1 The Satisfiability Problem

1. (0 points)
   (a) Let \( \sigma = \{U_1, \ldots, U_m\} \) be a relational vocabulary with \( k \) unary predicate symbols. Prove that sentences over this vocabulary satisfy the small model property: if \( \phi \) has a model, then it has a finite model of size \( \leq f(|\phi|) \), for some computable function \( f \). (Hint: given any infinite structure \( A \) and a number \( k \) describe a finite model \( B \) s.t. \( A \sim_k B \).)

   (b) Prove that the satisfiability problem for a relational vocabulary consisting of only unary predicate symbols is decidable.

   (c) Let \( \phi(x) \) be formula with a free variable \( x \), and \( R \) be a unary relational symbol. We say that \( \phi \) is monotone in a relational symbol \( R \) if for any two structures \( A, B \) with the same domain and satisfying \( R^A \subseteq R^B \), and \( S^A = S^B \) for every other relational symbol \( S \), we have \( \{a \in A \mid A \models \phi(a)\} \subseteq \{b \in B \mid B \models \phi(b)\} \). (Note: this is the semantic property needed for the least fixpoint, \( lfp_{R,x} \phi \).) Prove that, if the vocabulary includes at least one binary relational symbol other than \( R \), then the problem “given \( \phi \) check if it is monotone in \( R \) over all finite structures” is undecidable.
2 Descriptive Complexity

2. (0 points)

(a) Let $G = (V, E)$ be a finite graph, and consider the following query:

$$q(x) = \text{Lfp}_{S,x} (\forall y (E(x, y) \rightarrow S(y))) (x)$$

i. Which nodes $x$ does the query return on the graph below?

ii. Write an FO sentence (without fixpoints!) that is equivalent to $\forall x q(x)$.

iii. Consider these complexity classes: $AC^0$, $PTIME$, $NP$, $PSPACE$. Indicate the lowest complexity class to which $q$ belongs. You can just indicate the lowest complexity class, no need to prove that it’s not lower than that (but you are welcome to do so).

(b) Consider the vocabulary $(<, P_a, P_b, P_c)$ of strings over the alphabet $\Sigma = \{a, b, c\}$.

i. Write each of the regular expressions below in FO or in MSO. Use succ, $\leq$, min, max when needed, since these are expressible using $<$.  

$$E_1 = (a|b)^*c^* \quad E_2 = (a.b)^* \quad E_3 = (a.a.a)^*$$

ii. Write a regular expression describing the following language:

$$\forall S (\exists x (S(x) \land P_a(x))) \rightarrow (\exists y (S(y) \land P_b(y)))$$