]\)? \( c \).
- What is the cardinality of \( \mathbb{Q} \)?
Background: Cardinal Numbers

- 4 is a cardinal number, why? The equivalence class of \( \{a, b, c, d\} \).
- 4 < 7, why? \( \{a, b, c, d\} \rightarrow \{x, y, z, u, v, w, m\} \): \( a \mapsto x, b \mapsto y \) etc.
- \( \aleph_0 \) is the infinite countable cardinal; equivalence class of \( \mathbb{N} \).
- \( c \) is the cardinality of the continuum; equivalence class of \( \mathbb{R} \).
- What is the cardinality of the even numbers \( \{0, 2, 4, 6, \ldots\} \)？ \( \aleph_0 \).
- What is the cardinality of \([0, 1]\)? \( c \).
- What is the cardinality of \( \mathbb{Q} \)? \( \aleph_0 \).
Background: Cardinal Numbers

- 4 is a cardinal number, why? The equivalence class of \{a, b, c, d\}.
- 4 < 7, why? \{a, b, c, d\} → \{x, y, z, u, v, w, m\}: \ a \mapsto x, \ b \mapsto y \ etc.
- \aleph_0 \ is the \textit{infinite countable cardinal}; equivalence class of \mathbb{N}.
- \mathfrak{c} \ is the \textit{cardinality of the continuum}; equivalence class of \mathbb{R}.
- What is the cardinality of the even numbers \{0, 2, 4, 6, \ldots\}? \aleph_0.
- What is the cardinality of [0, 1]? \mathfrak{c}.
- What is the cardinality of \mathbb{Q}? \aleph_0
- Is there a cardinal number between \aleph_0 and \mathfrak{c}?
Background: Cardinal Numbers

- 4 is a cardinal number, why? The equivalence class of \( \{a, b, c, d\} \).
- 4 < 7, why? \( \{a, b, c, d\} \to \{x, y, z, u, v, w, m\}: a \mapsto x, b \mapsto y \) etc.
- \( \aleph_0 \) is the \textit{infinite countable cardinal}; equivalence class of \( \mathbb{N} \).
- \( c \) is the \textit{cardinality of the continuum}; equivalence class of \( \mathbb{R} \).
- What is the cardinality of the even numbers \( \{0, 2, 4, 6, \ldots\} \)? \( \aleph_0 \).
- What is the cardinality of \( [0, 1] \)? \( c \).
- What is the cardinality of \( \mathbb{Q} \)? \( \aleph_0 \).
- Is there a cardinal number between \( \aleph_0 \) and \( c \)? Either yes or no! (Recall Logicomix!)
Background: Cardinal Numbers

- 4 is a cardinal number, *why?* The equivalence class of \( \{a, b, c, d\} \).
- 4 < 7, *why?* \( \{a, b, c, d\} \rightarrow \{x, y, z, u, v, w, m\} : a \mapsto x, b \mapsto y \) etc.
- \( \aleph_0 \) is the *infinite countable cardinal*; equivalence class of \( \mathbb{N} \).
- \( c \) is the *cardinality of the continuum*; equivalence class of \( \mathbb{R} \).
- What is the cardinality of the even numbers \( \{0, 2, 4, 6, \ldots\} \)? \( \aleph_0 \).
- What is the cardinality of \( [0, 1] \)? \( c \).
- What is the cardinality of \( \mathbb{Q} \)? \( \aleph_0 \)
- Is there a cardinal number between \( \aleph_0 \) and \( c \)? Either yes or no! (Recall Logicomix!)
- What is the cardinality of the set of sentences over a finite vocabulary?
Background: Cardinal Numbers

- 4 is a cardinal number, why? The equivalence class of \( \{a, b, c, d\} \).
- 4 < 7, why? \( \{a, b, c, d\} \rightarrow \{x, y, z, u, v, w, m\}: a \mapsto x, b \mapsto y \) etc.
- \( \aleph_0 \) is the *infinite countable cardinal*; equivalence class of \( \mathbb{N} \).
- \( c \) is the *cardinality of the continuum*; equivalence class of \( \mathbb{R} \).
- What is the cardinality of the even numbers \( \{0, 2, 4, 6, \ldots\} \)? \( \aleph_0 \).
- What is the cardinality of \([0, 1]\)? \( c \).
- What is the cardinality of \( \mathbb{Q} \)? \( \aleph_0 \).
- Is there a cardinal number between \( \aleph_0 \) and \( c \)? Either yes or no! (Recall Logicomix!)
- What is the cardinality of the set of sentences over a finite vocabulary? \( \aleph_0 \).
Löwenheim-Skolem Theorem: Proof

Suppose the vocabulary $\sigma$ is finite or countable.

**Theorem**

*If $\Sigma$ admits an infinite model, then it admits a countable model.*

**Proof in four steps:**

- Write each sentence $\varphi \in \Sigma$ in prenex-normal form: $(\forall|\exists)^* \psi$.
- “Skolemize” $\Sigma$: replace each $\exists$ with a fresh “Skolem” function $f$, e.g.
  $$\forall x \exists y \forall z \exists u(\varphi) \rightarrow \forall x \forall z (\varphi[f_1(x)/y, f_2(x, z)/u])$$

  Let $\Sigma'$ be the set of Skolemized sentences.
- Property of Skolemization: $\Sigma$ satisfiable iff $\Sigma'$ satisfiable. In class
- Proof of Löwenheim-Skolem. Let $D \models \Sigma$; then $D \models \Sigma'$ (by interpreting the Skolem functions appropriately).
- Let: $D_0$ be any countable subset of $D$.
  $$D_{i+1} = \{ f^D(d_1, \ldots, d_k) \mid d_1, \ldots, d_k \in D_i, f \in \sigma \}.$$ 
  Then $\bigcup_i D_i$ is countable and a model of $\Sigma'$ why?.
Löwenheim-Skolem Theorem: Proof

Suppose the vocabulary $\sigma$ is finite or countable.

**Theorem**

*If $\Sigma$ admits an infinite model, then it admits a countable model.*

Proof in four steps:

- Write each sentence $\varphi \in \Sigma$ in prenex-normal form: $(\forall \exists)^* \psi$.
- "Skolemize" $\Sigma$: replace each $\exists$ with a fresh "Skolem" function $f$, e.g.

$$\forall x \exists y \forall z \exists u (\varphi) \rightarrow \forall x \forall z (\varphi[f_1(x)/y, f_2(x, z)/u])$$

Let $\Sigma'$ be the set of Skolemized sentences.
- Property of Skolemization: $\Sigma$ satisfiable iff $\Sigma'$ satisfiable. In class
- Proof of Löwenheim-Skolem. Let $D \models \Sigma$; then $D \models \Sigma'$ (by interpreting the Skolem functions appropriately).
- Let: $D_0$ be any countable subset of $D$,

$$D_{i+1} = \{ f^D (d_1, \ldots, d_k) \mid d_1, \ldots, d_k \in D_i, f \in \sigma \}.$$  

Then $\bigcup_i D_i$ is countable and a model of $\Sigma'$ why?.
Löwenheim-Skolem Theorem: Proof

Suppose the vocabulary $\sigma$ is finite or countable.

**Theorem**

*If $\Sigma$ admits an infinite model, then it admits a countable model.*

Proof in four steps:

- Write each sentence $\varphi \in \Sigma$ in prenex-normal form: $(\forall | \exists)^* \psi$.
- “Skolemize” $\Sigma$: replace each $\exists$ with a fresh “Skolem” function $f$, e.g.
  $$\forall x \exists y \forall z \exists u(\varphi) \rightarrow \forall x \forall z (\varphi[f_1(x)/y, f_2(x, z)/u])$$

Let $\Sigma'$ be the set of Skolemized sentences.

- Property of Skolemization: $\Sigma$ satisfiable iff $\Sigma'$ satisfiable. In class
- Proof of Löwenheim-Skolem. Let $D \models \Sigma$; then $D \models \Sigma'$ (by interpreting the Skolem functions appropriately).
- Let: $D_0$ be any countable subset of $D$, $D_{i+1} = \{ f^D(d_1, \ldots, d_k) \mid d_1, \ldots, d_k \in D_i, f \in \sigma \}$. Then $\bigcup_i D_i$ is countable and a model of $\Sigma'$ why?.
Löwenheim-Skolem Theorem: Proof

Suppose the vocabulary $\sigma$ is finite or countable.

**Theorem**

*If $\Sigma$ admits an infinite model, then it admits a countable model.*

**Proof in four steps:**

- Write each sentence $\varphi \in \Sigma$ in prenex-normal form: $(\forall|\exists)^* \psi$.
- "Skolemize" $\Sigma$: replace each $\exists$ with a fresh "Skolem" function $f$, e.g.

  $$\forall x \exists y \forall z \exists u (\varphi) \rightarrow \forall x \forall z (\varphi[f_1(x)/y, f_2(x,z)/u])$$

Let $\Sigma'$ be the set of Skolemized sentences.

- Property of Skolemization: $\Sigma$ satisfiable iff $\Sigma'$ satisfiable. **In class**
- Proof of Löwenheim-Skolem. Let $D \models \Sigma$; then $D \models \Sigma'$ (by interpreting the Skolem functions appropriately).
- Let: $D_0$ be any countable subset of $D$,

  $$D_{i+1} = \{ f^D(d_1, \ldots, d_k) \mid d_1, \ldots, d_k \in D_i, f \in \sigma \}.$$

Then $\bigcup_i D_i$ is countable and a model of $\Sigma'$ why?.

Dan Suciu

Finite Model Theory – Unit 1

Spring 2018 39 / 80
Löwenheim-Skolem Theorem: Proof

Suppose the vocabulary $\sigma$ is finite or countable.

**Theorem**

*If $\Sigma$ admits an infinite model, then it admits a countable model.*

Proof in four steps:

- Write each sentence $\varphi \in \Sigma$ in prenex-normal form: $(\forall|\exists)^* \psi$.
- "Skolemize" $\Sigma$: replace each $\exists$ with a fresh "Skolem" function $f$, e.g.

  $\forall x \exists y \forall z \exists u(\varphi) \rightarrow \forall x \forall z(\varphi[f_1(x)/y, f_2(x, z)/u])$

  Let $\Sigma'$ be the set of Skolemized sentences.

- Property of Skolemization: $\Sigma$ satisfiable iff $\Sigma'$ satisfiable. *In class*

- Proof of Löwenheim-Skolem. Let $D \models \Sigma$; then $D \models \Sigma'$ (by interpreting the Skolem functions appropriately).

- Let: $D_0$ be any countable subset of $D$,

  $D_{i+1} = \{ f^D(d_1, \ldots, d_k) \mid d_1, \ldots, d_k \in D_i, f \in \sigma \}$.

  Then $\bigcup_i D_i$ is countable and a model of $\Sigma'$ why?
Löwenheim-Skolem Theorem: Proof

Suppose the vocabulary $\sigma$ is finite or countable.

**Theorem**

*If $\Sigma$ admits an infinite model, then it admits a countable model.*

Proof in four steps:

- Write each sentence $\varphi \in \Sigma$ in prenex-normal form: $(\forall|\exists)^* \psi$.
- "Skolemize" $\Sigma$: replace each $\exists$ with a fresh "Skolem" function $f$, e.g.

$$\forall x \exists y \forall z \exists u (\varphi) \mapsto \forall x \forall z (\varphi[f_1(x)/y, f_2(x, z)/u])$$

Let $\Sigma'$ be the set of Skolemized sentences.

- Property of Skolemization: $\Sigma$ satisfiable iff $\Sigma'$ satisfiable. In class
- Proof of Löwenheim-Skolem. Let $D \models \Sigma$; then $D \models \Sigma'$ (by interpreting the Skolem functions appropriately).
- Let: $D_0$ be any countable subset of $D$,
  $$D_{i+1} = \{ f^D(d_1, \ldots, d_k) \mid d_1, \ldots, d_k \in D_i, f \in \sigma \}.$$  
Then $\cup_i D_i$ is countable and a model of $\Sigma'$ why?
Discussion

- We have assumed that $\sigma$ is finite, or countable.

- If $\sigma$ has cardinality $\kappa$, then the Löwenheim-Skolem Theorem says that there exists a model of cardinality $\kappa$.

- The *upwards* version of the Löwenheim-Skolem Theorem\(^3\) if $\Sigma$ has a model of infinite cardinality $\kappa$ and $\kappa < \kappa'$ then it also has a model of cardinality $\kappa'$.

Proof: add to $\sigma$ constants $c_k, k \in \kappa'$, add axioms $c_i \neq c_j$ for $i \neq j$. By compactness there is a model; then we repeat the previous proof of Löwenheim-Skolem.

\(^3\)Called: Löwenheim-Skolem-Tarski theorem.
The Los-Vaught Test

Simple observation: if $D_1, D_2$ are isomorphic then $\text{Th}(D_1) = \text{Th}(D_2)$.

Call $\Sigma \aleph_0$-categorical if any two countable models of $\Sigma$ are isomorphic.

Theorem (Los-Vaught Test)

If $\Sigma$ has no finite models and is $\aleph_0$ categorical then it is complete.

Proof. Suppose otherwise: there exists $\varphi$ s.t. $\Sigma \not\models \neg \varphi$ and $\Sigma \not\models \varphi$. Then:

- $\Sigma \cup \{\varphi\}$ has a model $D_1$; assume it is countable why can we?
- $\Sigma \cup \{\neg \varphi\}$ has a model $D_2$; assume it is countable.
- Then $D_1, D_2$ are isomorphic.
- Contradiction because $D_1 \models \varphi$ and $D_2 \models \neg \varphi$. 
The Los-Vaught Test

Simple observation: if $D_1, D_2$ are isomorphic then $\text{Th}(D_1) = \text{Th}(D_2)$.

Call $\Sigma \aleph_0$-categorical if any two countable models of $\Sigma$ are isomorphic.

**Theorem (Los-Vaught Test)**

*If $\Sigma$ has no finite models and is $\aleph_0$ categorical then it is complete.*

Proof. Suppose otherwise: there exists $\varphi$ s.t. $\Sigma \not\models \neg \varphi$ and $\Sigma \not\models \varphi$. Then:

- $\Sigma \cup \{ \varphi \}$ has a model $D_1$; assume it is countable (why can we?).
- $\Sigma \cup \{ \neg \varphi \}$ has a model $D_2$; assume it is countable.
- Then $D_1, D_2$ are isomorphic.
- Contradiction because $D_1 \models \varphi$ and $D_2 \models \neg \varphi$. 
The Los-Vaught Test

Simple observation: if $D_1, D_2$ are isomorphic then $\text{Th}(D_1) = \text{Th}(D_2)$.

Call $\Sigma$ $\aleph_0$-categorical if any two countable models of $\Sigma$ are isomorphic.

**Theorem (Los-Vaught Test)**

*If $\Sigma$ has no finite models and is $\aleph_0$ categorical then it is complete.*

Proof. Suppose otherwise: there exists $\varphi$ s.t. $\Sigma \not\models \neg \varphi$ and $\Sigma \not\models \varphi$. Then:

- $\Sigma \cup \{\varphi\}$ has a model $D_1$; assume it is countable why can we?
- $\Sigma \cup \{\neg \varphi\}$ has a model $D_2$; assume it is countable.
- Then $D_1, D_2$ are isomorphic.
- Contradiction because $D_1 \models \varphi$ and $D_2 \models \neg \varphi$. 

Dan Suciu
Finite Model Theory – Unit 1
Spring 2018
The Los-Vaught Test

Simple observation: if \( D_1, D_2 \) are isomorphic then \( \text{Th}(D_1) = \text{Th}(D_2) \).

Call \( \Sigma \) \( \aleph_0 \)-categorical if any two countable models of \( \Sigma \) are isomorphic.

**Theorem (Los-Vaught Test)**

*If \( \Sigma \) has no finite models and is \( \aleph_0 \) categorical then it is complete.*

Proof. Suppose otherwise: there exists \( \varphi \) s.t. \( \Sigma \not\models \neg \varphi \) and \( \Sigma \not\models \varphi \). Then:

- \( \Sigma \cup \{ \varphi \} \) has a model \( D_1 \); assume it is countable *why can we?*
- \( \Sigma \cup \{ \neg \varphi \} \) has a model \( D_2 \); assume it is countable.
- Then \( D_1, D_2 \) are isomorphic.
- Contradiction because \( D_1 \models \varphi \) and \( D_2 \models \neg \varphi \).
The Los-Vaught Test

Simple observation: if $D_1, D_2$ are isomorphic then $\text{Th}(D_1) = \text{Th}(D_2)$.

Call $\Sigma$ $\aleph_0$-categorical if any two countable models of $\Sigma$ are isomorphic.

**Theorem (Los-Vaught Test)**

*If $\Sigma$ has no finite models and is $\aleph_0$-categorical then it is complete.*

Proof. Suppose otherwise: there exists $\varphi$ s.t. $\Sigma \not\models \neg \varphi$ and $\Sigma \not\models \varphi$. Then:

- $\Sigma \cup \{\varphi\}$ has a model $D_1$; assume it is countable why can we?
- $\Sigma \cup \{\neg \varphi\}$ has a model $D_2$; assume it is countable.
- Then $D_1, D_2$ are isomorphic.
- Contradiction because $D_1 \models \varphi$ and $D_2 \models \neg \varphi$. 
Application of the Los-Vaught Test

The *theory of dense linear orders without endpoints* is complete.

\[
\forall x \forall y \neg((x < y) \land (y < x)) \\
\forall x \forall y((x < y) \lor (x = y) \lor (y < x)) \\
\forall x \forall y \forall z((x < y) \land (y < z) \rightarrow (x < z)) \\
\text{Dense: } \forall x \forall y(x < y \rightarrow \exists v(x < v < y)) \\
\text{W/o Endpoints: } \forall x \exists u \exists w (u < x < w)
\]

Note: just “total order” is not complete!

Proof: we apply the Los-Vaught test.

Let \( A, B \) be countable models. Construct inductively \( A_i \subseteq A, B_i \subseteq B \), and isomorphism \( f_i : (A_i, <) \rightarrow (B_i, <) \), using the Back and Forth argument.
Application of the Los-Vaught Test

The *theory of dense linear orders without endpoints* is complete.

\[
\begin{align*}
&\forall x \forall y \neg ((x < y) \land (y < x)) \\
&\forall x \forall y ((x < y) \lor (x = y) \lor (y < x)) \\
&\forall x \forall y \forall z ((x < y) \land (y < z) \rightarrow (x < z)) \\
&Dense: \quad \forall x \forall y (x < y \rightarrow \exists v (x < v < y)) \\
&W/o Endpoints: \quad \forall x \exists u \exists w (u < x < w)
\end{align*}
\]

Note: just “total order” is not complete!

Proof: we apply the Los-Vaught test.

Let \( A, B \) be countable models. Construct inductively \( A_i \subseteq A, B_i \subseteq B \), and isomorphism \( f_i : (A_i, <) \rightarrow (B_i, <) \), using the Back and Forth argument.
Application of the Los-Vaught Test

The theory of dense linear orders without endpoints is complete.

\[ \forall x \forall y \neg((x < y) \land (y < x)) \]

\[ \forall x \forall y ((x < y) \lor (x = y) \lor (y < x)) \]

\[ \forall x \forall y \forall z ((x < y) \land (y < z) \rightarrow (x < z)) \]

Dense: \[ \forall x \forall y (x < y \rightarrow \exists v (x < v < y)) \]

W/o Endpoints: \[ \forall x \exists u \exists w (u < x < w) \]

Note: just “total order” is not complete!

Proof: we apply the Los-Vaught test.

Let \( A, B \) be countable models. Construct inductively \( A_i \subseteq A, B_i \subseteq B \), and isomorphism \( f_i : (A_i, <) \rightarrow (B_i, <) \), using the Back and Forth argument.
The Back-and-Forth argument

\[ A = (\{a_1, a_2, \ldots\}, <), \quad B = (\{b_1, b_2, \ldots\}, <) \] are total orders w/o endpoints. Prove they are isomorphic.

\[ A_0 \overset{\text{def}}{=} \emptyset, \quad B_0 \overset{\text{def}}{=} \emptyset. \]

Assuming \((A_{i-1}, <) \cong (B_{i-1}, <)\), extend to \((A_i, <) \cong (B_i, <)\) as follows:

1. Add \(a_i\) and any \(b \in B\) s.t. \((A_{i-1} \cup \{a_i\}, <) \cong (B_{i-1} \cup \{b\})\).

2. Add \(b_i\) and any matching \(a \in A\).

Then \(A = \bigcup A_i, \quad B = \bigcup B_i\) and \((A, <) \cong (B, <)\).
The Back-and-Forth argument

\( \mathcal{A} = (\{a_1, a_2, \ldots\}, <) \), \( \mathcal{B} = (\{b_1, b_2, \ldots\}, <) \) are total orders w/o endpoints. Prove they are isomorphic.

\( A_0 \overset{\text{def}}{=} \emptyset, \ B_0 \overset{\text{def}}{=} \emptyset. \)

Assuming \( (A_{i-1}, <) \cong (B_{i-1}, <) \), extend to \( (A_i, <) \cong (B_i, <) \) as follows:

- Add \( a_i \) and any \( b \in B \) s.t. \( (A_{i-1} \cup \{a_i\}, <) \cong (B_{i-1} \cup \{b\}). \)

- Add \( b_i \) and any matching \( a \in A \).

Then \( A = \bigcup A_i, \ B = \bigcup B_i \) and \( (A, <) \cong (B, <) \).
The Back-and-Forth argument

\( A = (\{a_1, a_2, \ldots\}, <) \), \( B = (\{b_1, b_2, \ldots\}, <) \) are total orders w/o endpoints. Prove they are isomorphic.

\( A_0 \overset{\text{def}}{=} \emptyset \), \( B_0 \overset{\text{def}}{=} \emptyset \).

Assuming \((A_{i-1}, <) \cong (B_{i-1}, <)\), extend to \((A_i, <) \cong (B_i, <)\) as follows:

- Add \( a_i \) and any \( b \in B \) s.t. \((A_{i-1} \cup \{a_i\}, <) \cong (B_{i-1} \cup \{b\})\).

- Add \( b_i \) and any matching \( a \in A \).

Then \( A = \bigcup A_i \), \( B = \bigcup B_i \) and \((A, <) \cong (B, <)\).
The Back-and-Forth argument

\( A = (\{a_1, a_2, \ldots\}, <), \ B = (\{b_1, b_2, \ldots\}, <) \) are total orders w/o endpoints. Prove they are isomorphic.

\[ A_0 \overset{\text{def}}{=} \emptyset, \ B_0 \overset{\text{def}}{=} \emptyset. \]

Assuming \((A_{i-1}, <) \cong (B_{i-1}, <)\), extend to \((A_i, <) \cong (B_i, <)\) as follows:

- Add \(a_i\) and any \(b \in B\) s.t. \((A_{i-1} \cup \{a_i\}, <) \cong (B_{i-1} \cup \{b\})\).

- Add \(b_i\) and any matching \(a \in A\).

Then \( A = \bigcup A_i, \ B = \bigcup B_i \) and \((A, <) \cong (B, <)\).
The Back-and-Forth argument

$A = (\{a_1, a_2, \ldots\}, <), \ B = (\{b_1, b_2, \ldots\}, <)$ are total orders w/o endpoints. Prove they are isomorphic.

$A_0 \overset{\text{def}}{=} \emptyset, \ B_0 \overset{\text{def}}{=} \emptyset.$

Assuming $(A_{i-1}, <) \cong (B_{i-1}, <),$ extend to $(A_i, <) \cong (B_i, <)$ as follows:

- Add $a_i$ and any $b \in B$ s.t. $(A_{i-1} \cup \{a_i\}, <) \cong (B_{i-1} \cup \{b\}).$

- Add $b_i$ and any matching $a \in A.$

Then $A = \bigcup A_i, \ B = \bigcup B_i$ and $(A, <) \cong (B, <).$
Discussion

The Los-Vaught test applies to any cardinal number, as follows:

- If $\Sigma$ has no finite models and is categorical in some infinite cardinal $\kappa$ (meaning: any two models of cardinality $\kappa$ are isomorphic) then $\Sigma$ is complete.

Useful for your homework.
Recap: Three Classical Results in Model Theory

We proved:

- Compactness Theorem.
- Löwenheim-Skolem Theorem.
- Los-Vaught Test.

Next, we use them to prove Fagin’s 0/1 Law for First Order Logic.
Proof of the Zero-One Law: Plan

Zero-one Law: $\lim_{n \to \infty} \mu_n(\varphi)$ is 0 or 1, for every $\varphi$

For simplicity, assume vocabulary of graphs, i.e. only binary $E$.

- Define a set $\Sigma$ of extension axioms, $EA_{k,\Delta}$
- We prove that $\lim_{n} \mu_n(EA_{k,\Delta}) = 1$
- Hence $\Sigma$ is finitely satisfiable.
- By compactness: $\Sigma$ has a model.
- By Löwenheim-Skolem: has a countable model (called the Rado graph $R$, when undirected).
- We prove that all countable models of $\Sigma$ are isomorphic.
- By Los-Vaught: $\Sigma$ is complete.
- Then $\Sigma \models \varphi$ implies $\lim \mu_n(\varphi) = 1$ and $\Sigma \not\models \varphi$ implies $\lim \mu_n(\varphi) = 0$. 

Dan Suciu

Finite Model Theory – Unit 1

Spring 2018

46 / 80
The Extension Formulas and the Extension Axioms

For $k > 0$ denote $S_k = ([k] \times \{k\}) \cup (\{k\} \times [k])$ and $\Delta \subseteq S_k$.

\[
EF_{k,\Delta}(x_1, \ldots, x_{k-1}, x_k) = \bigwedge_{(i,j) \in \Delta} E(x_i, x_j) \land \bigwedge_{(i,j) \in S_k - \Delta} \neg E(x_i, x_j)
\]

\[
EA_{k,\Delta} = \forall x_1 \ldots \forall x_{k-1} \left( \bigwedge_{i<j<k} (x_i \neq x_j) \right) \rightarrow \exists x_k \left( \bigwedge_{i<k} (x_k \neq x_i) \land EF_{k,\Delta} \right)
\]

Intuition: we can extend the graph as prescribed by $\Delta$.

How many extension axioms are there for $k = 5$?

\[
E(x_1, x_5) \land \neg E(x_5, x_1) \land \\
E(x_2, x_5) \land E(x_5, x_2) \land \\
\neg E(x_3, x_5) \land \neg E(x_5, x_3) \land \\
\neg E(x_4, x_5) \land E(x_5, x_4) \land \\
E(x_5, x_5)
\]
The Extension Formulas and the Extension Axioms

For $k > 0$ denote $S_k = ([k] \times \{k\}) \cup (\{k\} \times [k])$ and $\Delta \subseteq S_k$.

$$EF_{k,\Delta}(x_1, \ldots, x_{k-1}, x_k) = \bigwedge_{(i,j) \in \Delta} E(x_i, x_j) \land \bigwedge_{(i,j) \in S_k - \Delta} \neg E(x_i, x_j)$$

$$EA_{k,\Delta} = \forall x_1 \ldots \forall x_{k-1}(\bigwedge_{i<j<k} (x_i \neq x_j)) \rightarrow \exists x_k(\bigwedge_{i<k} (x_k \neq x_i) \land EF_{k,\Delta})$$

Intuition: we can extend the graph as prescribed by $\Delta$.

How many extension axioms are there for $k = 5$?
The Extension Formulas and the Extension Axioms

For $k > 0$ denote $S_k = ([k] \times \{k\}) \cup (\{k\} \times [k])$ and $\Delta \subseteq S_k$.

\[
EF_{k,\Delta}(x_1, \ldots, x_{k-1}, x_k) = \bigwedge_{(i,j) \in \Delta} E(x_i, x_j) \land \bigwedge_{(i,j) \in S_k - \Delta} \neg E(x_i, x_j)
\]

\[
EA_{k,\Delta} = \forall x_1 \ldots \forall x_{k-1} \left( \bigwedge_{i<j<k} (x_i \neq x_j) \right) \rightarrow \exists x_k \left( \bigwedge_{i<k} (x_k \neq x_i) \land EF_{k,\Delta} \right)
\]

Intuition: we can extend the graph as prescribed by $\Delta$.

\[
E(x_1, x_5) \land \neg E(x_5, x_1) \land \\
E(x_2, x_5) \land E(x_5, x_2) \land \\
\neg E(x_3, x_5) \land \neg E(x_5, x_3) \land \\
\neg E(x_4, x_5) \land E(x_5, x_4) \land \\
E(x_5, x_5)
\]

How many extension axioms are there for $k = 5$?
Proof of \( \lim_{n} \mu_n(\mathit{EA}_{k,\Delta}) = 1 \)

\[
\begin{align*}
\mathit{EF}_{k,\Delta}(x_1, \ldots, x_{k-1}, x_k) &= \bigwedge_{(i,j) \in \Delta} E(x_i, x_j) \land \bigwedge_{(i,j) \in S_k - \Delta} \neg E(x_i, x_j) \\
\mathit{EA}_{k,\Delta} &= \forall x_1 \ldots \forall x_{k-1} \left( \bigwedge_{i < j < k} (x_i \neq x_j) \right) \rightarrow \exists x_k \left( \bigwedge_{i < k} (x_k \neq x_i) \land \mathit{EF}_{k,\Delta} \right)
\end{align*}
\]

\[
\mu_n(-\mathit{EA}_{k,\Delta}) = \mu_n \left( \exists x_1 \ldots \exists x_{k-1} \left( \bigwedge_{i \neq j} (x_i \neq x_j) \land \forall x_k \left( \bigwedge_{i < k} (x_k \neq x_i) \rightarrow \neg \mathit{EF}_{k,\Delta} \right) \right) \right)
\]

\[
\leq \sum_{a_1, \ldots, a_{k-1} \in [n], a_i \neq a_j} \mu_n \left( \bigwedge_{a_k \in [n] - \{a_1, \ldots, a_{k-1}\}} \neg \mathit{EF}_{k,\Delta}(a_1, \ldots, a_{k-1}, a_k) \right)
\]

\[
= \sum_{a_1, \ldots, a_{k-1} \in [n], a_i \neq a_j} \prod_{a_k \in [n] - \{a_1, \ldots, a_{k-1}\}} \mu_n(-\mathit{EF}_{k,\Delta}(a_1, \ldots, a_k)) \quad \text{why?}
\]

\[
= \sum_{a_1, \ldots, a_{k-1} \in [n], a_i \neq a_j} \prod_{a_k \in [n] - \{a_1, \ldots, a_{k-1}\}} c \quad \text{where } c = 1 - \frac{1}{2^{2k-1}} < 1
\]

\[
\leq n^{k-1} c^{n-k+1} \rightarrow 0
\]
Proof of \( \lim_n \mu_n(EA_{k,\Delta}) = 1 \)

\[
EF_{k,\Delta}(x_1, \ldots, x_{k-1}, x_k) = \bigwedge_{(i,j) \in \Delta} E(x_i, x_j) \land \bigwedge_{(i,j) \in S_k - \Delta} \neg E(x_i, x_j)
\]

\[
EA_{k,\Delta} = \forall x_1 \ldots \forall x_{k-1} \left( \bigwedge_{i < j < k} (x_i \neq x_j) \right) \rightarrow \exists x_k \left( \bigwedge_{i < k} (x_k \neq x_i) \land EF_{k,\Delta} \right)
\]

\[
\mu_n(-EA_{k,\Delta}) = \mu_n \left( \exists x_1 \ldots \exists x_{k-1} \left( \bigwedge (x_i \neq x_j) \land \forall x_k \left( \bigwedge (x_k \neq x_i) \rightarrow \neg EF_{k,\Delta} \right) \right) \right)
\]

\[
\leq \sum_{a_1,\ldots,a_{k-1} \in [n], a_i \neq a_j} \mu_n \left( \bigwedge_{a_k \in [n] - \{a_1,\ldots,a_{k-1}\}} \neg EF_{k,\Delta}(a_1,\ldots,a_{k-1},a_k) \right)
\]

\[
= \sum_{a_1,\ldots,a_{k-1} \in [n], a_i \neq a_j} \prod_{a_k \in [n] - \{a_1,\ldots,a_{k-1}\}} \mu_n(-EF_{k,\Delta}(a_1,\ldots,a_k)) \quad \text{why?}
\]

\[
= \sum_{a_1,\ldots,a_{k-1} \in [n], a_i \neq a_j} \prod_{a_k \in [n] - \{a_1,\ldots,a_{k-1}\}} c \quad \text{where } c = 1 - \frac{1}{2^{2k-1}} < 1
\]

\[
\leq n^{k-1} c^{n-k+1} \rightarrow 0
\]
Proof of \( \lim_n \mu_n(EA_{k,\Delta}) = 1 \)

\[
EF_{k,\Delta}(x_1, \ldots, x_{k-1}, x_k) = \bigwedge_{(i,j) \in \Delta} E(x_i, x_j) \land \bigwedge_{(i,j) \in S_k - \Delta} \neg E(x_i, x_j)
\]

\[
EA_{k,\Delta} = \forall x_1 \ldots \forall x_{k-1} \left( \bigwedge_{i<j<k} (x_i \neq x_j) \right) \rightarrow \exists x_k \left( \bigwedge_{i<k} (x_k \neq x_i) \land EF_{k,\Delta} \right)
\]

\[
\mu_n(\neg EA_{k,\Delta}) = \mu_n \left( \exists x_1 \ldots \exists x_{k-1} \left( \bigwedge (x_i \neq x_j) \land \forall x_k \left( \bigwedge (x_k \neq x_i) \rightarrow \neg EF_{k,\Delta} \right) \right) \right)
\]

\[
\leq \sum_{a_1,\ldots,a_{k-1} \in [n], a_i \neq a_j} \mu_n \left( \bigwedge_{a_k \in [n] - \{a_1,\ldots,a_{k-1}\}} \neg EF_{k,\Delta}(a_1, \ldots, a_{k-1}, a_k) \right)
\]

\[
= \sum_{a_1,\ldots,a_{k-1} \in [n], a_i \neq a_j} \prod_{a_k \in [n] - \{a_1,\ldots,a_{k-1}\}} \mu_n(\neg EF_{k,\Delta}(a_1, \ldots, a_k)) \quad \text{why?}
\]

\[
= \sum_{a_1,\ldots,a_{k-1} \in [n], a_i \neq a_j} \prod_{a_k \in [n] - \{a_1,\ldots,a_{k-1}\}} c \quad \text{where } c = 1 - \frac{1}{2^{2k-1}} < 1
\]

\[
\leq n^{k-1} c^{n-k+1} \rightarrow 0
\]
Proof of $\lim_n \mu_n(EA_{k,\Delta}) = 1$

$$EF_{k,\Delta}(x_1, \ldots, x_{k-1}, x_k) = \bigwedge_{(i,j)\in\Delta} E(x_i, x_j) \land \bigwedge_{(i,j)\in S_k-\Delta} \neg E(x_i, x_j)$$

$$EA_{k,\Delta} = \forall x_1 \ldots \forall x_{k-1} \left( \bigwedge_{i<j<k} (x_i \neq x_j) \right) \rightarrow \exists x_k \left( \bigwedge_{i<k} (x_k \neq x_i) \land EF_{k,\Delta} \right)$$

$$\mu_n(\neg EA_{k,\Delta}) = \mu_n \left( \exists x_1 \ldots \exists x_{k-1} \left( \bigwedge_{i<j} (x_i \neq x_j) \land \forall x_k \left( \bigwedge_{i<k} (x_k \neq x_i) \rightarrow \neg EF_{k,\Delta} \right) \right) \right)$$

\[
\leq \sum_{a_1,\ldots,a_{k-1}\in[n], a_i \neq a_j} \mu_n \left( \bigwedge_{a_k\in[n]-\{a_1,\ldots,a_{k-1}\}} \neg EF_{k,\Delta}(a_1, \ldots, a_{k-1}, a_k) \right)
\]

\[
= \sum_{a_1,\ldots,a_{k-1}\in[n], a_i \neq a_j} \prod_{a_k\in[n]-\{a_1,\ldots,a_{k-1}\}} \mu_n(\neg EF_{k,\Delta}(a_1, \ldots, a_k)) \quad \text{why?}
\]

\[
= \sum_{a_1,\ldots,a_{k-1}\in[n], a_i \neq a_j} \prod_{a_k\in[n]-\{a_1,\ldots,a_{k-1}\}} c \quad \text{where} \quad c = 1 - \frac{1}{2^{2k-1}} < 1
\]

\[
\leq n^{k-1} c^{n-k+1} \rightarrow 0
\]
Proof of \( \lim_{n} \mu_n( EA_{k,\Delta} ) = 1 \)

\[
EF_{k,\Delta}(x_1, \ldots, x_{k-1}, x_k) = \bigwedge_{(i,j)\in\Delta} E(x_i, x_j) \land \bigwedge_{(i,j)\in S_k - \Delta} \neg E(x_i, x_j)
\]

\[
EA_{k,\Delta} = \forall x_1 \ldots \forall x_{k-1} ( \bigwedge_{i<j<k} (x_i \neq x_j) ) \rightarrow \exists x_k ( \bigwedge_{i<k} (x_k \neq x_i) \land EF_{k,\Delta} )
\]

\[
\mu_n( \neg EA_{k,\Delta} ) = \mu_n( \exists x_1 \ldots \exists x_{k-1} ( \bigwedge (x_i \neq x_j) \land \forall x_k ( \bigwedge (x_k \neq x_i) \rightarrow \neg EF_{k,\Delta} ) ) )
\]

\[
\leq \sum_{a_1,\ldots,a_{k-1}\in[n],a_i\neq a_j} \mu_n \left( \bigwedge_{a_k\in[n]-\{a_1,\ldots,a_{k-1}\}} \neg EF_{k,\Delta}(a_1, \ldots, a_{k-1}, a_k) \right)
\]

\[
= \sum_{a_1,\ldots,a_{k-1}\in[n],a_i\neq a_j} \prod_{a_k\in[n]-\{a_1,\ldots,a_{k-1}\}} \mu_n( \neg EF_{k,\Delta}(a_1, \ldots, a_k) ) \quad \text{why?}
\]

\[
= \sum_{a_1,\ldots,a_{k-1}\in[n],a_i\neq a_j} \prod_{a_k\in[n]-\{a_1,\ldots,a_{k-1}\}} c \quad \text{where } c = 1 - \frac{1}{2^{2k-1}} < 1
\]

\[
\leq n^{k-1} c^{n-k+1} \rightarrow 0
\]
Proof of $\lim_n \mu_n(\text{EA}_{k,\Delta}) = 1$

$$EF_{k,\Delta}(x_1, \ldots, x_{k-1}, x_k) = \bigwedge_{(i,j) \in \Delta} E(x_i, x_j) \land \bigwedge_{(i,j) \in S_{k-\Delta}} \neg E(x_i, x_j)$$

$$\text{EA}_{k,\Delta} = \forall x_1 \ldots \forall x_{k-1} (\bigwedge_{i < j < k} (x_i \neq x_j)) \rightarrow \exists x_k (\bigwedge_{i < k} (x_k \neq x_i) \land EF_{k,\Delta})$$

$$\mu_n(\neg \text{EA}_{k,\Delta}) =\mu_n\left(\exists x_1 \ldots \exists x_{k-1} (\bigwedge (x_i \neq x_j) \land \forall x_k (\bigwedge (x_k \neq x_i) \rightarrow \neg EF_{k,\Delta})\right)$$

$$\leq \sum_{a_1,\ldots,a_{k-1} \in [n], a_i \neq a_j} \mu_n\left(\bigwedge_{a_k \in [n]-\{a_1,\ldots,a_{k-1}\}} \neg EF_{k,\Delta}(a_1,\ldots,a_{k-1},a_k)\right)$$

$$= \sum_{a_1,\ldots,a_{k-1} \in [n], a_i \neq a_j} \prod_{a_k \in [n]-\{a_1,\ldots,a_{k-1}\}} \mu_n(\neg EF_{k,\Delta}(a_1,\ldots,a_k)) \quad \text{why?}$$

$$= \sum_{a_1,\ldots,a_{k-1} \in [n], a_i \neq a_j} \prod_{a_k \in [n]-\{a_1,\ldots,a_{k-1}\}} c \quad \text{where } c = 1 - \frac{1}{2^{2k-1}} < 1$$

$$\leq n^{k-1} c^{n-k+1} \rightarrow 0$$
Extension Axioms Have a Countable Model

Let $\Sigma = \{ EA_{k,\Delta} \mid k > 0, \Delta \subseteq S_k \}$ be the set of extension axioms.

$\Sigma$ is finitely satisfiable why?

Because for all $\varphi_1, \ldots, \varphi_m \in \Sigma$, $\mu_n(\varphi_1 \land \cdots \land \varphi_m) \to 1$

Hence, when $n$ is large, there are many finite models for $\varphi_1, \ldots, \varphi_m$!

By compactness, $\Sigma$ has a model.

By Löwenheim-Skolem, $\Sigma$ has a countable model.
Extension Axioms Have a Countable Model

Let $\Sigma = \{ EA_{k,\Delta} \mid k > 0, \Delta \subseteq S_k \}$ be the set of extension axioms.

$\Sigma$ is finitely satisfiable why?

Because forall $\varphi_1, \ldots, \varphi_m \in \Sigma$, $\mu_n(\varphi_1 \land \cdots \land \varphi_m) \rightarrow 1$

Hence, when $n$ is large, there are many finite models for $\varphi_1, \ldots, \varphi_m$!

By compactness, $\Sigma$ has a model.

By L"owenheim-Skolem, $\Sigma$ has a countable model.
Extension Axioms Have a Countable Model

Let $\Sigma = \{ EA_{k,\Delta} \mid k > 0, \Delta \subseteq S_k \}$ be the set of extension axioms.

$\Sigma$ is finitely satisfiable why?

Because forall $\varphi_1, \ldots, \varphi_m \in \Sigma$, $\mu_n(\varphi_1 \land \cdots \land \varphi_m) \rightarrow 1$

Hence, when $n$ is large, there are many finite models for $\varphi_1, \ldots, \varphi_m$!

By compactness, $\Sigma$ has a model.

By Löwenheim-Skolem, $\Sigma$ has a countable model.
Extension Axioms Have a Countable Model

Let \( \Sigma = \{ EA_{k,\Delta} \mid k > 0, \Delta \subseteq S_k \} \) be the set of extension axioms.

\( \Sigma \) is finitely satisfiable why?

Because forall \( \varphi_1, \ldots, \varphi_m \in \Sigma \), \( \mu_n(\varphi_1 \land \cdots \land \varphi_m) \to 1 \)

Hence, when \( n \) is large, there are many finite models for \( \varphi_1, \ldots, \varphi_m \)!

By compactness, \( \Sigma \) has a model.

By Löwenheim-Skolem, \( \Sigma \) has a countable model.
Extension Axioms Have a Countable Model

Let \( \Sigma = \{ EA_{k, \Delta} \mid k > 0, \Delta \subseteq S_k \} \) be the set of extension axioms.

\( \Sigma \) is finitely satisfiable why?

Because for all \( \varphi_1, \ldots, \varphi_m \in \Sigma \), \( \mu_n(\varphi_1 \land \cdots \land \varphi_m) \to 1 \)

Hence, when \( n \) is large, there are many finite models for \( \varphi_1, \ldots, \varphi_m \)!

By compactness, \( \Sigma \) has a model.

By Löwenheim-Skolem, \( \Sigma \) has a countable model.
Extension Axioms have a **Unique Countable Model**

Need to prove: any two countable models $A, B$ of $\Sigma$ are isomorphic.

Will use the Back-and-Forth construction!

Let $A = \{a_1, a_2, \ldots\}$, $B = \{b_1, b_2, \ldots\}$.

By induction on $k$, construct $(A_k, E_k) \cong (B_k, E'_k)$, using the back-and-forth construction and the fact that both $A, B$ satisfy $\Sigma$.

Hence, there is a unique (up to isomorphism) countable model. Called *The Random Graph* or *Rado Graph*, $R$ for undirected graphs. See Libkin.
Extension Axioms have a **Unique Countable Model**

Need to prove: any two countable models \( A, B \) of \( \Sigma \) are isomorphic.

Will use the Back-and-Forth construction!

Let \( A = \{ a_1, a_2, \ldots \} \), \( B = \{ b_1, b_2, \ldots \} \).

By induction on \( k \), construct \( (A_k, E_k) \cong (B_k, E'_k) \), using the back-and-forth construction and the fact that both \( A, B \) satisfy \( \Sigma \).

Hence, there is a unique (up to isomorphism) countable model. Called *The Random Graph* or *Rado Graph*, \( R \) for undirected graphs. See Libkin.
Extension Axioms have a **Unique Countable Model**

Need to prove: any two countable models $A, B$ of $\Sigma$ are isomorphic.

Will use the Back-and-Forth construction!

Let $A = \{a_1, a_2, \ldots\}$, $B = \{b_1, b_2, \ldots\}$.

By induction on $k$, construct $(A_k, E_k) \cong (B_k, E'_k)$, using the back-and-forth construction and the fact that both $A, B$ satisfy $\Sigma$.

Hence, there is a unique (up to isomorphism) countable model. Called *The Random Graph* or *Rado Graph*, $R$ for undirected graphs. See Libkin.
Extension Axioms have a **Unique Countable Model**

Need to prove: any two countable models \( A, B \) of \( \Sigma \) are isomorphic.
Will use the Back-and-Forth construction!

Let \( A = \{ a_1, a_2, \ldots \} \), \( B = \{ b_1, b_2, \ldots \} \).

By induction on \( k \), construct \( (A_k, E_k) \cong (B_k, E'_k) \), using the back-and-forth construction and the fact that both \( A, B \) satisfy \( \Sigma \).

Hence, there is a unique (up to isomorphism) countable model. Called *The Random Graph* or *Rado Graph*, \( R \) for undirected graphs. See Libkin.
Proof of the Zero-One Law

Let $\varphi$ be any sentence: we’ll prove $\mu_n(\varphi)$ tends to either 0 or 1.

$\Sigma$ is complete, hence either $\Sigma \models \varphi$ or $\Sigma \models \neg \varphi$.

Assume $\Sigma \models \varphi$.

By compactness, then there exists a finite set $\{\psi_1, \ldots, \psi_m\} \models \varphi$.

Thus, $\mu_n(\varphi) \geq \mu_n(\psi_1 \land \cdots \land \psi_m) \to 1$ why?

Assume $\Sigma \models \neg \varphi$: then $\mu_n(\neg \varphi) \to 1$, hence $\mu_n(\varphi) \to 0$. 
Proof of the Zero-One Law

Let $\varphi$ be any sentence: we’ll prove $\mu_n(\varphi)$ tends to either 0 or 1.

$\Sigma$ is complete, hence either $\Sigma \models \varphi$ or $\Sigma \models \neg \varphi$.

Assume $\Sigma \models \varphi$.

By compactness, then there exists a finite set $\{\psi_1, \ldots, \psi_m\} \models \varphi$.

Thus, $\mu_n(\varphi) \geq \mu_n(\psi_1 \land \cdots \land \psi_m) \to 1$ why?

Assume $\Sigma \models \neg \varphi$: then $\mu_n(\neg \varphi) \to 1$, hence $\mu_n(\varphi) \to 0$. 
Proof of the Zero-One Law

Let $\varphi$ be any sentence: we’ll prove $\mu_n(\varphi)$ tends to either 0 or 1.

$\Sigma$ is complete, hence either $\Sigma \vDash \varphi$ or $\Sigma \vDash \neg \varphi$.

Assume $\Sigma \vDash \varphi$.

By compactness, then there exists a finite set $\{\psi_1, \ldots, \psi_m\} \vDash \varphi$.

Thus, $\mu_n(\varphi) \geq \mu_n(\psi_1 \land \cdots \land \psi_m) \to 1$ why?

Assume $\Sigma \vDash \neg \varphi$: then $\mu_n(\neg \varphi) \to 1$, hence $\mu_n(\varphi) \to 0$. 
Proof of the Zero-One Law

Let \( \varphi \) be any sentence: we’ll prove \( \mu_n(\varphi) \) tends to either 0 or 1.

\( \Sigma \) is complete, hence either \( \Sigma \models \varphi \) or \( \Sigma \models \neg \varphi \).

Assume \( \Sigma \models \varphi \).

By compactness, then there exists a finite set \( \{ \psi_1, \ldots, \psi_m \} \models \varphi \)

Thus, \( \mu_n(\varphi) \geq \mu_n(\psi_1 \land \cdots \land \psi_m) \to 1 \) why?

Assume \( \Sigma \models \neg \varphi \): then \( \mu_n(\neg \varphi) \to 1 \), hence \( \mu_n(\varphi) \to 0 \).
Proof of the Zero-One Law

Let \( \varphi \) be any sentence: we’ll prove \( \mu_n(\varphi) \) tends to either 0 or 1.

\( \Sigma \) is complete, hence either \( \Sigma \models \varphi \) or \( \Sigma \models \neg \varphi \).

Assume \( \Sigma \models \varphi \).

By compactness, then there exists a finite set \( \{ \psi_1, \ldots, \psi_m \} \models \varphi \)

Thus, \( \mu_n(\varphi) \geq \mu_n(\psi_1 \wedge \cdots \wedge \psi_m) \to 1 \) why?

Assume \( \Sigma \models \neg \varphi \): then \( \mu_n(\neg \varphi) \to 1 \), hence \( \mu_n(\varphi) \to 0 \).
Proof of the Zero-One Law

Let $\varphi$ be any sentence: we’ll prove $\mu_n(\varphi)$ tends to either 0 or 1.

$\Sigma$ is complete, hence either $\Sigma \vdash \varphi$ or $\Sigma \vdash \neg \varphi$.

Assume $\Sigma \vdash \varphi$.

By compactness, then there exists a finite set $\{\psi_1, \ldots, \psi_m\} \vdash \varphi$

Thus, $\mu_n(\varphi) \geq \mu_n(\psi_1 \land \cdots \land \psi_m) \rightarrow 1 \text{ why?}$

Assume $\Sigma \vdash \neg \varphi$: then $\mu_n(\neg \varphi) \rightarrow 1$, hence $\mu_n(\varphi) \rightarrow 0$. 
Discussion

- The 0/1 law does not hold if there constants:
e.g. \( \lim \mu_n R(a, b) = 1/2 \) (neither 0 nor 1).
  Where in the proof did we use this fact? (Homework!)

- The Random Graph \( R \) satisfies precisely those sentences for which
  \( \lim \mu_n(\varphi) = 1 \).

- We proved the 0/1 law when every edge \( E(i, j) \) has probability
  \( p = 1/2 \).
  The same proof also holds when every edge has probability \( p \in (0, 1) \)
  (independent of \( n \)).

- The Erdös-Rényi random graph \( G(n, p) \) allows \( p \) to depend on \( n \).
  0/1 law for FO may or may not hold. discuss more in class
A Cool Application: Non-standard Analysis

“Infinitesimals” have been used in calculus since Leibniz and Newton.

But they are not rigorous! Recall Logicomix.

Example: compute the derivative of $x^2$:

$$\frac{dx^2}{dx} = \frac{(x + dx)^2 - x^2}{dx} = \frac{2 \cdot x \cdot dx + (dx)^2}{dx} = 2x + dx \approx 2x$$

because $dx$ is “infinitely small”, hence $dx \approx 0$.

Robinson in 1961 showed that how to define infinitesimals rigorously (and easily) using the compactness theorem!
The Nonstandard Reals

\( \mathbb{R} = \) the true real numbers.

- Let \( \sigma \) be the vocabulary of all numbers, functions, relations:
  - Every number in \( \mathbb{R} \) has a symbol: 0, \(-5\), \(\pi\), ...
  - Every function \( \mathbb{R}^k \to \mathbb{R} \) has a symbol: +, \(*\), −, sin, ...
  - Every relation \( \subseteq \mathbb{R}^k \) has a symbol: <, \(\geq\), ...

- Let \( \text{Th}(\mathbb{R}) \) all true sentences, e.g.:
  \[
  \forall x(x^2 \geq 0) \\
  \forall x \forall y(|x + y| \leq |x| + |y|) \\
  \forall x(\sin(x + \pi) = -\sin(x))
  \]

- Let \( \Omega \) be a new constant, and \( \Sigma \overset{\text{def}}{=} \text{Th}(\mathbb{R}) \cup \{n < \Omega \mid n \in \mathbb{N}\} \).
  “\( \Omega \) is bigger than everything”.

- \( \Sigma \) has a model \( \mathbb{R}^* \). WHY?

What exactly is \( \mathbb{R}^* \)???
The Nonstandard Reals

\( \mathbb{R} = \) the true real numbers.

- Let \( \sigma \) be the vocabulary of all numbers, functions, relations:
  - Every number in \( \mathbb{R} \) has a symbol: 0, \(-5\), \(\pi\), …
  - Every function \( \mathbb{R}^k \to \mathbb{R} \) has a symbol: \(+\), \(*\), \(-\), \(\sin\), …
  - Every relation \( \subseteq R^k \) has a symbol: \(<\), \(\geq\), …

- Let \( \text{Th}(\mathbb{R}) \) all true sentences, e.g.:
  \[
  \forall x (x^2 \geq 0) \\
  \forall x \forall y (|x + y| \leq |x| + |y|) \\
  \forall x (\sin(x + \pi) = -\sin(x))
  \]

- Let \( \Omega \) be a new constant, and \( \Sigma \overset{\text{def}}{=} \text{Th}(\mathbb{R}) \cup \{ n < \Omega \mid n \in \mathbb{N} \} \).
  "\( \Omega \) is bigger than everything".

- \( \Sigma \) has a model \( ^*\mathbb{R} \). WHY?

What exactly is \( ^*\mathbb{R} \)??
The Nonstandard Reals

\( \mathbb{R} = \) the true real numbers.

- Let \( \sigma \) be the vocabulary of all numbers, functions, relations:
  - Every number in \( \mathbb{R} \) has a symbol: 0, −5, π, ...
  - Every function \( \mathbb{R}^k \to \mathbb{R} \) has a symbol: +, *, −, sin, ...
  - Every relation \( \subseteq \mathbb{R}^k \) has a symbol: <, ≥, ...

- Let \( \text{Th}(\mathbb{R}) \) all true sentences, e.g.:
  \[
  \forall x (x^2 \geq 0)
  \]
  \[
  \forall x \forall y (|x + y| \leq |x| + |y|)
  \]
  \[
  \forall x (\sin(x + \pi) = -\sin(x))
  \]

- Let \( \Omega \) be a new constant, and \( \Sigma \stackrel{\text{def}}{=} \text{Th}(\mathbb{R}) \cup \{ n < \Omega \mid n \in \mathbb{N} \} \).
  “\( \Omega \) is bigger than everything”.

- \( \Sigma \) has a model \( ^*\mathbb{R} \). \textbf{WHY?}

\textbf{What exactly is} \( ^*\mathbb{R} \)????
The Nonstandard Reals

\( \mathbb{R} = \) the true real numbers.

- Let \( \sigma \) be the vocabulary of all numbers, functions, relations:
  - Every number in \( \mathbb{R} \) has a symbol: 0, -5, \( \pi \), ...
  - Every function \( \mathbb{R}^k \to \mathbb{R} \) has a symbol: +, \( \times \), -, \( \sin \), ...
  - Every relation \( \subseteq \mathbb{R}^k \) has a symbol: <, \( \geq \), ...

- Let \( \text{Th}(\mathbb{R}) \) all true sentences, e.g.:
  \[ \forall x (x^2 \geq 0) \]
  \[ \forall x \forall y (|x + y| \leq |x| + |y|) \]
  \[ \forall x (\sin(x + \pi) = -\sin(x)) \]

- Let \( \Omega \) be a new constant, and \( \Sigma \overset{\text{def}}{=} \text{Th}(\mathbb{R}) \cup \{ n < \Omega \mid n \in \mathbb{N} \} \). “\( \Omega \) is bigger than everything”.
- \( \Sigma \) has a model \( \mathbb{R}^* \). \textbf{WHY?}

What exactly is \( \mathbb{R}^* \)???
The Nonstandard Reals

\( \mathbb{R} = \) the true real numbers.

- Let \( \sigma \) be the vocabulary of all numbers, functions, relations:
  - Every number in \( \mathbb{R} \) has a symbol: 0, –5, \( \pi \), ...
  - Every function \( \mathbb{R}^k \to \mathbb{R} \) has a symbol: +, *, –, \( \sin \), ...
  - Every relation \( \subseteq \mathbb{R}^k \) has a symbol: <, \( \geq \), ...

- Let \( \text{Th}(\mathbb{R}) \) all true sentences, e.g.:
  
  \[
  \forall x \left( x^2 \geq 0 \right) \\
  \forall x \forall y \left( |x + y| \leq |x| + |y| \right) \\
  \forall x \left( \sin(x + \pi) = -\sin(x) \right)
  \]

- Let \( \Omega \) be a new constant, and \( \Sigma \overset{\text{def}}{=} \text{Th}(\mathbb{R}) \cup \{ n < \Omega \mid n \in \mathbb{N} \} \).
  "\( \Omega \) is bigger than everything".

- \( \Sigma \) has a model \( \mathbb{R}^\ast \). \textbf{WHY?}

\textbf{What exactly is} \( \mathbb{R}^\ast \)???
The Nonstandard Reals

- Every number in \( \mathbb{R} \) also exists in \( \mathbb{R}^* \): 0, –5, \( \pi \), …
- Every function \( \mathbb{R}^k \rightarrow \mathbb{R} \) has an extension \( (\mathbb{R}^*)^k \rightarrow \mathbb{R}^* \).
- Every relation \( \subseteq \mathbb{R}^k \) has a corresponding \( \subseteq (\mathbb{R}^*)^k \).
- \( \omega \overset{\text{def}}{=} 1/\Omega \); the, \( 0 < \omega < c \) forall real \( c > 0 \). Infinitezimal! others?

The infinitezimals are \( \mathcal{I} \overset{\text{def}}{=} \{ v \in \mathbb{R}^* \mid \forall c \in \mathbb{R}, c > 0 : |v| < c \} \)
The finite elements are \( \mathcal{F} \overset{\text{def}}{=} \{ v \in \mathbb{R}^* \mid \exists c \in \mathbb{R}, |v| < c \} \)
- \( 2\omega, \omega^3, \sin(\omega) \) are infinitezimals; 0.001 is not.
- \( \pi, 0.001, 10^{10^{10}} \) are finite; \( \Omega, \Omega/1000, \Omega^\Omega \) are not.
The Nonstandard Reals

- Every number in $\mathbb{R}$ also exists in $\ast\mathbb{R}$: $0, -5, \pi, \ldots$

- Every function $\mathbb{R}^k \rightarrow \mathbb{R}$ has an extension $(\ast\mathbb{R})^k \rightarrow \ast\mathbb{R}$.

- Every relation $\subseteq \mathbb{R}^k$ has a corresponding $\subseteq (\ast\mathbb{R})^k$.

- $\omega \overset{\text{def}}{=} 1/\Omega$; the, $0 < \omega < c$ forall real $c > 0$. Infinitezimal! others?

- The infinitezimals are $\mathcal{I} \overset{\text{def}}{=} \{ \nu \in \ast\mathbb{R} \mid \forall c \in \mathbb{R}, c > 0 : |\nu| < c \}$

- The finite elements are $\mathcal{F} \overset{\text{def}}{=} \{ \nu \in \ast\mathbb{R} \mid \exists c \in \mathbb{R}, |\nu| < c \}$

- $2\omega, \omega^3, \sin(\omega)$ are infinitezimals; $0.001$ is not.

- $\pi, 0.001, 10^{10^{10}}$ are finite; $\Omega, \Omega/1000, \Omega^{\Omega^{\Omega}}$ are not.
The Nonstandard Reals

- Every number in $\mathbb{R}$ also exists in $\mathbb{R}^*$: $0, -5, \pi, \ldots$
- Every function $\mathbb{R}^k \rightarrow \mathbb{R}$ has an extension $(\mathbb{R}^*)^k \rightarrow \mathbb{R}^*$.
- Every relation $\subseteq \mathbb{R}^k$ has a corresponding $\subseteq (\mathbb{R}^*)^k$.

- $\omega \overset{\text{def}}{=} 1/\Omega$; the, $0 < \omega < c$ forall real $c > 0$. Infinitezimal! others?

- The infinitezimals are $\mathcal{I} \overset{\text{def}}{=} \{ v \in \mathbb{R}^* | \forall c \in \mathbb{R}, c > 0 : |v| < c \}$
- The finite elements are $\mathcal{F} \overset{\text{def}}{=} \{ v \in \mathbb{R}^* | \exists c \in \mathbb{R}, |v| < c \}$

- $2\omega, \omega^3, \sin(\omega)$ are infinitezimals; $0.001$ is not.
- $\pi, 0.001, 10^{10^{10}}$ are finite; $\Omega, \Omega/1000, \Omega^{\Omega}$ are not.
The Nonstandard Reals

- Every number in $\mathbb{R}$ also exists in $\mathbb{R}^*$: $0, -5, \pi, \ldots$
- Every function $\mathbb{R}^k \to \mathbb{R}$ has an extension $(\mathbb{R}^*)^k \to \mathbb{R}^*$.
- Every relation $\subseteq \mathbb{R}^k$ has a corresponding $\subseteq (\mathbb{R}^*)^k$.
- $\omega \overset{\text{def}}{=} 1/\Omega$; the, $0 < \omega < c$ forall real $c > 0$. Infinitezimal! others?

- The infinitezimals are $\mathcal{I} \overset{\text{def}}{=} \{ v \in \mathbb{R}^* | \forall c \in \mathbb{R}, c > 0 : |v| < c \}$
- The finite elements are $\mathcal{F} \overset{\text{def}}{=} \{ v \in \mathbb{R}^* | \exists c \in \mathbb{R}, |v| < c \}$
- $2 \omega, \omega^3, \sin(\omega)$ are infinitezimals; $0.001$ is not.
- $\pi, 0.001, 10^{10^{10}}$ are finite; $\Omega, \Omega/1000, \Omega^{\Omega}$ are not.
The Nonstandard Reals

- Every number in $\mathbb{R}$ also exists in $\^{\mathbb{R}}$: $0, -5, \pi, \ldots$
- Every function $\mathbb{R}^k \rightarrow \mathbb{R}$ has an extension $(\^{\mathbb{R}})^k \rightarrow \^{\mathbb{R}}$.
- Every relation $\subseteq \mathbb{R}^k$ has a corresponding $\subseteq (\^{\mathbb{R}})^k$.

$\omega \overset{\text{def}}{=} 1/\Omega$; the, $0 < \omega < c$ forall real $c > 0$. **Infinitezimal!** others?

- The infinitezimals are $\mathcal{I} \overset{\text{def}}{=} \{ v \in \^{\mathbb{R}} | \forall c \in \mathbb{R}, c > 0 : |v| < c \}$
- The finite elements are $\mathcal{F} \overset{\text{def}}{=} \{ v \in \^{\mathbb{R}} | \exists c \in \mathbb{R}, |v| < c \}$

- $2\omega, \omega^3, \sin(\omega)$ are infinitezimals; $0.001$ is not.
- $\pi, 0.001, 10^{10^{10}}$ are finite; $\Omega, \Omega/1000, \Omega^{\Omega}$ are not.
The Nonstandard Reals

- Every number in \( \mathbb{R} \) also exists in \( \mathbb{R}^* \): 0, \(-5\), \(\pi\), \ldots
- Every function \( \mathbb{R}^k \rightarrow \mathbb{R} \) has an extension \( (\mathbb{R}^*)^k \rightarrow \mathbb{R}^* \).
- Every relation \( \subseteq \mathbb{R}^k \) has a corresponding \( \subseteq (\mathbb{R}^*)^k \).
- \( \omega \overset{\text{def}}{=} 1/\Omega \); the, \( 0 < \omega < c \) for all real \( c > 0 \). Infinitezimal! others?
- The infinitezimals are \( \mathcal{I} \overset{\text{def}}{=} \{ v \in \mathbb{R}^* | \forall c \in \mathbb{R}, c > 0 : |v| < c \} \)
- The finite elements are \( \mathcal{F} \overset{\text{def}}{=} \{ v \in \mathbb{R}^* | \exists c \in \mathbb{R}, |v| < c \} \)
- \( 2\omega, \omega^3, \sin(\omega) \) are infinitezimals; 0.001 is not.
- \( \pi, 0.001, 10^{10^{10}} \) are finite; \( \Omega, \Omega/1000, \Omega^\Omega \) are not.
The Nonstandard Reals

- Every number in $\mathbb{R}$ also exists in $\mathbb{R}^*$: 0, $-5$, $\pi$, ...
- Every function $\mathbb{R}^k \rightarrow \mathbb{R}$ has an extension $(\mathbb{R}^*)^k \rightarrow \mathbb{R}^*$.
- Every relation $\subseteq \mathbb{R}^k$ has a corresponding $\subseteq (\mathbb{R}^*)^k$.

$\omega \overset{\text{def}}{=} 1/\Omega$; the, $0 < \omega < c$ for all real $c > 0$. Infinitezimal! others?

- The infinitezimals are $\mathcal{I} \overset{\text{def}}{=} \{ v \in \mathbb{R}^* | \forall c \in \mathbb{R}, c > 0 : |v| < c \}$
- The finite elements are $\mathcal{F} \overset{\text{def}}{=} \{ v \in \mathbb{R}^* | \exists c \in \mathbb{R}, |v| < c \}$

- $2\omega, \omega^3, \sin(\omega)$ are infinitezimals; 0.001 is not.
- $\pi, 0.001, 10^{10^{10}}$ are finite; $\Omega, \Omega/1000, \Omega^{\Omega^\Omega}$ are not.
The Nonstandard Reals

Infinitezimals closed under $+, -, \cdot$; $x, y \in \mathcal{I}$ implies $x + y, x - y, x \cdot y \in \mathcal{I}$

Finite elements closed under $+, -, \cdot$; $x, y \in \mathcal{F}$ implies $x + y, x - y, x \cdot y \in \mathcal{F}$

Call $x, y \in \ast \mathbb{R}$ infinitely close if $x - y \in \mathcal{I}$; write $x \simeq y$.

Fact: $\simeq$ is an equivalence relation. Exercise!

Now we can work with infinitezimals rigorously:

$$
\frac{dx^2}{dx} = \frac{(x + dx)^2 - x^2}{dx} = \frac{2 \cdot x \cdot dx + (dx)^2}{dx} = 2x + dx \simeq 2x
$$
The Nonstandard Reals

Infinitezimals closed under $+, -, *; \ x, y \in \mathcal{I}$ implies $x + y, x - y, x \cdot y \in \mathcal{I}$

Finite elements closed under $+, -, *; \ x, y \in \mathcal{F}$ implies $x + y, x - y, x \cdot y \in \mathcal{F}$

Call $x, y \in \mathbb{R}^*$ infinitely close if $x - y \in \mathcal{I}$; write $x \sim y$.

Fact: $\sim$ is an equivalence relation. Exercise!

Now we can work with infinitezimals rigorously:

$$\frac{dx^2}{dx} = \frac{(x + dx)^2 - x^2}{dx} = \frac{2 \cdot x \cdot dx + (dx)^2}{dx} = 2x + dx \sim 2x$$
The Nonstandard Reals

Infinitezimals closed under $+, -, *; \ x, y \in \mathcal{I}$ implies $x + y, x - y, x \ast y \in \mathcal{I}$

Finite elements closed under $+, -, *; \ x, y \in \mathcal{F}$ implies $x + y, x - y, x \ast y \in \mathcal{F}$

Call $x, y \in \ast \mathbb{R}$ infinitely close if $x - y \in \mathcal{I}$; write $x \simeq y$.

Fact: $\simeq$ is an equivalence relation. Exercise!

Now we can work with infinitezimals rigorously:

$$\frac{dx^2}{dx} = \frac{(x + dx)^2 - x^2}{dx} = \frac{2 \cdot x \cdot dx + (dx)^2}{dx} = 2x + dx \simeq 2x$$
The Nonstandard Reals

Infinitezimals closed under $+, -, *$; $x, y \in I$ implies $x + y, x - y, x \cdot y \in I$

Finite elements closed under $+, -, *$; $x, y \in F$ implies $x + y, x - y, x \cdot y \in F$

Call $x, y \in \mathbb{R}$ infinitely close if $x - y \in I$; write $x \simeq y$.

Fact: $\simeq$ is an equivalence relation. Exercise!

Now we can work with infinitezimals rigorously:

$$\frac{d x^2}{d x} = \frac{(x + dx)^2 - x^2}{dx} = \frac{2 \cdot x \cdot dx + (dx)^2}{dx} = 2x + dx \simeq 2x$$
Two Other Classical Theorem (which everyone should know!)

- Gödel’s completeness theorem.

- Gödel’s incompleteness theorem.

We discuss them next
Gödel’s Completeness Theorem

- Part of Gödel’s PhD Thesis. (We need to raise the bar at UW too.)

- It says that, using some reasonable axioms:
  \( \Sigma \models \varphi \iff \text{there exists a syntactic proof of } \varphi \text{ from } \Sigma \).

- Completeness \( \iff \) Compactness (⇒ is immediate; ⇐ no easy proof).

- Instead, proof of Completeness Theorem is similar to Compactness.

- The Completeness Theorem depends on the rather ad-hoc choice of axioms, hence mathematicians consider it less deep than compactness.
Axioms

There are dozens of choices\(^4\) for the axioms\(^5\). Recall \(\neg \varphi\) is \(\varphi \rightarrow F\).

\[
\begin{align*}
A_1 & : \varphi \rightarrow (\psi \rightarrow \varphi) \\
A_2 & : (\varphi \rightarrow (\psi \rightarrow \gamma)) \rightarrow ((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \gamma)) \\
A_3 & : \neg\neg \varphi \rightarrow \varphi \\
A_4 & : \forall x \varphi \rightarrow \varphi[t/x] \\
A_5 & : (\forall x(\varphi \rightarrow \psi)) \rightarrow (\forall x(\varphi) \rightarrow \forall x(\psi)) \\
A_6 & : \varphi \rightarrow \forall x(\varphi) \\
A_7 & : x = x \\
A_8 & : (x = y) \rightarrow (\varphi \rightarrow \varphi[y/x])
\end{align*}
\]

These are axiom schemas: each \(A_i\) defines an infinite set of formulas.

\(^4\)\(A_1 - A_8\) are a combination of axioms from Barnes&Mack and Enderton.
\(^5\)Fans of the Curry-Howard isomorphisms will recognize typed \(\lambda\)-calculus in \(A_1, A_2\).
Proofs

Let $\Sigma$ be a set of formulas.

**Definition**

A *proof* or a *deduction* is a sequence $\varphi_1, \varphi_2, \ldots, \varphi_n$ such that\(^{\text{a}}\), for every $i$:

- $\varphi_i$ is an Axiom, or $\varphi_i \in \Sigma$ or,
- $\varphi_i$ is obtained by modus ponens from earlier $\varphi_j, \varphi_k$ ($\varphi_k \equiv (\varphi_j \rightarrow \varphi_i)$).

\(^{\text{a}}\)There is no Generalization Rule since it follows from $A_6$ (Enderton).

**Definition**

We say that $\varphi$ is *provable*, or *deducible* from $\Sigma$, and write $\Sigma \vdash \varphi$, if there exists a proof sequence ending in $\varphi$.

If $\vdash \varphi$ then we call $\varphi$ a *theorem*.

$\text{Ded}(\Sigma)$ is the set of formulas $\varphi$ provable from $\Sigma$. 
Proofs

Let $\Sigma$ be a set of formulas.

**Definition**

A *proof* or a *deduction* is a sequence $\varphi_1, \varphi_2, \ldots, \varphi_n$ such that\(^a\), for every $i$:

- $\varphi_i$ is an Axiom, or $\varphi_i \in \Sigma$ or,
- $\varphi_i$ is obtained by modus ponens from earlier $\varphi_j, \varphi_k$ ($\varphi_k \equiv (\varphi_j \to \varphi_i)$).

\(^a\)There is no Generalization Rule since it follows from $A_6$ (Enderton).

**Definition**

We say that $\varphi$ is *provable*, or *deducible* from $\Sigma$, and write $\Sigma \vdash \varphi$, if there exists a proof sequence ending in $\varphi$.

If $\vdash \varphi$ then we call $\varphi$ a *theorem*.

$\text{Ded}(\Sigma)$ is the set of formulas $\varphi$ provable from $\Sigma$. 
Discussion

- $\Sigma \models \varphi$ is semantics: it says something about truth.
- $\Sigma \vdash \varphi$ is syntactic: an application of ad-hoc rules.

Example: prove that $\varphi \rightarrow \varphi$:

$$A_1 : \varphi \rightarrow ((\varphi \rightarrow \varphi) \rightarrow \varphi)$$
$$A_2 : (\varphi \rightarrow ((\varphi \rightarrow \varphi) \rightarrow \varphi)) \rightarrow ((\varphi \rightarrow (\varphi \rightarrow \varphi)) \rightarrow (\varphi \rightarrow \varphi))$$
$$MP : (\varphi \rightarrow (\varphi \rightarrow \varphi)) \rightarrow (\varphi \rightarrow \varphi)$$
$$A_1 : (\varphi \rightarrow (\varphi \rightarrow \varphi))$$
$$MP : (\varphi \rightarrow \varphi)$$

Prove at home $F \rightarrow \varphi$ and $\varphi \rightarrow \psi, \psi \rightarrow \omega \vdash \varphi \rightarrow \omega$. 
Discussion

- $\Sigma \models \varphi$ is semantics: it says something about truth.
- $\Sigma \vdash \varphi$ is syntactic: an application of ad-hoc rules.
- **Example:** prove that $\varphi \rightarrow \varphi$:

  \[
  A_1 : \varphi \rightarrow ((\varphi \rightarrow \varphi) \rightarrow \varphi) \\
  A_2 : (\varphi \rightarrow ((\varphi \rightarrow \varphi) \rightarrow \varphi)) \rightarrow ((\varphi \rightarrow (\varphi \rightarrow \varphi)) \rightarrow (\varphi \rightarrow \varphi)) \\
  MP : (\varphi \rightarrow (\varphi \rightarrow \varphi)) \rightarrow (\varphi \rightarrow \varphi) \\
  A_1 : (\varphi \rightarrow (\varphi \rightarrow \varphi)) \\
  MP : (\varphi \rightarrow \varphi)
  \]

- **Prove at home** $F \rightarrow \varphi$ and $\varphi \rightarrow \psi, \psi \rightarrow \omega \vdash \varphi \rightarrow \omega$. 
Discussion

- $\Sigma = \varphi$ is semantics: it says something about truth.
- $\Sigma \vdash \varphi$ is syntactic: an application of ad-hoc rules.

Example: prove that $\varphi \rightarrow \varphi$:

\[
A_1 : \varphi \rightarrow ((\varphi \rightarrow \varphi) \rightarrow \varphi) \\
A_2 : (\varphi \rightarrow ((\varphi \rightarrow \varphi) \rightarrow \varphi)) \rightarrow ((\varphi \rightarrow (\varphi \rightarrow \varphi)) \rightarrow (\varphi \rightarrow \varphi))
\]

MP : $(\varphi \rightarrow (\varphi \rightarrow \varphi)) \rightarrow (\varphi \rightarrow \varphi)$

A1 : $(\varphi \rightarrow (\varphi \rightarrow \varphi))$

MP : $(\varphi \rightarrow \varphi)$

Prove at home $F \rightarrow \varphi$ and $\varphi \rightarrow \psi, \psi \rightarrow \omega \vdash \varphi \rightarrow \omega$. 
Discussion

- $\Sigma \models \varphi$ is semantics: it says something about truth.
- $\Sigma \vdash \varphi$ is syntactic: an application of ad-hoc rules.
- Example: prove that $\varphi \rightarrow \varphi$:

\[
A_1 : \varphi \rightarrow ((\varphi \rightarrow \varphi) \rightarrow \varphi) \\
A_2 : (\varphi \rightarrow ((\varphi \rightarrow \varphi) \rightarrow \varphi)) \rightarrow ((\varphi \rightarrow (\varphi \rightarrow \varphi)) \rightarrow (\varphi \rightarrow \varphi)) \\
\text{MP} : (\varphi \rightarrow (\varphi \rightarrow \varphi)) \rightarrow (\varphi \rightarrow \varphi) \\
A_1 : (\varphi \rightarrow (\varphi \rightarrow \varphi)) \\
\text{MP} : (\varphi \rightarrow \varphi)
\]

- Prove at home $F \rightarrow \varphi$ and $\varphi \rightarrow \psi, \psi \rightarrow \omega \vdash \varphi \rightarrow \omega$. 
Discussion

- $\Sigma \models \varphi$ is semantics: it says something about truth.
- $\Sigma \vdash \varphi$ is syntactic: an application of ad-hoc rules.
- **Example:** prove that $\varphi \rightarrow \varphi$:

  $A_1 : \varphi \rightarrow ((\varphi \rightarrow \varphi) \rightarrow \varphi)$
  $A_2 : (\varphi \rightarrow ((\varphi \rightarrow \varphi) \rightarrow \varphi)) \rightarrow (((\varphi \rightarrow (\varphi \rightarrow \varphi)) \rightarrow (\varphi \rightarrow \varphi))$  
  $\text{MP} : (\varphi \rightarrow (\varphi \rightarrow \varphi)) \rightarrow (\varphi \rightarrow \varphi)$
  $A_1 : (\varphi \rightarrow (\varphi \rightarrow \varphi))$
  $\text{MP} : (\varphi \rightarrow \varphi)$

- Prove at home $F \rightarrow \varphi$ and $\varphi \rightarrow \psi, \psi \rightarrow \omega \vdash \varphi \rightarrow \omega$.  

Dan Suciu
Finite Model Theory – Unit 1
Spring 2018 61 / 80
Discussion

- $\Sigma \models \varphi$ is semantics: it says something about truth.
- $\Sigma \vdash \varphi$ is syntactic: an application of ad-hoc rules.
- Example: prove that $\varphi \rightarrow \varphi$:

  $A_1 : \varphi \rightarrow ((\varphi \rightarrow \varphi) \rightarrow \varphi)$
  $A_2 : (\varphi \rightarrow ((\varphi \rightarrow \varphi) \rightarrow \varphi)) \rightarrow ((\varphi \rightarrow (\varphi \rightarrow \varphi)) \rightarrow (\varphi \rightarrow \varphi))$
  $\text{MP} : (\varphi \rightarrow (\varphi \rightarrow \varphi)) \rightarrow (\varphi \rightarrow \varphi)$
  $A_1 : (\varphi \rightarrow (\varphi \rightarrow \varphi))$
  $\text{MP} : (\varphi \rightarrow \varphi)$

- Prove at home $F \rightarrow \varphi$ and $\varphi \rightarrow \psi, \psi \rightarrow \omega \vdash \varphi \rightarrow \omega$. 

Dan Suciu  
Finite Model Theory – Unit 1  
Spring 2018  
61 / 80
Discussion

- $\Sigma \models \varphi$ is semantics: it says something about truth.
- $\Sigma \vdash \varphi$ is syntactic: an application of ad-hoc rules.

**Example:** prove that $\varphi \rightarrow \varphi$:

\[ A_1 : \varphi \rightarrow ((\varphi \rightarrow \varphi) \rightarrow \varphi) \]
\[ A_2 : (\varphi \rightarrow ((\varphi \rightarrow \varphi) \rightarrow \varphi)) \rightarrow ((\varphi \rightarrow (\varphi \rightarrow \varphi)) \rightarrow (\varphi \rightarrow \varphi)) \]
\[ MP : (\varphi \rightarrow (\varphi \rightarrow \varphi)) \rightarrow (\varphi \rightarrow \varphi) \]
\[ A_1 : (\varphi \rightarrow (\varphi \rightarrow \varphi)) \]
\[ MP : (\varphi \rightarrow \varphi) \]

**Prove at home** $F \rightarrow \varphi$ and $\varphi \rightarrow \psi, \psi \rightarrow \omega \vdash \varphi \rightarrow \omega$. 
Definition

Σ is called inconsistent if Σ ⊬ F. Otherwise we say Σ is consistent.

Σ is inconsistent iff for every ϕ, Σ ⊬ ϕ
Proof: ⊬ F → ϕ.

Σ is inconsistent iff there exists ϕ s.t. both Σ ⊬ ϕ and Σ ⊬ ¬ϕ
Proof: ϕ, ¬ϕ ⊬ F.
Consistency

Definition

Σ is called inconsistent if Σ ⊬ F. Otherwise we say Σ is consistent.

Σ is inconsistent iff for every φ, Σ ⊬ φ
Proof: ⊬ F → φ.

Σ is inconsistent iff there exists φ s.t. both Σ ⊬ φ and Σ ⊬ ¬φ
Proof: φ, ¬φ ⊬ F.
Consistency

**Definition**

\( \Sigma \) is called **inconsistent** if \( \Sigma \vdash \neg F \). Otherwise we say \( \Sigma \) is **consistent**.

\( \Sigma \) is inconsistent iff for every \( \varphi \), \( \Sigma \vdash \varphi \)

Proof: \( \vdash F \rightarrow \varphi \).

\( \Sigma \) is inconsistent iff there exists \( \varphi \) s.t. both \( \Sigma \vdash \varphi \) and \( \Sigma \vdash \neg \varphi \)

Proof: \( \varphi, \neg \varphi \vdash F \).
Consistency

Definition

Σ is called inconsistent if Σ ⊭ F. Otherwise we say Σ is consistent.

Σ is inconsistent iff for every ϕ, Σ ⊨ ϕ
Proof: ⊬ F → ϕ.

Σ is inconsistent iff there exists ϕ s.t. both Σ ⊨ ϕ and Σ ⊨ ¬ϕ
Proof: ϕ, ¬ϕ ⊨ F.
Consistency

Definition

\( \Sigma \) is called **inconsistent** if \( \Sigma \not\vdash F \). Otherwise we say \( \Sigma \) is **consistent**.

\( \Sigma \) is inconsistent iff for every \( \varphi \), \( \Sigma \vdash \varphi \)
Proof: \( \vdash F \to \varphi \).

\( \Sigma \) is inconsistent iff there exists \( \varphi \) s.t. both \( \Sigma \vdash \varphi \) and \( \Sigma \vdash \neg \varphi \)
Proof: \( \varphi, \neg \varphi \vdash F \).
Soundness and Completeness

**Theorem (Soundness)**

*If \( \Sigma \) is satisfiable (i.e. \( \Sigma \not\models F \)), then it is consistent (i.e. \( \Sigma \not\vdash F \)).*

Equivalent formulation: if \( \Sigma \vdash \varphi \) then \( \Sigma \models \varphi \).

Prove and discuss in class

**Theorem (Gödel's Completeness Theorem)**

*If \( \Sigma \) is consistent \((\Sigma \not\vdash F)\), then it has a model \((\Sigma \not\models F)\).*

Equivalent formulation: if \( \Sigma \models \varphi \) then \( \Sigma \vdash \varphi \).

The Completeness Theorem immediately implies the Compactness Theorem why?.
Soundness and Completeness

**Theorem (Soundness)**

If $\Sigma$ is satisfiable (i.e. $\Sigma \not\models F$), then it is consistent (i.e. $\Sigma \not\models F$).

Equivalent formulation: if $\Sigma \vdash \varphi$ then $\Sigma \models \varphi$.

Prove and discuss in class

**Theorem (Gödel's Completeness Theorem)**

If $\Sigma$ is consistent ($\Sigma \not\models F$), then it has a model ($\Sigma \not\models F$).

Equivalent formulation: if $\Sigma \models \varphi$ then $\Sigma \vdash \varphi$.

The Completeness Theorem immediately implies the Compactness Theorem why?
Soundness and Completeness

**Theorem (Soundness)**

*If* $\Sigma$ *is satisfiable (i.e. $\Sigma \not\vdash F$), then it is consistent (i.e. $\Sigma \vdash \neg F$).*

Equivalent formulation: if $\Sigma \vdash \varphi$ then $\Sigma \models \varphi$.

Prove and discuss in class

**Theorem (Gödel’s Completeness Theorem)**

*If* $\Sigma$ *is consistent ($\Sigma \not\vdash F$), then it has a model ($\Sigma \not\models F$).*

Equivalent formulation: if $\Sigma \models \varphi$ then $\Sigma \vdash \varphi$.

The Completeness Theorem immediately implies the Compactness Theorem why?
Soundness and Completeness

Theorem (Soundness)

If $\Sigma$ is satisfiable (i.e. $\Sigma \not\models F$), then it is consistent (i.e. $\Sigma \models \neg F$).

Equivalent formulation: if $\Sigma \models \varphi$ then $\Sigma \models \varphi$.

Prove and discuss in class

Theorem (Gödel’s Completeness Theorem)

If $\Sigma$ is consistent ($\Sigma \not\models F$), then it has a model ($\Sigma \models \neg F$).

Equivalent formulation: if $\Sigma \models \varphi$ then $\Sigma \models \varphi$.

The Completeness Theorem immediately implies the Compactness Theorem why?
Soundness and Completeness

**Theorem (Soundness)**

*If $\Sigma$ is satisfiable (i.e. $\Sigma \not\models F$), then it is consistent (i.e. $\Sigma \models F$).*

Equivalent formulation: if $\Sigma \models \varphi$ then $\Sigma \models \varphi$.

Prove and discuss in class

**Theorem (Gödel’s Completeness Theorem)**

*If $\Sigma$ is consistent (i.e. $\Sigma \not\models F$), then it has a model (i.e. $\Sigma \models F$).*

Equivalent formulation: if $\Sigma \models \varphi$ then $\Sigma \models \varphi$.

The Completeness Theorem immediately implies the Compactness Theorem *why?*. 

Proof of Gödel’s Completeness Theorem

Follow exactly the steps of the compactness theorem.

- Extend a consistent $\Sigma$ to a consistent $\tilde{\Sigma}$ that is complete and witness-complete

- Use the Inductive Structure of a complete and witness-complete set.
Two Lemmas

Lemma (The Deduction Lemma)

If $\Sigma, \varphi \vdash \psi$ then $\Sigma \vdash \varphi \rightarrow \psi$.

Proof: induction on the length of $\Sigma, \varphi \vdash \psi$. Note: converse is trivial.

Lemma (Excluded Middle)

If $\Sigma, \varphi \vdash \psi$ and $\Sigma, (\varphi \rightarrow \mathsf{F}) \vdash \psi$ then $\Sigma \vdash \psi$.

$\Sigma \vdash \varphi \rightarrow \psi$

$\Sigma, \psi \rightarrow \mathsf{F} \vdash \varphi \rightarrow \mathsf{F}$

$\Sigma \vdash (\varphi \rightarrow \mathsf{F}) \rightarrow \psi$

$\Sigma, \psi \rightarrow \mathsf{F} \vdash (\varphi \rightarrow \mathsf{F}) \rightarrow \mathsf{F}$

$\Sigma, \psi \rightarrow \mathsf{F} \vdash \mathsf{F}$

$\Sigma \vdash (\psi \rightarrow \mathsf{F}) \rightarrow \mathsf{F}$

$\Sigma \vdash \psi$

Deduction Lemma

by $\varphi \rightarrow \psi, \psi \rightarrow \mathsf{F} \vdash \varphi \rightarrow \mathsf{F}$

Deduction Lemma

As above

MP: $\varphi \rightarrow \mathsf{F}, (\varphi \rightarrow \mathsf{F}) \rightarrow \mathsf{F} \vdash \mathsf{F}$

Deduction Lemma

by $A_3$
**Two Lemmas**

**Lemma (The Deduction Lemma)**

*If* $\Sigma, \varphi \vdash \psi$ *then* $\Sigma \vdash \varphi \rightarrow \psi$.

**Proof:** induction on the length of $\Sigma, \varphi \vdash \psi$. Note: converse is trivial.

**Lemma (Excluded Middle)**

*If* $\Sigma, \varphi \vdash \psi$ *and* $\Sigma, (\varphi \rightarrow F) \vdash \psi$ *then* $\Sigma \vdash \psi$.

\[
\begin{align*}
\Sigma \vdash \varphi \rightarrow \psi & \quad \text{Deduction Lemma} \\
\Sigma, \psi \rightarrow F \vdash \varphi \rightarrow F & \quad \text{by } \varphi \rightarrow \psi, \psi \rightarrow F \vdash \varphi \rightarrow F \\
\Sigma \vdash (\varphi \rightarrow F) \rightarrow \psi & \quad \text{Deduction Lemma} \\
\Sigma, \psi \rightarrow F \vdash (\varphi \rightarrow F) \rightarrow F & \quad \text{As above} \\
\Sigma, \psi \rightarrow F \vdash F & \quad \text{MP: } \varphi \rightarrow F, (\varphi \rightarrow F) \rightarrow F \vdash F \\
\Sigma \vdash (\psi \rightarrow F) \rightarrow F & \quad \text{Deduction Lemma} \\
\Sigma \vdash \psi & \quad \text{by } A_3
\end{align*}
\]
Two Lemmas

Lemma (The Deduction Lemma)

If $\Sigma, \varphi \vdash \psi$ then $\Sigma \vdash \varphi \rightarrow \psi$.

Proof: induction on the length of $\Sigma, \varphi \vdash \psi$. Note: converse is trivial.

Lemma (Excluded Middle)

If $\Sigma, \varphi \vdash \psi$ and $\Sigma, (\varphi \rightarrow F) \vdash \psi$ then $\Sigma \vdash \psi$.

\[
\begin{align*}
\Sigma \vdash & \varphi \rightarrow \psi \\
\Sigma, \psi \rightarrow F \vdash & \varphi \rightarrow F \\
\Sigma \vdash & (\varphi \rightarrow F) \rightarrow \psi \\
\Sigma, \psi \rightarrow F \vdash & (\varphi \rightarrow F) \rightarrow F \\
\Sigma, \psi \rightarrow F \vdash & F \\
\Sigma \vdash & (\psi \rightarrow F) \rightarrow F \\
\Sigma \vdash & \psi
\end{align*}
\]

Deduction Lemma by $\varphi \rightarrow \psi, \psi \rightarrow F \vdash \varphi \rightarrow F$

Deduction Lemma

As above

MP: $\varphi \rightarrow F, (\varphi \rightarrow F) \rightarrow F \vdash F$

Deduction Lemma by $A_3$
Step 1: Extend $\Sigma$ to a (witness-) complete $\bar{\Sigma}$

Enumerate all formulas $\varphi_1, \varphi_2, \ldots$, and define:

$$\Sigma_0 = \Sigma \quad \Sigma_{i+1} = \begin{cases} 
\Sigma_i \cup \{\varphi_i\} & \text{if } \Sigma_i \cup \{\varphi_i\} \text{ is consistent} \\
\Sigma_i \cup \neg \varphi_i & \text{if } \Sigma_i \cup \neg \varphi_i \text{ is consistent}
\end{cases}$$

At least one set is consistent, otherwise:
$\Sigma_i, \varphi_i \vdash F$ and $\Sigma_i, \neg \varphi_i \vdash F$, thus $\Sigma_i \vdash F$ by Excluded Middle.

To make it witness-complete, add countably many fresh constants $c_1, c_2, \ldots$, and repeatedly add $\neg \varphi[c_i/x]$ to $\Sigma$ whenever $\neg \forall x(\varphi) \in \Sigma$; must show that we still have a consistent set (omitted).
Step 1: Extend $\Sigma$ to a (witness-) complete $\bar{\Sigma}$

Enumerate all formulas $\varphi_1, \varphi_2, \ldots$, and define:

$$
\Sigma_0 = \Sigma \quad \Sigma_{i+1} = \begin{cases} 
\Sigma_i \cup \{\varphi_i\} & \text{if } \Sigma_i \cup \{\varphi_i\} \text{ is consistent} \\
\Sigma_i \cup \{\neg \varphi_i\} & \text{if } \Sigma_i \cup \{\neg \varphi_i\} \text{ is consistent}
\end{cases}
$$

At least one set is consistent, otherwise:
$\Sigma_i, \varphi_i \vdash F$ and $\Sigma_i, \neg \varphi_i \vdash F$, thus $\Sigma_i \vdash F$ by Excluded Middle.

To make it witness-complete, add countably many fresh constants $c_1, c_2, \ldots$, and repeatedly add $\neg \varphi[c_i/x]$ to $\Sigma$ whenever $\neg \forall x(\varphi) \in \Sigma$; must show that we still have a consistent set (omitted).
Step 1: Extend $\Sigma$ to a (witness-) complete $\bar{\Sigma}$

Enumerate all formulas $\varphi_1, \varphi_2, \ldots$, and define:

$$
\Sigma_0 = \Sigma \\
\Sigma_{i+1} = \begin{cases} 
\Sigma_i \cup \{\varphi_i\} & \text{if } \Sigma_i \cup \{\varphi_i\} \text{ is consistent} \\
\Sigma_i \cup \{\neg \varphi_i\} & \text{if } \Sigma_i \cup \{\neg \varphi_i\} \text{ is consistent}
\end{cases}
$$

At least one set is consistent, otherwise:
$\Sigma_i, \varphi_i \vdash F$ and $\Sigma_i, \neg \varphi_i \vdash F$, thus $\Sigma_i \vdash F$ by Excluded Middle.

To make it witness-complete, add countably many fresh constants $c_1, c_2, \ldots$, and repeatedly add $\neg \varphi[c_i/x]$ to $\Sigma$ whenever $\neg \forall x(\varphi) \in \Sigma$; must show that we still have a consistent set (omitted).
Step 2: Inductive Structure of a (Witness-)Complete Set

Lemma

If $\Sigma$ is complete, witness-complete, and consistent, then:

- $\varphi \rightarrow \psi \in \Sigma$ iff $\varphi \notin \Sigma$ or both $\varphi, \psi \in \Sigma$.
- $\neg \varphi \in \Sigma$ iff $\varphi \notin \Sigma$.
- $\neg \forall x(\varphi) \in \Sigma$ iff there exists a constant s.t. $\neg \varphi[c/x] \in \Sigma$.

Sketch of the Proof in class

Now we can prove Gödel’s completeness theorem:

- If $\Sigma$ is consistent ($\Sigma \not\vdash F$), then it has a model.

Simply construct a model of $\Sigma$ exactly the same way as in the compactness theorem.
Step 2: Inductive Structure of a (Witness-)Complete Set

Lemma

If $\bar{\Sigma}$ is complete, witness-complete, and consistent, then:

- $\varphi \to \psi \in \bar{\Sigma}$ iff $\varphi \notin \bar{\Sigma}$ or both $\varphi, \psi \in \bar{\Sigma}$.
- $\neg \varphi \in \bar{\Sigma}$ iff $\varphi \notin \bar{\Sigma}$.
- $\neg \forall x(\varphi) \in \bar{\Sigma}$ iff there exists a constant s.t. $\neg \varphi[c/x] \in \bar{\Sigma}$.

Sketch of the Proof in class

Now we can prove Gödel’s completeness theorem:

- If $\Sigma$ is consistent ($\Sigma \not\vdash F$), then it has a model.

Simply construct a model of $\bar{\Sigma}$ exactly the same way as in the compactness theorem.
Gödel’s completeness theorem is very strong: everything that is true has a syntactic proof.

In particular, Con(Σ) is r.e.

If, furthermore, Σ is complete, then Con(Σ) is decidable!

Gödel’s completeness theorem is also very weak: it does not tell us how to prove sentences that hold in a particular structure $D$.

Gödel’s incompleteness proves that this is unavoidable, if the structure is rich enough.
Application to Decidability

Corollary

If $\Sigma$ is r.e. and complete (meaning: $\Sigma \models \varphi$ or $\Sigma \models \neg \varphi$ forall $\varphi$), then $\text{Con}(\Sigma)$ is decidable.

why?

Proof: given $\varphi$, simply enumerate all theorems from $\Sigma$:

$$\Sigma \models \varphi_1, \varphi_2, \varphi_3, \ldots$$

Eventually, either $\varphi$ or $\neg \varphi$ will appear in the list.

Example 1: total, dense linear order without fixpoint is decidable

Example 2: Th($\mathbb{N}$, 0, succ) is decidable (on your homework).
Application to Decidability

Corollary

If $\Sigma$ is r.e. and complete (meaning: $\Sigma \models \varphi$ or $\Sigma \models \neg \varphi$ for all $\varphi$), then $\text{Con}(\Sigma)$ is decidable.

why?

Proof: given $\varphi$, simply enumerate all theorems from $\Sigma$:

$$\Sigma \vdash \varphi_1, \varphi_2, \varphi_3, \ldots$$

Eventually, either $\varphi$ or $\neg \varphi$ will appear in the list.

Example 1: total, dense linear order without fixpoint is decidable

Example 2: $\text{Th}(\mathbb{N}, 0, \text{succ})$ is decidable (on your homework).
Application to Decidability

**Corollary**

*If* $\Sigma$ *is r.e. and complete (meaning: $\Sigma \models \varphi$ or $\Sigma \models \neg \varphi$ for all $\varphi$), then $\text{Con}(\Sigma)$ *is decidable.*

**why?**

**Proof:** given $\varphi$, simply enumerate all theorems from $\Sigma$:

$$\Sigma \vdash \varphi_1, \varphi_2, \varphi_3, \ldots$$

Eventually, either $\varphi$ or $\neg \varphi$ will appear in the list.

**Example 1:** total, dense linear order without fixpoint is decidable

**Example 2:** $\text{Th}(\mathbb{N}, 0, \text{succ})$ is decidable (on your homework).
Application to Decidability

Corollary

If $\Sigma$ is r.e. and complete (meaning: $\Sigma \models \varphi$ or $\Sigma \models \neg \varphi$ for all $\varphi$), then $\text{Con}(\Sigma)$ is decidable.

why?

Proof: given $\varphi$, simply enumerate all theorems from $\Sigma$:

$$\Sigma \vdash \varphi_1, \varphi_2, \varphi_3, \ldots$$

Eventually, either $\varphi$ or $\neg \varphi$ will appear in the list.

Example 1: total, dense linear order without fixpoint is decidable

Example 2: $\text{Th}(\mathbb{N}, 0, \text{succ})$ is decidable (on your homework).
Gödel’s Incompleteness Theorem

- Proven by Gödel in 1931 (after his thesis).

- It says that no r.e. $\Sigma$ exists that is both consistent and can prove all true sentences in $(\mathbb{N}, +, \ast)$.

- The proof is actually not very hard for someone who knows programming (anyone in the audience?).

- What is absolutely remarkable is that Gödel proved it before programming, and even computation, had been invented.

- Turing published his *Turing Machine* only in 1937, to explain what goes on in Gödel’s proof.

- … and 81 years later, we have Deep Learning!
Gödel’s Incompleteness Theorem

**Theorem**

*Let \( \Sigma \) be any r.e. set of axioms for \((\mathbb{N}, +, \ast)\). If \( \Sigma \) is consistent, then it is not complete.*

What if \( \Sigma \) is not consistent?

In particular, there exists a sentence \( A \) s.t. \((\mathbb{N}, +, \ast) \models A\) but \( \Sigma \not\models A \).

We will prove it, by simplifying the (already simple!) proof by Arindama Singh [https://mat.iitm.ac.in/home/samy/public_html/mnl-v22-Dec2012-i3.pdf](https://mat.iitm.ac.in/home/samy/public_html/mnl-v22-Dec2012-i3.pdf)
Gödel’s Incompleteness Theorem

**Theorem**

Let \( \Sigma \) be any r.e. set of axioms for \((\mathbb{N}, +, \times)\). If \( \Sigma \) is consistent, then it is not complete.

What if \( \Sigma \) is not consistent?

In particular, there exists a sentence \( A \) s.t. \((\mathbb{N}, +, \times) \models A \) but \( \Sigma \not\models A \).

We will prove it, by simplifying the (already simple!) proof by Arindama Singh [here](https://mat.iitm.ac.in/home/samy/public_html/mnl-v22-Dec2012-i3.pdf).
Computing in \((\mathbb{N}, +, \cdot)\)

**Lemma**

*Fact:* for every Turing computable function \(f : \mathbb{N} \to \mathbb{N}\) there exists a sentence \(\varphi(x, y)\) s.t. \(\forall m, n \in \mathbb{N}, \mathbb{N} \models \varphi(m, n)\iff f(m) = n.\)

In other words, \(\varphi\) represents \(f\).

The proof requires a lot of sweat, but it’s not that hard.

Sketch on the next slide.
Computing in \((\mathbb{N}, +, \times)\)

- Express exponentiation: \(\mathbb{N} \models \varphi(m, n, p) \iff p = m^n\). This is hard, lots of math. Some books give up and assume exp is given: \((\mathbb{N}, +, \times, E)\).

- Encode a sequence \([n_1, n_2, \ldots, n_k]\) as powers of primes: \(2^{n_1}3^{n_2}5^{n_3}\ldots\). You might prefer: a sequence is just bits, hence just a number.

- Encode the API: concatenate, get \(i\)'th position, check membership.

- For any Turing Machine \(T\), write a sentence \(\varphi_T(x, y, z)\) that says\(^6\): “the sequence of tape contents \(z\) is a correct computation of output \(y\) from input \(x\).”

- The function computed by \(T\) is \(\exists z(\varphi_T(x, y, z))\).

\(^6\)We will do this in detail in Unit 3.
Computing in \((\mathbb{N}, +, \times)\)

- **Express exponentiation**: \(\mathbb{N} \models \varphi(m, n, p) \iff p = m^n\). This is hard, lots of math. Some books give up and assume \(\exp\) is given: \((\mathbb{N}, +, \times, \mathbb{E})\).

- **Encode a sequence** \([n_1, n_2, \ldots, n_k]\) as powers of primes: \(2^{n_1}3^{n_2}5^{n_3}\ldots\). You might prefer: a sequence is just bits, hence just a number.

- **Encode the API**: concatenate, get \(i^{th}\) position, check membership.

- **For any Turing Machine** \(T\), write a sentence \(\varphi_T(x, y, z)\) that says\(^6\): "the sequence of tape contents \(z\) is a correct computation of output \(y\) from input \(x\)."

- **The function computed by** \(T\) is \(\exists z(\varphi_T(x, y, z))\).

\(^6\)We will do this in detail in Unit 3.
Computing in \((\mathbb{N}, +, \ast)\)

- Express exponentiation: \(\mathbb{N} \models \varphi(m, n, p) \iff p = m^n\). This is hard, lots of math. Some books give up and assume exp is given: \((\mathbb{N}, +, \ast, E)\).

- Encode a sequence \([n_1, n_2, \ldots, n_k]\) as powers of primes: \(2^{n_1}3^{n_2}5^{n_3}\ldots\). You might prefer: a sequence is just bits, hence just a number.

- Encode the API: concatenate, get \(i\)'th position, check membership.

  - For any Turing Machine \(T\), write a sentence \(\varphi_T(x, y, z)\) that says\(^6\): “the sequence of tape contents \(z\) is a correct computation of output \(y\) from input \(x\).”

  - The function computed by \(T\) is \(\exists z(\varphi_T(x, y, z))\).

\(^6\)We will do this in detail in Unit 3.
Computing in \((\mathbb{N}, +, \times)\)

- Express exponentiation: \(\mathbb{N} \models \varphi(m, n, p) \iff p = m^n\). This is hard, lots of math. Some books give up and assume exp is given: \((\mathbb{N}, +, \times, E)\).

- Encode a sequence \([n_1, n_2, \ldots, n_k]\) as powers of primes: \(2^{n_1}3^{n_2}5^{n_3}\ldots\). You might prefer: a sequence is just bits, hence just a number.

- Encode the API: concatenate, get \(i\)'th position, check membership.

- For any Turing Machine \(T\), write a sentence \(\varphi_T(x, y, z)\) that says\(^6\): “the sequence of tape contents \(z\) is a correct computation of output \(y\) from input \(x\).”

- The function computed by \(T\) is \(\exists z(\varphi_T(x, y, z))\).

---

\(^6\)We will do this in detail in Unit 3.
The Checker and the Prover

Fix an r.e. set of axioms\(^7\), \((\mathbb{N}, +, \times) \models \Sigma\). Construct two sentences s.t.:

- \((\mathbb{N}, +, \times) \models \text{Checker}(x, y, z)\) iff
  - \(x\) encodes a formula \(\varphi\),
  - \(y\) encodes a sequence \([\varphi_1, \varphi_2, \ldots, \varphi_k]\),
  - \(z\) encodes a finite set \(\Sigma_{\text{fin}}\), and
  - \([\varphi_1, \varphi_2, \ldots, \varphi_k]\) is proof of \(\Sigma_{\text{fin}} \vdash \varphi\).

- \(\text{Prover}_\Sigma(x) \equiv \exists y \exists z ("z encodes \(\Sigma_{\text{fin}} \subseteq \Sigma" \wedge \text{Checker}(x, y, z)).\)

Here we assume \(\Sigma\) is r.e.

By Soundness, \((\mathbb{N}, +, \times) \models \text{Prover}_\Sigma(\varphi)\) implies \(\Sigma \vdash \varphi\).

---

\(^7\)E.g. Enderton pp. 203 describes 11 axioms
Gödel’s Sentence

- Let $\varphi_1(x), \varphi_2(x), \ldots$ be an enumeration of all formulas with one free variable.

- Consider the formula $\neg \text{Prover}_\Sigma(\varphi_x(x))$ this requires some thinking!

- It has a single variable $x$, hence it is in the list, say on position $k$: $\varphi_k(x) \equiv \neg \text{Prover}_\Sigma(\varphi_x(x))$.

- Denote $\alpha \equiv \varphi_k(k)$.

- In other words: $\alpha \equiv \neg \text{Prover}_\Sigma(\alpha)$ (syntactic identity)

- $\alpha$ says “I am not provable”!

- Next: prove two lemmas which imply Gödel’s theorem.

---

8 Computable!
**Lemma 1**

\[ \alpha \equiv \neg \text{Prover}_\Sigma(\alpha) \] (syntactic identity)

**Lemma (1)**

\[ \Sigma \vdash \text{Prover}_\Sigma(\alpha) \rightarrow \text{Prover}_\Sigma(\neg \alpha) \]

Proof. Assume \( \Sigma \) is rich enough to prove:

1. \( P_1 : \Sigma \vdash \varphi \) implies \( \Sigma \vdash \text{Prover}_\Sigma(\varphi) \)
2. \( P_2 : \Sigma \vdash (\text{Prover}_\Sigma(\varphi \rightarrow \psi)) \rightarrow (\text{Prover}_\Sigma(\varphi) \rightarrow \text{Prover}_\Sigma(\psi)) \)
3. \( P_3 : \Sigma \vdash \text{Prover}_\Sigma(\varphi) \rightarrow \text{Prover}_\Sigma(\text{Prover}_\Sigma(\varphi)) \)

The lemma follows from the last two lines:

\[ \vdash \neg \neg \text{Prover}_\Sigma(\alpha) \rightarrow \neg \alpha \]
by \( \varphi \rightarrow \varphi \)

\[ \neg \text{Prover}_\Sigma(\alpha) \rightarrow \neg \alpha \]
by \( \psi \rightarrow \neg \neg \psi \)

\[ \Sigma \vdash \text{Prover}_\Sigma(\text{Prover}_\Sigma(\alpha) \rightarrow \neg \alpha) \]
\( P_1 \)

\[ \Sigma \vdash \text{Prover}_\Sigma(\text{Prover}_\Sigma(\alpha)) \rightarrow \text{Prover}_\Sigma(\neg \alpha) \]
\( P_2 \)

\[ \Sigma \vdash \text{Prover}_\Sigma(\alpha) \rightarrow \text{Prover}_\Sigma(\text{Prover}_\Sigma(\alpha)) \]
\( P_3 \)
Lemma 1

\[ \alpha \equiv \neg \text{Prover}_\Sigma(\alpha) \] (syntactic identity)

Lemma (1)

\[ \Sigma \vdash \text{Prover}_\Sigma(\alpha) \rightarrow \text{Prover}_\Sigma(\neg \alpha) \]

Proof. Assume \( \Sigma \) is rich enough to prove:

\[ P_1 : \Sigma \vdash \varphi \text{ implies } \Sigma \vdash \text{Prover}_\Sigma(\varphi) \]

\[ P_2 : \Sigma \vdash (\text{Prover}_\Sigma(\varphi \rightarrow \psi)) \rightarrow (\text{Prover}_\Sigma(\varphi) \rightarrow \text{Prover}_\Sigma(\psi)) \]

\[ P_3 : \Sigma \vdash \text{Prover}_\Sigma(\varphi) \rightarrow \text{Prover}_\Sigma(\text{Prover}_\Sigma(\varphi)) \]

The lemma follows from the last two lines:

\[ \vdash \neg \neg \text{Prover}_\Sigma(\alpha) \rightarrow \neg \alpha \quad \text{by } \varphi \rightarrow \varphi \]

\[ \vdash \text{Prover}_\Sigma(\alpha) \rightarrow \neg \alpha \quad \psi \rightarrow \neg \neg \psi \]

\[ \Sigma \vdash \text{Prover}_\Sigma(\text{Prover}_\Sigma(\alpha) \rightarrow \neg \alpha) \quad P_1 \]

\[ \Sigma \vdash \text{Prover}_\Sigma(\text{Prover}_\Sigma(\alpha)) \rightarrow \text{Prover}_\Sigma(\neg \alpha) \quad P_2 \]

\[ \Sigma \vdash \text{Prover}_\Sigma(\alpha) \rightarrow \text{Prover}_\Sigma(\text{Prover}_\Sigma(\alpha)) \quad P_3 \]
Lemma 1
\[ \alpha \equiv \neg \text{Prover}_\Sigma(\alpha) \text{ (syntactic identity)} \]

Lemma (1)
\[ \Sigma \vdash \text{Prover}_\Sigma(\alpha) \rightarrow \text{Prover}_\Sigma(\neg \alpha) \]

Proof. Assume \( \Sigma \) is rich enough to prove:

\[ P_1 : \Sigma \vdash \varphi \text{ implies } \Sigma \vdash \text{Prover}_\Sigma(\varphi) \]
\[ P_2 : \Sigma \vdash (\text{Prover}_\Sigma(\varphi \rightarrow \psi)) \rightarrow (\text{Prover}_\Sigma(\varphi) \rightarrow \text{Prover}_\Sigma(\psi)) \]
\[ P_3 : \Sigma \vdash \text{Prover}_\Sigma(\varphi) \rightarrow \text{Prover}_\Sigma(\text{Prover}_\Sigma(\varphi)) \]

The lemma follows from the last two lines:

\[ \vdash \neg \neg \text{Prover}_\Sigma(\alpha) \rightarrow \neg \alpha \quad \text{by } \varphi \rightarrow \varphi \]
\[ \vdash \text{Prover}_\Sigma(\alpha) \rightarrow \neg \alpha \quad \psi \rightarrow \neg \neg \psi \]
\[ \Sigma \vdash \text{Prover}_\Sigma(\text{Prover}_\Sigma(\alpha)) \rightarrow \neg \alpha \quad P_1 \]
\[ \Sigma \vdash \text{Prover}_\Sigma(\text{Prover}_\Sigma(\alpha)) \rightarrow \text{Prover}_\Sigma(\neg \alpha) \quad P_2 \]
\[ \Sigma \vdash \text{Prover}_\Sigma(\alpha) \rightarrow \text{Prover}_\Sigma(\text{Prover}_\Sigma(\alpha)) \quad P_3 \]
Lemma 1
\[ \alpha \equiv \neg \text{Prover}_\Sigma(\alpha) \] (syntactic identity)

Lemma (1)
\[ \Sigma \vdash \text{Prover}_\Sigma(\alpha) \rightarrow \text{Prover}_\Sigma(\neg \alpha) \]

Proof. Assume \( \Sigma \) is rich enough to prove:

\[ P_1 : \Sigma \vdash \varphi \text{ implies } \Sigma \vdash \text{Prover}_\Sigma(\varphi) \]
\[ P_2 : \Sigma \vdash (\text{Prover}_\Sigma(\varphi \rightarrow \psi)) \rightarrow (\text{Prover}_\Sigma(\varphi) \rightarrow \text{Prover}_\Sigma(\psi)) \]
\[ P_3 : \Sigma \vdash \text{Prover}_\Sigma(\varphi) \rightarrow \text{Prover}_\Sigma(\text{Prover}_\Sigma(\varphi)) \]

The lemma follows from the last two lines:

\[ \vdash \neg \neg \text{Prover}_\Sigma(\alpha) \rightarrow \neg \alpha \quad \text{by } \varphi \rightarrow \varphi \]
\[ \vdash \text{Prover}_\Sigma(\alpha) \rightarrow \neg \alpha \quad \psi \rightarrow \neg \neg \psi \]
\[ \Sigma \vdash \text{Prover}_\Sigma(\text{Prover}_\Sigma(\alpha) \rightarrow \neg \alpha) \quad P_1 \]
\[ \Sigma \vdash \text{Prover}_\Sigma(\text{Prover}_\Sigma(\alpha)) \rightarrow \text{Prover}_\Sigma(\neg \alpha) \quad P_2 \]
\[ \Sigma \vdash \text{Prover}_\Sigma(\alpha) \rightarrow \text{Prover}_\Sigma(\text{Prover}_\Sigma(\alpha)) \quad P_3 \]
Lemma 1

\[ \alpha \equiv \neg \text{Prover}_\Sigma (\alpha) \text{ (syntactic identity)} \]

Lemma (1)

\[ \Sigma \vdash \text{Prover}_\Sigma (\alpha) \rightarrow \text{Prover}_\Sigma (\neg \alpha) \]

Proof. Assume \( \Sigma \) is rich enough to prove:

\[ P_1 : \Sigma \vdash \varphi \text{ implies } \Sigma \vdash \text{Prover}_\Sigma (\varphi) \]
\[ P_2 : \Sigma \vdash (\text{Prover}_\Sigma (\varphi \rightarrow \psi)) \rightarrow (\text{Prover}_\Sigma (\varphi) \rightarrow \text{Prover}_\Sigma (\psi)) \]
\[ P_3 : \Sigma \vdash \text{Prover}_\Sigma (\varphi) \rightarrow \text{Prover}_\Sigma (\text{Prover}_\Sigma (\varphi)) \]

The lemma follows from the last two lines:

\[ \vdash \neg \neg \text{Prover}_\Sigma (\alpha) \rightarrow \neg \alpha \quad \text{by } \varphi \rightarrow \varphi \]
\[ \vdash \text{Prover}_\Sigma (\alpha) \rightarrow \neg \alpha \quad \psi \rightarrow \neg \neg \psi \]
\[ \Sigma \vdash \text{Prover}_\Sigma (\text{Prover}_\Sigma (\alpha) \rightarrow \neg \alpha) \quad P_1 \]
\[ \Sigma \vdash \text{Prover}_\Sigma (\text{Prover}_\Sigma (\alpha)) \rightarrow \text{Prover}_\Sigma (\neg \alpha) \quad P_2 \]
\[ \Sigma \vdash \text{Prover}_\Sigma (\alpha) \rightarrow \text{Prover}_\Sigma (\text{Prover}_\Sigma (\alpha)) \quad P_3 \]
Lemma 1

\[ \alpha \equiv \neg \text{Prover}_\Sigma(\alpha) \] (syntactic identity)

Lemma (1)

\[ \Sigma \vdash \text{Prover}_\Sigma(\alpha) \rightarrow \text{Prover}_\Sigma(\neg \alpha) \]

Proof. Assume \( \Sigma \) is rich enough to prove:

\[ P_1 : \Sigma \vdash \varphi \text{ implies } \Sigma \vdash \text{Prover}_\Sigma(\varphi) \]
\[ P_2 : \Sigma \vdash (\text{Prover}_\Sigma(\varphi \rightarrow \psi)) \rightarrow (\text{Prover}_\Sigma(\varphi) \rightarrow \text{Prover}_\Sigma(\psi)) \]
\[ P_3 : \Sigma \vdash \text{Prover}_\Sigma(\varphi) \rightarrow \text{Prover}_\Sigma(\text{Prover}_\Sigma(\varphi)) \]

The lemma follows from the last two lines:

\[ \vdash \neg \neg \text{Prover}_\Sigma(\alpha) \rightarrow \neg \alpha \quad \text{by } \varphi \rightarrow \varphi \]
\[ \vdash \text{Prover}_\Sigma(\alpha) \rightarrow \neg \alpha \quad \psi \rightarrow \neg \neg \psi \]
Lemma 1

\[ \alpha \equiv \neg \text{Prover}_\Sigma(\alpha) \] (syntactic identity)

Lemma (1)

\[ \Sigma \vdash \text{Prover}_\Sigma(\alpha) \to \text{Prover}_\Sigma(\neg \alpha) \]

Proof. Assume \( \Sigma \) is rich enough to prove:

\[ P_1 : \Sigma \vdash \varphi \text{ implies } \Sigma \vdash \text{Prover}_\Sigma(\varphi) \]
\[ P_2 : \Sigma \vdash (\text{Prover}_\Sigma(\varphi \to \psi)) \to (\text{Prover}_\Sigma(\varphi) \to \text{Prover}_\Sigma(\psi)) \]
\[ P_3 : \Sigma \vdash \text{Prover}_\Sigma(\varphi) \to \text{Prover}_\Sigma(\text{Prover}_\Sigma(\varphi)) \]

The lemma follows from the last two lines:

\[ \vdash \neg \neg \text{Prover}_\Sigma(\alpha) \to \neg \alpha \]
\[ \vdash \text{Prover}_\Sigma(\alpha) \to \neg \alpha \]
\[ \Sigma \vdash \text{Prover}_\Sigma(\text{Prover}_\Sigma(\alpha) \to \neg \alpha) \]

\[ \Sigma \vdash \text{Prover}_\Sigma(\text{Prover}_\Sigma(\alpha)) \to \text{Prover}_\Sigma(\neg \alpha) \]
\[ \Sigma \vdash \text{Prover}_\Sigma(\alpha) \to \text{Prover}_\Sigma(\text{Prover}_\Sigma(\alpha)) \]
Lemma 1

\[ \alpha \equiv \neg \text{Prover}_\Sigma (\alpha) \] (syntactic identity)

Lemma (1)

\[ \Sigma \vdash \text{Prover}_\Sigma (\alpha) \rightarrow \text{Prover}_\Sigma (\neg \alpha) \]

Proof. Assume \( \Sigma \) is rich enough to prove:

\[ P_1 : \Sigma \vdash \varphi \text{ implies } \Sigma \vdash \text{Prover}_\Sigma (\varphi) \]
\[ P_2 : \Sigma \vdash (\text{Prover}_\Sigma (\varphi \rightarrow \psi)) \rightarrow (\text{Prover}_\Sigma (\varphi) \rightarrow \text{Prover}_\Sigma (\psi)) \]
\[ P_3 : \Sigma \vdash \text{Prover}_\Sigma (\varphi) \rightarrow \text{Prover}_\Sigma (\text{Prover}_\Sigma (\varphi)) \]

The lemma follows from the last two lines:

\[ \vdash \neg \neg \text{Prover}_\Sigma (\alpha) \rightarrow \neg \alpha \quad \text{by } \varphi \rightarrow \varphi \]
\[ \vdash \text{Prover}_\Sigma (\alpha) \rightarrow \neg \alpha \quad \psi \rightarrow \neg \neg \psi \]
\[ \Sigma \vdash \text{Prover}_\Sigma (\text{Prover}_\Sigma (\alpha) \rightarrow \neg \alpha) \quad P_1 \]
\[ \Sigma \vdash \text{Prover}_\Sigma (\text{Prover}_\Sigma (\alpha)) \rightarrow \text{Prover}_\Sigma (\neg \alpha) \quad P_2 \]
Lemma 1

\[ \alpha \equiv \neg \text{Prover}_\Sigma (\alpha) \] (syntactic identity)

Lemma (1)

\[ \Sigma \vdash \text{Prover}_\Sigma (\alpha) \rightarrow \text{Prover}_\Sigma (\neg \alpha) \]

Proof. Assume \( \Sigma \) is rich enough to prove:

\[ P_1 : \Sigma \vdash \varphi \text{ implies } \Sigma \vdash \text{Prover}_\Sigma (\varphi) \]
\[ P_2 : \Sigma \vdash (\text{Prover}_\Sigma (\varphi \rightarrow \psi)) \rightarrow (\text{Prover}_\Sigma (\varphi) \rightarrow \text{Prover}_\Sigma (\psi)) \]
\[ P_3 : \Sigma \vdash \text{Prover}_\Sigma (\varphi) \rightarrow \text{Prover}_\Sigma (\text{Prover}_\Sigma (\varphi)) \]

The lemma follows from the last two lines:

\[ \vdash \neg \neg \text{Prover}_\Sigma (\alpha) \rightarrow \neg \alpha \quad \text{by } \varphi \rightarrow \varphi \]
\[ \vdash \text{Prover}_\Sigma (\alpha) \rightarrow \neg \alpha \quad \psi \rightarrow \neg \neg \psi \]
\[ \Sigma \vdash \text{Prover}_\Sigma (\text{Prover}_\Sigma (\alpha) \rightarrow \neg \alpha) \quad P_1 \]
\[ \Sigma \vdash \text{Prover}_\Sigma (\text{Prover}_\Sigma (\alpha)) \rightarrow \text{Prover}_\Sigma (\neg \alpha) \quad P_2 \]
\[ \Sigma \vdash \text{Prover}_\Sigma (\alpha) \rightarrow \text{Prover}_\Sigma (\text{Prover}_\Sigma (\alpha)) \quad P_3 \]
Lemma 2

\[ \alpha \equiv \neg \text{Prover}_\Sigma(\alpha) \text{ (syntax)} \quad \Sigma \vdash \text{Prover}_\Sigma(\alpha) \rightarrow \text{Prover}_\Sigma(\neg \alpha) \text{ (Lemma 1)} \]

Lemma (2)

\[ \Sigma \vdash \text{Prover}_\Sigma(\alpha) \rightarrow \text{Prover}_\Sigma(F) \]

Assume \( \Sigma \) is rich enough to also prove:

\[ P_4 : \Sigma \vdash \text{Prover}_\Sigma(\varphi) \land \text{Prover}_\Sigma(\psi) \rightarrow \text{Prover}_\Sigma(\varphi \land \psi) \]

Lemma 2 follows from the last line:

\[ \Sigma, \text{Prover}_\Sigma(\alpha) \vdash \text{Prover}_\Sigma(\neg \alpha) \quad \text{Lemma 1 and Deduction Lemma} \]

\[ \Sigma, \text{Prover}_\Sigma(\alpha) \vdash \text{Prover}_\Sigma(\neg \alpha \land \alpha) \quad P_4 \]

\[ \Sigma, \text{Prover}_\Sigma(\alpha) \vdash \text{Prover}_\Sigma(F) \quad \neg \alpha \land \alpha \rightarrow F \]
Lemma 2

\[ \alpha \equiv \neg \text{Prover}_\Sigma (\alpha) \text{ (syntax)} \quad \Sigma \vdash \text{Prover}_\Sigma (\alpha) \to \text{Prover}_\Sigma (\neg \alpha) \text{ (Lemma 1)} \]

Lemma (2)

\[ \Sigma \vdash \text{Prover}_\Sigma (\alpha) \to \text{Prover}_\Sigma (F) \]

Assume \( \Sigma \) is rich enough to also prove:

\[ P_4 : \Sigma \vdash \text{Prover}_\Sigma (\varphi) \land \text{Prover}_\Sigma (\psi) \to \text{Prover}_\Sigma (\varphi \land \psi) \]

Lemma 2 follows from the last line:

\[ \Sigma, \text{Prover}_\Sigma (\alpha) \vdash \text{Prover}_\Sigma (\neg \alpha) \text{ Lemma 1 and Deduction Lemma} \]
\[ \Sigma, \text{Prover}_\Sigma (\alpha) \vdash \text{Prover}_\Sigma (\neg \alpha \land \alpha) \text{ } P_4 \]
\[ \Sigma, \text{Prover}_\Sigma (\alpha) \vdash \text{Prover}_\Sigma (F) \text{ } \neg \alpha \land \alpha \to F \]
Lemma 2

\[ \alpha \equiv \neg \text{Prover}_\Sigma(\alpha) \text{ (syntax)} \quad \Sigma \vdash \text{Prover}_\Sigma(\alpha) \rightarrow \text{Prover}_\Sigma(\neg \alpha) \text{ (Lemma 1)} \]

Lemma (2)

\[ \Sigma \vdash \text{Prover}_\Sigma(\alpha) \rightarrow \text{Prover}_\Sigma(F) \]

Assume \( \Sigma \) is rich enough to also prove:

\[ P_4 : \Sigma \vdash \text{Prover}_\Sigma(\varphi) \land \text{Prover}_\Sigma(\psi) \rightarrow \text{Prover}_\Sigma(\varphi \land \psi) \]

Lemma 2 follows from the last line:

\[ \Sigma, \text{Prover}_\Sigma(\alpha) \vdash \text{Prover}_\Sigma(\neg \alpha) \quad \text{Lemma 1 and Deduction Lemma} \]

\[ \Sigma, \text{Prover}_\Sigma(\alpha) \vdash \text{Prover}_\Sigma(\neg \alpha \land \alpha) \quad P_4 \]

\[ \Sigma, \text{Prover}_\Sigma(\alpha) \vdash \text{Prover}_\Sigma(F) \quad \neg \alpha \land \alpha \rightarrow F \]
Lemma 2

\[ \alpha \equiv \neg \text{Prover}_\Sigma(\alpha) \]  (syntax)  \[ \Sigma \vdash \text{Prover}_\Sigma(\alpha) \rightarrow \text{Prover}_\Sigma(\neg \alpha) \]  (Lemma 1)

Lemma (2)

\[ \Sigma \vdash \text{Prover}_\Sigma(\alpha) \rightarrow \text{Prover}_\Sigma(\textbf{F}) \]

Assume \( \Sigma \) is rich enough to also prove:

\[ P_4 : \Sigma \vdash \text{Prover}_\Sigma(\varphi) \land \text{Prover}_\Sigma(\psi) \rightarrow \text{Prover}_\Sigma(\varphi \land \psi) \]

Lemma 2 follows from the last line:

\[ \Sigma, \text{Prover}_\Sigma(\alpha) \vdash \text{Prover}_\Sigma(\neg \alpha) \]  Lemma 1 and Deduction Lemma
\[ \Sigma, \text{Prover}_\Sigma(\alpha) \vdash \text{Prover}_\Sigma(\neg \alpha \land \alpha) \]  \( P_4 \)
\[ \Sigma, \text{Prover}_\Sigma(\alpha) \vdash \text{Prover}_\Sigma(\textbf{F}) \]  \( \neg \alpha \land \alpha \rightarrow \textbf{F} \)
Lemma 2

\[ \alpha \equiv \neg \text{Prover}_\Sigma(\alpha) \quad \text{(syntax)} \quad \Sigma \vdash \text{Prover}_\Sigma(\alpha) \rightarrow \text{Prover}_\Sigma(\neg \alpha) \quad \text{(Lemma 1)} \]

**Lemma (2)**

\[ \Sigma \vdash \text{Prover}_\Sigma(\alpha) \rightarrow \text{Prover}_\Sigma(F) \]

Assume \( \Sigma \) is rich enough to also prove:

\[ P_4 : \Sigma \vdash \text{Prover}_\Sigma(\varphi) \land \text{Prover}_\Sigma(\psi) \rightarrow \text{Prover}_\Sigma(\varphi \land \psi) \]

Lemma 2 follows from the last line:

\[
\begin{align*}
\Sigma, \text{Prover}_\Sigma(\alpha) & \vdash \text{Prover}_\Sigma(\neg \alpha) & \text{Lemma 1 and Deduction Lemma} \\
\Sigma, \text{Prover}_\Sigma(\alpha) & \vdash \text{Prover}_\Sigma(\neg \alpha \land \alpha) & P_4 \\
\Sigma, \text{Prover}_\Sigma(\alpha) & \vdash \text{Prover}_\Sigma(F) & \neg \alpha \land \alpha \rightarrow F
\end{align*}
\]
Proof of Gödel’s First Incompleteness Theorems

\[ \alpha \equiv \neg \text{Prover}_\Sigma(\alpha) \text{ (syntax)} \quad \Sigma \vdash \text{Prover}_\Sigma(\alpha) \rightarrow \text{Prover}_\Sigma(F) \text{ (Lemma 2)} \]

**Theorem (\( \Sigma \) Is Not Complete)**

*If \( \Sigma \) is consistent \( (\Sigma \nvdash F) \), then \( \Sigma \nvdash \alpha \) and \( \Sigma \nvdash \neg \alpha \).*

Proof:

Suppose \( \Sigma \vdash \alpha \):

- \( \Sigma \vdash \text{Prover}_\Sigma(\alpha) \) \( P_1 \)
- \( \Sigma \vdash \neg \text{Prover}_\Sigma(\alpha) \) \( \text{syntax} \)
- \( \Sigma \vdash F \) \( \varphi, \neg \varphi \vdash F \)

Suppose \( \Sigma \vdash \neg \alpha \):

- \( \Sigma \vdash \neg \neg \text{Prover}_\Sigma(\alpha) \) \( \text{syntax} \)
- \( \Sigma \vdash \text{Prover}_\Sigma(\alpha) \) \( A_3 \)
- \( \Sigma \vdash \text{Prover}_\Sigma(F) \) \( \text{Lemma 2} \)
- \( \Sigma \vdash F \) \( \text{soundness} \)
Proof of Gödel’s First Incompleteness Theorems

\[ \alpha \equiv \neg \text{Prover}_\Sigma(\alpha) \quad (\text{syntax}) \quad \Sigma \vdash \text{Prover}_\Sigma(\alpha) \rightarrow \text{Prover}_\Sigma(F) \quad (\text{Lemma 2}) \]

**Theorem (\(\Sigma\) Is Not Complete)**

*If \(\Sigma\) is consistent (\(\Sigma \not\vdash F\)), then \(\Sigma \not\vdash \alpha\) and \(\Sigma \not\vdash \neg \alpha\).*

**Proof:**

Suppose \(\Sigma \vdash \alpha\):

\[ \Sigma \vdash \text{Prover}_\Sigma(\alpha) \quad P_1 \]

\[ \Sigma \vdash \neg \text{Prover}_\Sigma(\alpha) \quad \text{syntax} \]

\[ \Sigma \vdash F \quad \varphi, \neg \varphi \vdash F \]

Suppose \(\Sigma \vdash \neg \alpha\):

\[ \Sigma \vdash \neg \neg \text{Prover}_\Sigma(\alpha) \quad \text{syntax} \]

\[ \Sigma \vdash \text{Prover}_\Sigma(\alpha) \quad A_3 \]

\[ \Sigma \vdash \text{Prover}_\Sigma(F) \quad \text{Lemma 2} \]

\[ \Sigma \vdash F \quad \text{soundness} \]
Proof of Gödel’s First Incompleteness Theorems

\[ \alpha \equiv \neg \text{Prover}_\Sigma(\alpha) \] (syntax) \quad \Sigma \vdash \text{Prover}_\Sigma(\alpha) \to \text{Prover}_\Sigma(F) \quad \text{(Lemma 2)}

**Theorem (\Sigma \text{ Is Not Complete})**

*If \( \Sigma \) is consistent (\( \Sigma \not\vdash F \)), then \( \Sigma \not\vdash \alpha \) and \( \Sigma \not\vdash \neg \alpha \).*

Proof:

Suppose \( \Sigma \vdash \alpha \):

\[ \Sigma \vdash \text{Prover}_\Sigma(\alpha) \quad P_1 \]
\[ \text{syntax} \]
\[ \text{syntax} \]
\[ \varphi, \neg \varphi \vdash F \]

Suppose \( \Sigma \vdash \neg \alpha \):

\[ \Sigma \vdash \neg \neg \text{Prover}_\Sigma(\alpha) \quad \text{syntax} \]
\[ \Sigma \vdash \text{Prover}_\Sigma(\alpha) \quad A_3 \]
\[ \Sigma \vdash \text{Prover}_\Sigma(F) \quad \text{Lemma 2} \]
\[ \text{soundness} \]
Proof of Gödel’s First Incompleteness Theorems

\[ \alpha \equiv \neg \text{Prover}_\Sigma (\alpha) \text{ (syntax)} \quad \Sigma \vdash \text{Prover}_\Sigma (\alpha) \rightarrow \text{Prover}_\Sigma (F) \text{ (Lemma 2)} \]

**Theorem (\(\Sigma\) Is Not Complete)**

*If \(\Sigma\) is consistent (\(\Sigma \not\vdash F\)), then \(\Sigma \not\vdash \alpha\) and \(\Sigma \not\vdash \neg \alpha\).*

**Proof:**

Suppose \(\Sigma \vdash \alpha\):

\[ \Sigma \vdash \text{Prover}_\Sigma (\alpha) \quad P_1 \]
\[ \Sigma \vdash \neg \text{Prover}_\Sigma (\alpha) \text{ syntax} \]
\[ \Sigma \vdash F \quad \varphi, \neg \varphi \vdash F \]

Suppose \(\Sigma \vdash \neg \alpha\):

\[ \Sigma \vdash \neg \neg \text{Prover}_\Sigma (\alpha) \text{ syntax} \]
\[ \Sigma \vdash \text{Prover}_\Sigma (\alpha) \text{ A}_3 \]
\[ \Sigma \vdash \text{Prover}_\Sigma (F) \text{ Lemma 2 soundness} \]
\[ \Sigma \vdash F \]
Proof of Gödel’s First Incompleteness Theorems

\[ \alpha \equiv \neg \text{Prover}_\Sigma(\alpha) \quad \text{(syntax)} \quad \Sigma \vdash \text{Prover}_\Sigma(\alpha) \rightarrow \text{Prover}_\Sigma(F) \quad \text{(Lemma 2)} \]

**Theorem (\(\Sigma\) Is Not Complete)**

*If \(\Sigma\) is consistent (\(\Sigma \not\vdash F\)), then \(\Sigma \not\vdash \alpha\) and \(\Sigma \not\vdash \neg \alpha\).*

**Proof:**

Suppose \(\Sigma \vdash \alpha\):

\[ \Sigma \vdash \text{Prover}_\Sigma(\alpha) \quad P_1 \]
\[ \Sigma \not\vdash \neg \text{Prover}_\Sigma(\alpha) \quad \text{syntax} \]
\[ \Sigma \vdash F \quad \varphi, \neg \varphi \vdash F \]

Suppose \(\Sigma \vdash \neg \alpha\):

\[ \Sigma \vdash \neg \neg \text{Prover}_\Sigma(\alpha) \quad \text{syntax} \]
\[ \Sigma \vdash \text{Prover}_\Sigma(\alpha) \quad A_3 \]
\[ \Sigma \vdash \text{Prover}_\Sigma(F) \quad \text{Lemma 2} \]
\[ \Sigma \vdash F \quad \text{soundness} \]
Proof of Gödel’s First Incompleteness Theorems

\[ \alpha \equiv \neg \text{Prover}_\Sigma(\alpha) \quad (\text{syntax}) \quad \Sigma \vdash \text{Prover}_\Sigma(\alpha) \rightarrow \text{Prover}_\Sigma(F) \quad (\text{Lemma 2}) \]

**Theorem (\(\Sigma\) Is Not Complete)**

*If \(\Sigma\) is consistent (\(\Sigma \not\vdash F\)), then \(\Sigma \not\vdash \alpha\) and \(\Sigma \not\vdash \neg \alpha\).*

**Proof:**

Suppose \(\Sigma \vdash \alpha\):

\[ \Sigma \vdash \text{Prover}_\Sigma(\alpha) \quad P_1 \]

\[ \Sigma \vdash \neg \text{Prover}_\Sigma(\alpha) \quad \text{syntax} \]

\[ \Sigma \vdash F \quad \varphi, \neg \varphi \vdash F \]

Suppose \(\Sigma \vdash \neg \alpha\):

\[ \Sigma \vdash \neg \neg \text{Prover}_\Sigma(\alpha) \quad \text{syntax} \]

\[ \Sigma \vdash \text{Prover}_\Sigma(\alpha) \quad A_3 \]

\[ \Sigma \vdash \text{Prover}_\Sigma(F) \quad \text{Lemma 2} \]

\[ \Sigma \vdash \neg \text{Prover}_\Sigma(F) \quad \text{soundness} \]
Proof of Gödel’s First Incompleteness Theorems

\[ \alpha \equiv \neg \text{Prover}_\Sigma(\alpha) \text{ (syntax)} \quad \Sigma \vdash \text{Prover}_\Sigma(\alpha) \rightarrow \text{Prover}_\Sigma(F) \text{ (Lemma 2)} \]

**Theorem (\(\Sigma\) Is Not Complete)**

*If \(\Sigma\) is consistent \((\Sigma \not\vdash F)\), then \(\Sigma \not\vdash \alpha\) and \(\Sigma \not\vdash \neg \alpha\).*

Proof:

Suppose \(\Sigma \vdash \alpha\):

\[
\begin{align*}
\Sigma &\vdash \text{Prover}_\Sigma(\alpha) & P_1 \\
\Sigma &\vdash \neg \text{Prover}_\Sigma(\alpha) & \text{syntax} \\
\Sigma &\vdash F & \varphi, \neg \varphi \vdash F
\end{align*}
\]

Suppose \(\Sigma \vdash \neg \alpha\):

\[
\begin{align*}
\Sigma &\vdash \neg \neg \text{Prover}_\Sigma(\alpha) & \text{syntax} \\
\Sigma &\vdash \text{Prover}_\Sigma(\alpha) & A_3 \\
\Sigma &\vdash \text{Prover}_\Sigma(F) & \text{Lemma 2} \\
\Sigma &\vdash F & \text{soundness}
\end{align*}
\]
Proof of Gödel’s First Incompleteness Theorems

\[ \alpha \equiv \neg \text{Prover}_\Sigma(\alpha) \text{ (syntax)} \quad \Sigma \vdash \text{Prover}_\Sigma(\alpha) \rightarrow \text{Prover}_\Sigma(F) \text{ (Lemma 2)} \]

**Theorem (\(\Sigma\) Is Not Complete)**

*If \(\Sigma\) is consistent (\(\Sigma \not\models F\)), then \(\Sigma \not\models \alpha\) and \(\Sigma \not\models \neg \alpha\).*

Proof:

Suppose \(\Sigma \vdash \alpha\):

\[ \Sigma \vdash \text{Prover}_\Sigma(\alpha) \quad P_1 \]
\[ \Sigma \vdash \neg \text{Prover}_\Sigma(\alpha) \quad \text{syntax} \]
\[ \Sigma \vdash F \quad \varphi, \neg \varphi \vdash F \]

Suppose \(\Sigma \vdash \neg \alpha\):

\[ \Sigma \vdash \neg \neg \text{Prover}_\Sigma(\alpha) \quad \text{syntax} \]
\[ \Sigma \vdash \text{Prover}_\Sigma(\alpha) \quad A_3 \]
\[ \Sigma \vdash \text{Prover}_\Sigma(F) \quad \text{Lemma 2} \]
\[ \Sigma \vdash F \quad \text{soundness} \]
Proof of Gödel’s First Incompleteness Theorems

\[ \alpha \equiv \neg \text{Prover}_\Sigma(\alpha) \quad \text{(syntax)} \]
\[ \Sigma \vdash \text{Prover}_\Sigma(\alpha) \rightarrow \text{Prover}_\Sigma(F) \quad \text{(Lemma 2)} \]

**Theorem (Σ Is Not Complete)**

*If Σ is consistent (Σ \( \not\vdash F \)), then Σ \( \not\vdash \alpha \) and Σ \( \not\vdash \neg \alpha \).*

**Proof:**

Suppose \( \Sigma \vdash \alpha \):

\[ \Sigma \vdash \text{Prover}_\Sigma(\alpha) \quad P_1 \]
\[ \Sigma \vdash \neg \text{Prover}_\Sigma(\alpha) \quad \text{syntax} \]
\[ \Sigma \vdash F \quad \varphi, \neg \varphi \vdash F \]

Suppose \( \Sigma \vdash \neg \alpha \):

\[ \Sigma \vdash \neg \neg \text{Prover}_\Sigma(\alpha) \quad \text{syntax} \]
\[ \Sigma \vdash \text{Prover}_\Sigma(\alpha) \quad A_3 \]
\[ \Sigma \vdash \text{Prover}_\Sigma(F) \quad \text{Lemma 2} \]
\[ \Sigma \vdash F \quad \text{soundness} \]
Proof of Gödel’s First Incompleteness Theorems

\[ \alpha \equiv \neg \text{Prover}_\Sigma(\alpha) \] (syntax) \quad \Sigma \vdash \text{Prover}_\Sigma(\alpha) \to \text{Prover}_\Sigma(F) \quad (\text{Lemma 2})

**Theorem (\(\Sigma\) Is Not Complete)**

*If \(\Sigma\) is consistent \((\Sigma \not\vdash F)\), then \(\Sigma \not\vdash \alpha\) and \(\Sigma \not\vdash \neg \alpha\).*

Proof:

Suppose \(\Sigma \vdash \alpha\):

\[ \Sigma \vdash \text{Prover}_\Sigma(\alpha) \quad P_1 \]
\[ \Sigma \vdash \neg \text{Prover}_\Sigma(\alpha) \quad \text{syntax} \]
\[ \Sigma \vdash F \quad \varphi, \neg \varphi \vdash F \]

Suppose \(\Sigma \vdash \neg \alpha\):

\[ \Sigma \vdash \neg \neg \text{Prover}_\Sigma(\alpha) \quad \text{syntax} \]
\[ \Sigma \vdash \text{Prover}_\Sigma(\alpha) \quad A_3 \]
\[ \Sigma \vdash \text{Prover}_\Sigma(F) \quad \text{Lemma 2} \]
\[ \Sigma \vdash F \quad \text{soundness} \]
Proof of Gödel’s Second Incompleteness Theorems

\[ \alpha \equiv \neg \text{Prover}_\Sigma(\alpha) \] (syntax) \quad \Sigma \vdash \text{Prover}_\Sigma(\alpha) \rightarrow \text{Prover}_\Sigma(F) \] (Lemma 2)

**Theorem (\(\Sigma\) Cannot Prove its Own Consistency)**

\[ \Sigma \not\vdash \neg \text{Prover}_\Sigma(F) \]

Proof: suppose \(\Sigma \vdash \neg \text{Prover}_\Sigma(F)\)

\[ \begin{align*}
\Sigma & \vdash \neg \text{Prover}_\Sigma(F) \rightarrow \neg \text{Prover}_\Sigma(\alpha) \quad \text{(Lemma 2)} \\
\Sigma & \vdash \neg \text{Prover}_\Sigma(\alpha) \quad \text{Modus Ponens} \\
\Sigma & \vdash \alpha \quad \text{Syntax} \\
\Sigma & \vdash F \quad \text{First Incompleteness Theorem}
\end{align*} \]
Proof of Gödel’s Second Incompleteness Theorems

\[ \alpha \equiv \neg \text{Prover}_\Sigma (\alpha) \text{ (syntax)} \quad \Sigma \vdash \text{Prover}_\Sigma (\alpha) \rightarrow \text{Prover}_\Sigma (F) \text{ (Lemma 2)} \]

**Theorem** (**Σ Cannot Prove its Own Consistency**)  
\[ \Sigma \not\vdash \neg \text{Prover}_\Sigma (F) \]

**Proof:** suppose \( \Sigma \vdash \neg \text{Prover}_\Sigma (F) \)

\[ \begin{align*}
\Sigma & \vdash \neg \text{Prover}_\Sigma (F) \rightarrow \neg \text{Prover}_\Sigma (\alpha) \quad \text{Lemma 2} \\
\Sigma & \vdash \neg \text{Prover}_\Sigma (\alpha) \quad \text{Modus Ponens} \\
\Sigma & \vdash \alpha \quad \text{Syntax} \\
\Sigma & \vdash F \quad \text{First Incompleteness Theorem}
\end{align*} \]
Proof of Gödel’s Second Incompleteness Theorems

\[ \alpha \equiv \neg \text{Prover}_\Sigma(\alpha) \] (syntax) \hspace{1cm} \Sigma \vdash \text{Prover}_\Sigma(\alpha) \rightarrow \text{Prover}_\Sigma(F) \] (Lemma 2)

**Theorem (\(\Sigma\) Cannot Prove its Own Consistency)**

\[ \Sigma \not\vdash \neg \text{Prover}_\Sigma(F) \]

Proof: suppose \(\Sigma \vdash \neg \text{Prover}_\Sigma(F)\)

\[ \begin{align*}
\Sigma & \vdash \neg \text{Prover}_\Sigma(F) \rightarrow \neg \text{Prover}_\Sigma(\alpha) \\
\Sigma & \vdash \neg \text{Prover}_\Sigma(\alpha) \\
\Sigma & \vdash \alpha \\
\Sigma & \vdash F
\end{align*} \]

Lemma 2

Modus Ponens

Syntax

First Incompleteness Theorem
Proof of Gödel’s Second Incompleteness Theorems

\[ \alpha \equiv \neg \text{Prover}_\Sigma(\alpha) \] (syntax) \hspace{1cm} \Sigma \vdash \text{Prover}_\Sigma(\alpha) \rightarrow \text{Prover}_\Sigma(F) \] (Lemma 2)

**Theorem (Σ Cannot Prove its Own Consistency)**

\[ \Sigma \nvdash \neg \text{Prover}_\Sigma(F) \]

Proof: suppose \( \Sigma \vdash \neg \text{Prover}_\Sigma(F) \)

\[ \Sigma \vdash \neg \text{Prover}_\Sigma(F) \rightarrow \neg \text{Prover}_\Sigma(\alpha) \] (Lemma 2)

\[ \Sigma \vdash \neg \text{Prover}_\Sigma(\alpha) \] (Modus Ponens)

\[ \Sigma \vdash \alpha \] (Syntax)

\[ \Sigma \vdash F \] (First Incompleteness Theorem)
Proof of Gödel’s Second Incompleteness Theorems

\( \alpha \equiv \neg \text{Prover}_\Sigma(\alpha) \) (syntax) \hspace{1cm} \Sigma \vdash \text{Prover}_\Sigma(\alpha) \rightarrow \text{Prover}_\Sigma(F) \) (Lemma 2)

**Theorem (\( \Sigma \) Cannot Prove its Own Consistency)**

\( \Sigma \nvdash \neg \text{Prover}_\Sigma(F) \)

Proof: suppose \( \Sigma \vdash \neg \text{Prover}_\Sigma(F) \)

\[
\begin{align*}
\Sigma & \vdash \neg \text{Prover}_\Sigma(F) \rightarrow \neg \text{Prover}_\Sigma(\alpha) & \text{Lemma 2} \\
\Sigma & \vdash \neg \text{Prover}_\Sigma(\alpha) & \text{Modus Ponens} \\
\Sigma & \vdash \alpha & \text{Syntax} \\
\Sigma & \vdash F & \text{First Incompleteness Theorem}
\end{align*}
\]
Proof of Gödel’s Second Incompleteness Theorems

\[ \alpha \equiv \neg \text{Prover}_\Sigma(\alpha) \quad \text{(syntax)} \quad \Sigma \vdash \text{Prover}_\Sigma(\alpha) \rightarrow \text{Prover}_\Sigma(F) \quad \text{(Lemma 2)} \]

**Theorem (Σ Cannot Prove its Own Consistency)**

\[ \Sigma \nvdash \neg \text{Prover}_\Sigma(F) \]

**Proof:** suppose \( \Sigma \vdash \neg \text{Prover}_\Sigma(F) \)

\[ \begin{align*}
\Sigma & \vdash \neg \text{Prover}_\Sigma(F) \rightarrow \neg \text{Prover}_\Sigma(\alpha) & \text{Lemma 2} \\
\Sigma & \vdash \neg \text{Prover}_\Sigma(\alpha) & \text{Modus Ponens} \\
\Sigma & \vdash \alpha & \text{Syntax} \\
\Sigma & \vdash F & \text{First Incompleteness Theorem}
\end{align*} \]
We only proved that neither \( \alpha \) nor \( \neg \alpha \) is provable. Can we get a complete theory by adding \( \alpha \) or \( \neg \alpha \) to \( \Sigma \) (whichever is true)? In class.

Not all theories of \( \mathbb{N} \) are undecidable. Examples:\(^9\):

- \( (\mathbb{N}, 0, \text{succ}) \) is decidable (homework!).
- \( (\mathbb{N}, 0, \text{succ}, <) \) is decidable; can define finite and co-finite sets.
- \( (\mathbb{N}, 0, \text{succ}, +, <) \) is decidable and called Presburger Arithmetic; can define eventually periodic sets.
- \( (\mathbb{N}, 0, \text{succ}, +, *, <) \) is undecidable (Gödel).
- \( (\mathbb{C}, 0, 1, +, *) \) is decidable.

\(^9\)Enderton pp. 187, 197, 158