

How to Solve a Cubic Equation

Part 1 – The Shape of the Discriminant

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The problem

Last time, in the March/April 2006 issue, we finished up the discussion of solving quadratic equations. This time we will begin a discussion of solving cubic equations. The conventional version of this problem is to find the roots x of the equation

$$f(x) = Ax^3 + 3Bx^2 + 3Cx + D = 0$$

But what I am really interested in is the solution to homogeneous cubics:

$$f(x, w) = Ax^3 + 3Bx^2w + 3Cw^2x + Dw^3 = 0 \tag{0.1}$$

These roots will consist of $[x, w]$ pairs, where any scalar multiple represents the same root. Before we can solve this it's important to first understand what types of solutions can occur and how they relate to the values of the four coefficients $[A, B, C, D]$. That's primarily what we will do this time. The actual solution process will occur next time.

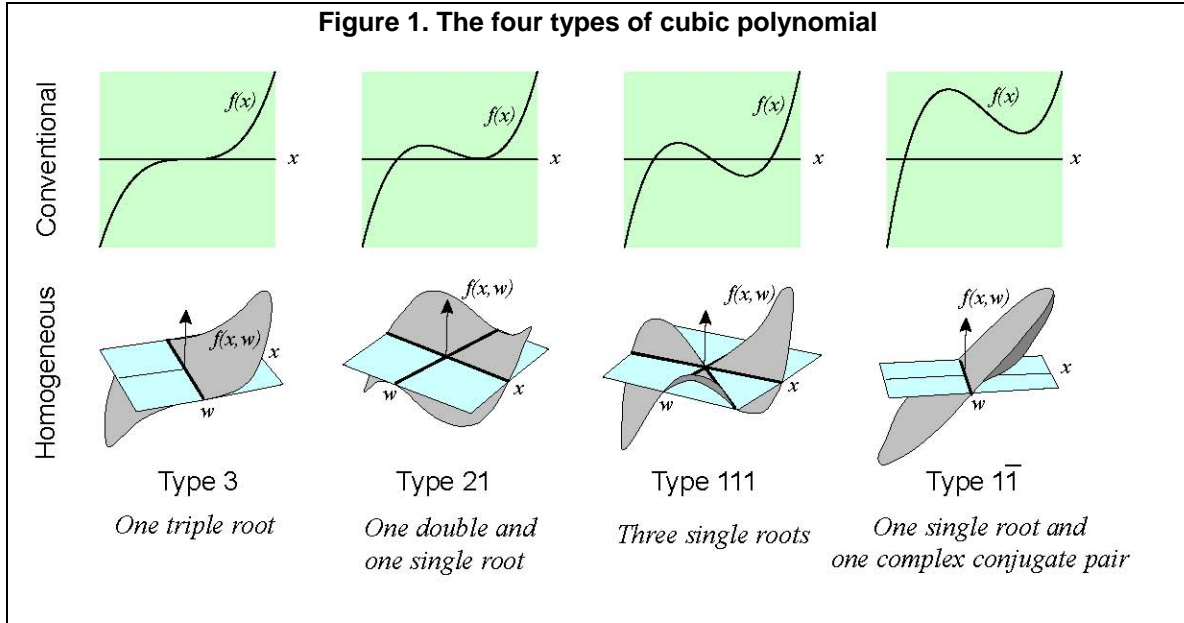
Types of solutions

There are four things that can happen with cubic polynomial roots. These consist of the various permutations of multiple roots and imaginary roots. There can be:

- a triple root
- a double root and a single root,
- three distinct real roots
- one real and one complex conjugate pair of roots.

Figure 1 gives an example of each of these situations both in the conventional and the homogeneous versions. The notation for the type code is a concatenation of the multiplicity counts for the various roots, with an overbar representing a complex conjugate pair. For example Type 3 has one triple root, Type 21 has one double and one single root, etc.

We want to see how the values of A , B , C and D correspond to these different situations. We can visualize this by thinking of cubic polynomials as points in a four dimensional space with coordinates $[A, B, C, D]$. At first it may seem like it would require four dimensional plots to understand this space. But this is a homogeneous four dimensional space; any scalar multiple of $[A, B, C, D]$ gives the same cubic polynomial. This means that we can use some of our computer graphics intuition about homogeneous coordinates to project the homogeneous four dimensional space into a three dimensional space that includes a plane at infinity.



Basic categorization

The basic relationship between $[A,B,C,D]$ and the four cubic function types was covered in an earlier article [1]. That article showed this relationship in terms of the four quantities

$$\begin{aligned}
 \delta_1 &= AC - B^2 \\
 \delta_2 &= AD - BC \\
 \delta_3 &= BD - C^2 \\
 \Delta &= 4\delta_1\delta_3 - \delta_2^2 \\
 &= -A^2D^2 + 6ABCD - 4AC^3 - 4B^3D + 3B^2C^2
 \end{aligned}
 \tag{0.2}$$

The relation between these values and the root structure appears in Table 1 (ignore column four for now). The value of Δ is the discriminant of the cubic and the equation $\Delta = 0$ defines a surface in $[A,B,C,D]$ space. My main goal in this article is to give an understanding of the shape of that surface. We will look at this surface from various viewpoints (literally).

Table 1. The four types of cubic polynomials			
Type	Algebraic Condition	Roots of Cubic	Roots of Hessian
3	$\delta_1 = \delta_2 = \delta_3 = 0$	One triple root	H=0
21	$\Delta = 0$	One Double and one Single root	Double root (that equals the double root of the cubic)
111	$\Delta > 0$	Three distinct real roots	Complex conjugate pair
1 $\bar{1}$	$\Delta < 0$	One real root and one complex conjugate pair	Two real roots

Transformations

When we fiddled with quadratics one of the interesting insight-building tools was the coordinate transformation. Suppose that we transform the polynomial parameters via

$$[x \quad w] = [\tilde{x} \quad \tilde{w}] \begin{bmatrix} t & u \\ s & v \end{bmatrix} \quad (0.3)$$

Looking at the shapes in the second row of Figure 1 we can see that, as long as the transformation is nonsingular, the transformation can rotate or shear the shape, but it cannot change the number or multiplicity of roots. In fact, any two cubics of a particular type can always be connected by the transformation in equation (0.3). We can see this by reviewing a property of 2x2 transformations described in [2]: given any three pairs of distinct input and output points it is always possible to construct a 2x2 transformation that maps the inputs to (a homogeneous factor times) the outputs. In other words, we can always find a matrix that satisfies

$$\begin{bmatrix} \tilde{x}_1 & \tilde{w}_1 \\ \tilde{x}_2 & \tilde{w}_2 \\ \tilde{x}_3 & \tilde{w}_3 \end{bmatrix} \begin{bmatrix} t & u \\ s & v \end{bmatrix} = \begin{bmatrix} \lambda_1 [x_1 & w_1] \\ \lambda_2 [x_2 & w_2] \\ \lambda_3 [x_3 & w_3] \end{bmatrix} \quad (0.4)$$

We do not get to pick the λ_i values, they are determined by the x and w values; but that's OK since homogeneous scale factors don't bother us. The meaning of this for our situation is that any cubic polynomial with, say three real roots, can be transformed to any other cubic with three real roots by some 2x2 matrix.

When we do the transform of Equation (0.3) we generate a different cubic polynomial. What are the coefficients for this new polynomial? We answered this question last time for quadratic polynomials and the process is the same here: plug the expressions for x and w of Equation (0.3) into Equation (0.1), expand out all terms, collect the terms that multiply the new \tilde{x}^3 and call this \tilde{A} , etc. Write this all out in matrix form and we get

$$\begin{bmatrix} \tilde{A} \\ \tilde{B} \\ \tilde{C} \\ \tilde{D} \end{bmatrix} = \begin{bmatrix} t^3 & 3t^2u & 3tu^2 & u^3 \\ t^2s & 2tus + t^2v & u^2s + 2tuv & u^2v \\ ts^2 & us^2 + 2tsv & 2usv + tv^2 & uv^2 \\ s^3 & 3s^2v & 3sv^2 & v^3 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} \quad (0.5)$$

We will get a lot of mileage out of this equation.

The Line of Triple Roots

Let's begin by looking at the locus of points in $[A,B,C,D]$ space that correspond to triple-root polynomials. Saying that a cubic has a triple root is the same as saying that it is the cube of a linear function

$$(ax + bw)^3 = a^3x^3 + 3a^2bx^2w + 3ab^2xw^2 + d^3w^3$$

The coefficients A,B,C,D must then follow the pattern $A = a^3, B = a^2b, C = ab^2, D = b^3$. In other words, they have a constant ratio to each other

$$\frac{A}{B} = \frac{B}{C} = \frac{C}{D}$$

This is actually three equations, and the condition $\delta_1 = \delta_2 = \delta_3 = 0$ is the same thing as clearing fractions from the three which gives

$$\begin{aligned} AC - B^2 &= 0 \\ AD - BC &= 0 \\ BD - C^2 &= 0 \end{aligned} \tag{0.6}$$

There is (almost) a redundancy in these three equations; any two imply the other one. But this works only if all of A, B, C, D are nonzero. We'll see this visually in a moment.

The Hessian

The values $\delta_1, \delta_2, \delta_3$ are the coefficients of what is called the Hessian of the cubic. The Hessian is officially defined as the determinant of the matrix formed from the various second derivatives of f with respect to x and w :

$$\text{Hessian}(f) = \det \begin{bmatrix} f_{xx} & f_{xw} \\ f_{xw} & f_{ww} \end{bmatrix} \tag{0.7}$$

If you work out the derivatives for the cubic you find:

$$\begin{aligned} \text{Hessian}(f) &= \det \begin{bmatrix} 6Ax + 6Bw & 6Bx + 6Cw \\ 6Bx + 6Cw & 6Cx + 6Dw \end{bmatrix} \\ &= 36 \begin{pmatrix} (AC - B^2)x^2 \\ + (AD - BC)xw \\ + (BD - C^2)w^2 \end{pmatrix} \\ &= 36(\delta_1 x^2 + \delta_2 xw + \delta_3 w^2) \end{aligned}$$

It is often useful to write this in matrix form

$$\begin{aligned} \text{Hessian}(f) &= \\ 18 \begin{bmatrix} x & w \end{bmatrix} & \begin{bmatrix} 2(AC - B^2) & AD - BC \\ AD - BC & 2(BD - C^2) \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} \end{aligned}$$

I'll pretty much ignore constant factor of 18 from now on, and I'll give a name to matrix

$$\mathbf{H} = \begin{bmatrix} 2\delta_1 & \delta_2 \\ \delta_2 & 2\delta_3 \end{bmatrix} = \begin{bmatrix} 2(AC - B^2) & AD - BC \\ AD - BC & 2(BD - C^2) \end{bmatrix} \tag{0.8}$$

I will sometimes talk interchangeably about the matrix \mathbf{H} and the quadratic equation that it represents.

So the condition that Equation (0.1) is the cube of a linear function is the same as that its Hessian is identically zero. In fact, this test works for any polynomial of degree n ; just calculate the Hessian as defined by equation (0.7) and if it's identically zero the polynomial is the n^{th} power of a linear function. (The Hessian of a quadratic is just the discriminant, a scalar. If this one value is zero there is a double root and the quadratic is the square of a linear function.)

So each cubic polynomial f has an associated quadratic polynomial $\text{Hessian}(f)$. This Hessian has an important property. Suppose you transform a cubic and then calculate its Hessian (giving $\tilde{\delta}_i = \tilde{A}\tilde{B} - \tilde{C}^2$ etc). Now compare this with the Hessian of the original cubic. In other words, mash together equations and (0.5) and (0.8). When the dust settles you find

$$\begin{bmatrix} 2\tilde{\delta}_1 \\ \tilde{\delta}_2 \\ 2\tilde{\delta}_3 \end{bmatrix} = (tv - su)^2 \begin{bmatrix} t^2 & 2tu & u^2 \\ ts & tv + us & uv \\ s^2 & 2sv & v^2 \end{bmatrix} \begin{bmatrix} 2\delta_1 \\ \delta_2 \\ 2\delta_3 \end{bmatrix} \tag{0.9}$$

Another way to write this is

$$\begin{bmatrix} 2\tilde{\delta}_1 & \tilde{\delta}_2 \\ \tilde{\delta}_2 & 2\tilde{\delta}_3 \end{bmatrix} = (tv - su)^2 \begin{bmatrix} t & u \\ s & v \end{bmatrix} \begin{bmatrix} 2\delta_1 & \delta_2 \\ \delta_2 & 2\delta_3 \end{bmatrix} \begin{bmatrix} t & s \\ u & v \end{bmatrix}$$

This is just the transformation equation for a quadratic polynomial. (There is a more elegant derivation of this in [3] as well as in Part 3 of this series.) In other words when you transform a cubic polynomial by equation (0.3) the Hessian transforms (as a quadratic) by the same matrix. Such a quantity is called a *covariant* of f . The discriminant Δ is a scalar and is called an *invariant* of f since it transforms according to

$$\tilde{\Delta} = (tv - su)^6 \Delta$$

Note that the determinant of H is the discriminant of the cubic but is minus the discriminant of the Hessian quadratic. What this means is that if the cubic has one real root then the Hessian has two real roots. If the cubic has three real roots the Hessian has no real roots. This is sort of a conservation-of-real-roots effect. It also gives you three handles on any Type 1 $\bar{1}$ cubic (one from the cubic and two from the Hessian) to map using equation (0.4). Finally, if the discriminant is zero, both the Hessian and the cubic have a double root, and as it happens, they are the same. Now you can go back and understand the final column of Table 1.

Algebraic form of the triple root line

Equations(0.6) define three surfaces in homogeneous $[A,B,C,D]$ space. The intersection of these three surfaces is the locus of coefficients that generate Type 3 polynomials. What does this look like? Two of the surfaces ($\delta_1 = 0$ and $\delta_3 = 0$) are cones and the other ($\delta_2 = 0$) is a hyperbolic paraboloid. This is made more obvious by writing them as

$$\begin{aligned} \delta_1 &= \left(\frac{A+C}{2}\right)^2 - \left(\frac{A-C}{2}\right)^2 - B^2 \\ \delta_2 &= \left(\frac{A+D}{2}\right)^2 - \left(\frac{A-D}{2}\right)^2 - \left(\frac{B+C}{2}\right)^2 + \left(\frac{B-C}{2}\right)^2 \\ \delta_3 &= \left(\frac{B+D}{2}\right)^2 - \left(\frac{B-D}{2}\right)^2 - C^2 \end{aligned}$$

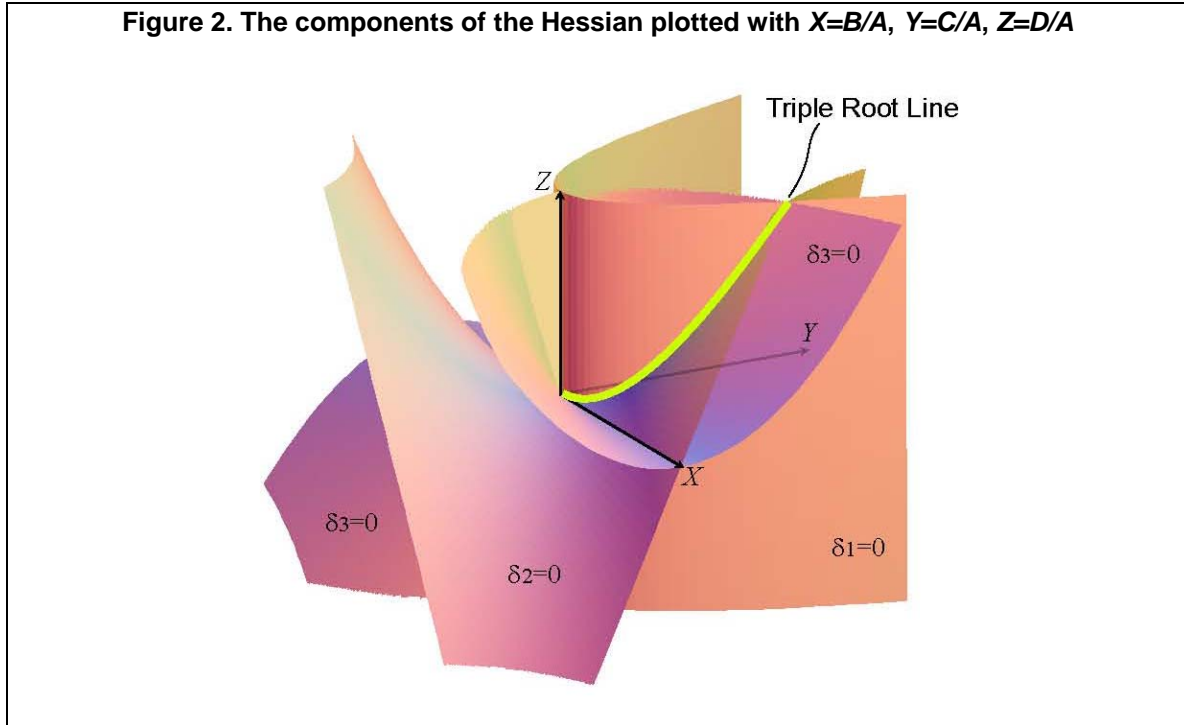
We can plot these surfaces by considering $[A,B,C,D]$ as homogeneous coordinates and picking A as the coordinate to divide out to project into three dimensions. Thus we have

$$X = \frac{B}{A}, Y = \frac{C}{A}, Z = \frac{D}{A}$$

And the three surfaces are

$$\begin{aligned} Y - X^2 &= 0 \\ Z - XY &= 0 \\ XZ - Y^2 &= 0 \end{aligned}$$

These appear in Figure 2. The surface $\delta_3 = 0$ is a cone with apex at the origin, the surface $\delta_2 = 0$ is a saddle point, and the surface $\delta_1 = 0$ is a cone with apex at infinity so that it looks like a parabolic cylinder. You can see that all three surfaces intersect in a curve. This is our triple-root locus. You can also see that each pair of surfaces intersects along another straight line that does not represent a triple-root cubic. This happens on the X -axis ($\delta_2 = \delta_3 = 0, \delta_1 \neq 0$), on the Z -axis ($\delta_1 = \delta_3 = 0, \delta_2 \neq 0$), and on the line-at-infinity in the YZ plane ($\delta_1 = \delta_2 = 0, \delta_3 \neq 0$). This is why we need all three quantities to be zero to determine a Type 3 cubic.



Parametric form of the triple root line

There is another way to describe the triple-root curve. We can use the fact that all triple-root cubics are transforms of each other. We pick one simple example and then plug it into equation (0.5) and see what we get for all possible values of (t,s,u,v) . For example, starting with the cubic $f(x,w) = x^3$ where $[A,B,C,D]=[1,0,0,0]$ we get

$$\begin{bmatrix} \tilde{A} \\ \tilde{B} \\ \tilde{C} \\ \tilde{D} \end{bmatrix}_{Type3} = \begin{bmatrix} t^3 \\ t^2s \\ ts^2 \\ s^3 \end{bmatrix} \quad (0.10)$$

When we do the homogeneous division for plotting purposes we get the parametric equation for the triple-root curve in terms of a single parameter, $\frac{s}{t}$.

$$\begin{aligned} X &= \frac{B}{A} = \left(\frac{s}{t}\right) \\ Y &= \frac{C}{A} = \left(\frac{s}{t}\right)^2 \\ Z &= \frac{D}{A} = \left(\frac{s}{t}\right)^3 \end{aligned}$$

Equation (0.10) is homogeneous in the variables (t,s) ; any nonzero scalar of both t and s give the same $[A,B,C,D]$ vector (times the scalar cubed). So the vector (t,s) is a homogeneous parameter pair with basically just one degree of freedom. A good way to generate all useful values for this pair is the trigonometric substitution.

$$t = \cos \theta, \quad s = \sin \theta$$

so

$$\begin{bmatrix} \tilde{A} \\ \tilde{B} \\ \tilde{C} \\ \tilde{D} \end{bmatrix}_{Type3} = \begin{bmatrix} \cos^3 \theta \\ \cos^2 \theta \sin \theta \\ \cos \theta \sin^2 \theta \\ \sin^3 \theta \end{bmatrix}$$

As θ goes from 0 to 90 degrees the curve winds its way through space with positive XYZ . At 90 degrees we pass through $[A,B,C,D]=[0,0,0,1]$, the point at infinity on the Z axis. (The fact that this point is on the curve is not immediately apparent from figure 2. Also non-obvious is the fact that the curve is tangent to the plane-at-infinity at that point. The tangency is third order so it looks like the conventional curve in Figure 1 type 3.) Past 90 degrees the point wraps around infinity to come back in the negative XZ –positive Y octant. At 180 degrees we are back where we started, but with a global sign flip on $[A,B,C,D]$. But global sign flips don't bother us here as they still represent the same cubic polynomial.

The Surface of Double Roots

Now let's look at the locus of points that represent cubics of type 21. These will be all points $[A,B,C,D]$ that satisfy the equation $\Delta=0$ and will be a surface in our projected 3-space. We expect that the triple-root line will lie in this surface since a triple root is a special case of a double root. As before there are two ways to describe this surface.

Algebraic form of discriminant surface

The algebraic form of the surface is, from Equation (0.2),

$$-A^2D^2 + 6ABCD - 4AC^3 - 4B^3D + 3B^2C^2 = 0$$

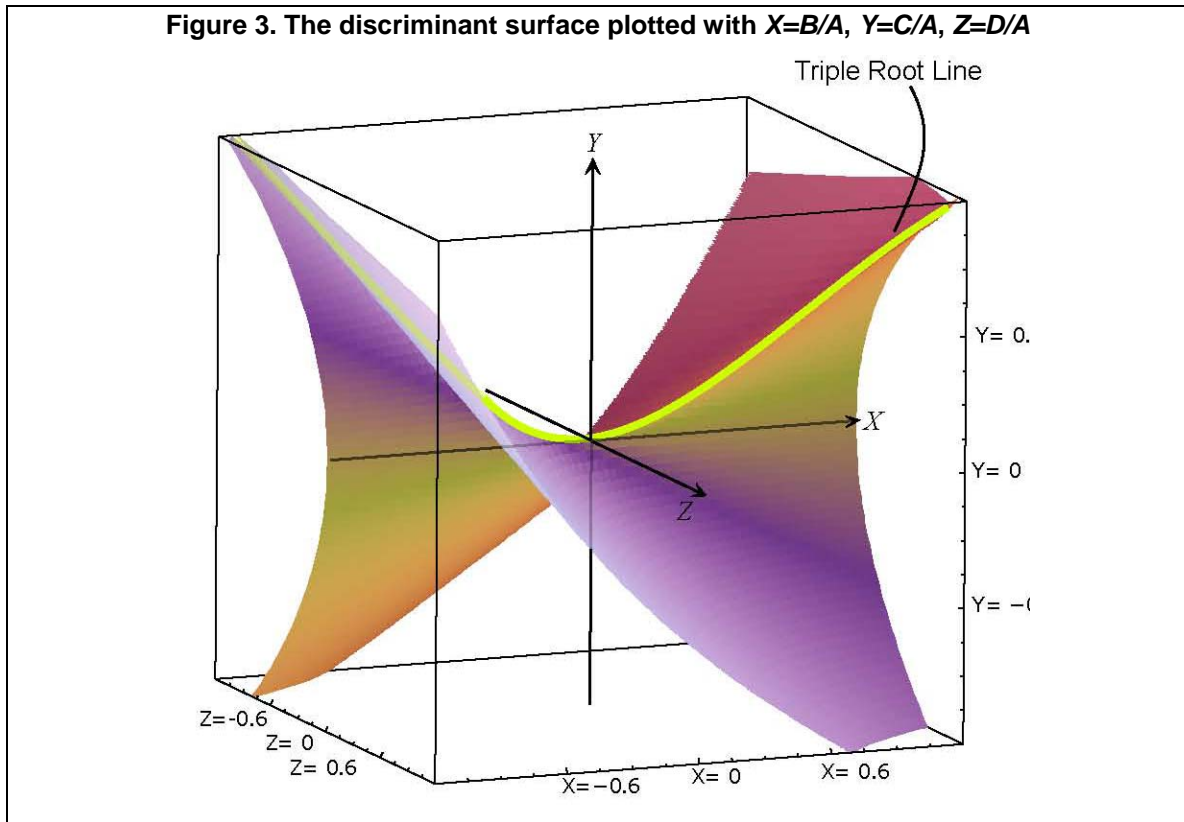
For plotting purposes we divide by A^4 and use our mapping $X = \frac{B}{A}, Y = \frac{C}{A}, Z = \frac{D}{A}$ to get

$$-Z^2 + 6XYZ - 4Y^3 - 4X^3Z + 3X^2Y^2 = 0$$

This is quadratic in Z so we can solve for it

$$Z = -2X^3 + 3XY \mp 2\sqrt{(X^2 - Y)^3}$$

This has two branches for the two signs of the square root. They join together along the curve where the radical is zero, where $X^2 - Y = 0$. This seam is just the triple root line. A plot appears in figure 3.



Parametric form of discriminant

As with the triple-root curve we can generate a parametric form for the double-root surface by starting with a canonical double-root cubic like $f(x, w) = x^2w$, which has the coefficient vector $[A, B, C, D] = [0, 1, 0, 0]$, and applying all possible transformations to it via equation (0.5). This gives us:

$$\begin{bmatrix} \tilde{A} \\ \tilde{B} \\ \tilde{C} \\ \tilde{D} \end{bmatrix}_{Type21} = \begin{bmatrix} 3t^2u \\ 2tsu + t^2v \\ s^2u + 2tsv \\ 3s^2v \end{bmatrix} \quad (0.11)$$

This appears scary at first as it looks like the points in $[A, B, C, D]$ space are generated by four parameters t, s, u and v . But a closer look reveals a simplification; the expressions in equation (0.11) are homogeneous in the parameter pair (t, s) and are separately homogeneous in the parameter pair (u, v) . In other words the values of $[\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}]$ are functions of just two homogeneous parameter pairs. This two-parameter dependence generates a surface in $[A, B, C, D]$ space as expected. We can generate the two parameter pairs, again, by trigonometry as $(t, s) = (\cos \theta, \sin \theta)$ and $(u, v) = (\cos \phi, \sin \phi)$. We now have the two parameter surface in terms of two angles which, given how they wrap around, means that the surface is topologically a torus.

Another look at equation (0.11) reveals another interesting property of the discriminant surface. The function is linear in the parameter pair (u, v) . So we can write

$$\begin{bmatrix} \tilde{A} \\ \tilde{B} \\ \tilde{C} \\ \tilde{D} \end{bmatrix}_{Type21} = u \begin{bmatrix} 3t^2 \\ 2ts \\ s^2 \\ 0 \end{bmatrix} + v \begin{bmatrix} 0 \\ t^2 \\ 2ts \\ 3s^2 \end{bmatrix} \quad (0.12)$$

If we fix the pair (t,s) and allow (u,v) to vary, the result will trace out a straight line in $[A,B,C,D]$ space. In other words the discriminant surface is a ruled surface. It's swept out by a line whose endpoints follow the paths described by the two column vectors on the right of equation (0.12) as (s,t) vary. For example in figure 3 the X axis is such a ruling line; it is completely embedded in the surface. For $X>0$ it is in the front branch, and for $X<0$ it's in the back branch on the left. It's formed by fixing $(t,s)=(1,0)$ and varying (u,v) .

Each line that rules the surface crosses the triple-root crease exactly once, when the values of t,s,u,v satisfy $tv - su = 0$. This is the condition that the transformation that we are using to sweep out the surface is singular. This makes sense since applying a singular transformation in $[x,w]$ space squashes all the roots onto one line giving a triple-root cubic.

What else can we say about these ruling lines? Consider the following. The mapping from cubic curves to their Hessian is a mapping from a 4 dimensional space $[A,B,C,D]$ to a 3 dimensional space $[\delta_1, \delta_2, \delta_3]$. That means that more than one cubic must map to the same Hessian. So for a particular Hessian what is the locus of all cubics that map to it? For cubics on the $\Delta = 0$ surface (cubics with double roots) that locus is just one of these ruling lines. We can see this by plugging the expressions for $[A,B,C,D]$ from equation (0.11) into equation(0.2). After some cancelling and factoring you will get

$$\mathbf{H} = -2(tv - su)^2 \begin{bmatrix} t^2 & ts \\ ts & s^2 \end{bmatrix}$$

In other words, any dependence on (u,v) has factored out into the scalar factor $(tv - us)^2$. For any value (t,s) the cubic in equation (0.11) has the same Hessian independent of (u,v) . The shared double root between the cubic and its Hessian is $[x \ w]_{double} = [-s \ t]$ and the single root of the cubic is $[x \ w]_{single} = [-v \ u]$

The Volumes of Distinct Roots

The surface $\Delta = 0$ divides $[A,B,C,D]$ space into two disjoint volumes. One contains the Type 111 polynomials (in figure 3 this is the region below the surface) and the other contains the Type 1 $\bar{1}$ polynomials (this is the region above the surface).

We can get a parametric formula for type 111 cubics by starting with a simple three-root cubic $f(x,w) = xw(x+w)$ whose coefficient vector is $[A,B,C,D]=[0,1,1,0]$ and apply equation (0.5). We get

$$\begin{bmatrix} \tilde{A} \\ \tilde{B} \\ \tilde{C} \\ \tilde{D} \end{bmatrix}_{Type111} = \begin{bmatrix} 3t^2u + 3tu^2 \\ 2tus + t^2v + u^2s + 2tuv \\ us^2 + 2tsv + 2usv + tv^2 \\ 3s^2v + 3sv^2 \end{bmatrix}$$

This is no longer separately homogeneous in the two parameter pairs (t,s) and (u,v) . But it is homogeneous in t,s,u,v together. We could, for example, divide all the expressions by t^3 and get formulas in terms of the three parameters $(s/t, u/t, v/t)$. Three parameters means that this sweeps out a volume.

We can similarly get a parametric formula for type 1 \bar{I} cubics by starting with the simple one-root cubic $f(x, w) = x^3 + w^3$ whose coefficient vector is $[A, B, C, D] = [1, 0, 0, 1]$ and applying equation (0.5). We get

$$\begin{bmatrix} \tilde{A} \\ \tilde{B} \\ \tilde{C} \\ \tilde{D} \end{bmatrix}_{\text{type1 } \bar{I}} = \begin{bmatrix} t^3 + u^3 \\ t^2s + u^2v \\ ts^2 + uv^2 \\ s^3 + v^3 \end{bmatrix}$$

Constant Hessian loci

What can we say about the loci of constant Hessian in either of these two regions? To generate such a locus we need to pick a cubic and then apply a transformation that leaves the Hessian unchanged, while sliding the $[A, B, C, D]$ of the cubic along the locus.

Staying in one place

How can this be? A transformation that changes the cubic but leaves the quadratic alone? Well, there are lots of them and we can construct them in one of two ways. First, we can construct a transformation that has eigenvectors along the roots of \mathbf{H} . We found last time that one such transformation is (in our current notation):

$$\begin{bmatrix} t & u \\ s & v \end{bmatrix} = \begin{bmatrix} -\delta_2 & 2\delta_1 \\ -2\delta_3 & \delta_2 \end{bmatrix}$$

More generally, if we add any scalar amount of the identity matrix to this it will not change the eigenvectors but only the eigenvalues. So a more general \mathbf{H} -maintaining transform is the homogeneous blend

$$\begin{aligned} \begin{bmatrix} t & u \\ s & v \end{bmatrix} &= \alpha \begin{bmatrix} -\delta_2 & 2\delta_1 \\ -2\delta_3 & \delta_2 \end{bmatrix} + \beta \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} -\alpha\delta_2 + \beta & 2\alpha\delta_1 \\ -2\alpha\delta_3 & \alpha\delta_2 + \beta \end{bmatrix} \end{aligned} \tag{0.13}$$

If you plug these values for t, s, u, v into equation (0.9) you will find that the transformed matrix is indeed just the original matrix times a scalar factor equal to the determinant of the transformation. This means that if you plot the coefficients δ_i as a point in 3D space (like we did last time) and see what happens to it under various choices of (α, β) you will see that it just scales in and out from the origin. And equation (0.13) still works even if \mathbf{H} has no real roots, in other words when the transformation only has complex eigenvalues.

Another transformation that keeps \mathbf{H} intact is one that exchanges the roots. Two examples of such a matrix are

$$\begin{bmatrix} t & u \\ s & v \end{bmatrix} = \begin{bmatrix} -\delta_3 & \delta_2 \\ 0 & \delta_3 \end{bmatrix} \text{ and } \begin{bmatrix} \delta_1 & 0 \\ \delta_2 & -\delta_1 \end{bmatrix}$$

To see that these have the desired property, first note that the square of each matrix is δ_3^2 or δ_1^2 times the identity. This makes sense; exchanging twice gets you back to the original. Now let's see what happens to the roots of \mathbf{H} , which are $[-\delta_2 \pm \sqrt{-\Delta} \quad 2\delta_1]$. Plug in one root to one of the matrices and, sure enough, we get a scalar times the other root.

$$\begin{bmatrix} -\delta_2 + \sqrt{-\Delta} & 2\delta_1 \end{bmatrix} \begin{bmatrix} \delta_1 & 0 \\ \delta_2 & -\delta_1 \end{bmatrix} = -\delta_1 \begin{bmatrix} -\delta_2 - \sqrt{-\Delta} & 2\delta_1 \end{bmatrix}$$

You can test for yourself that the other matrix works too. Any linear combination of these two matrices is also a root-swapping transformation. This gives us the general continuum of transformations:

$$\begin{aligned} \begin{bmatrix} t & u \\ s & v \end{bmatrix} &= \gamma \begin{bmatrix} -\delta_3 & \delta_2 \\ 0 & \delta_3 \end{bmatrix} + \varepsilon \begin{bmatrix} \delta_1 & 0 \\ \delta_2 & -\delta_1 \end{bmatrix} \\ &= \begin{bmatrix} -\gamma\delta_3 + \varepsilon\delta_1 & \gamma\delta_2 \\ \varepsilon\delta_2 & \gamma\delta_3 - \varepsilon\delta_1 \end{bmatrix} \end{aligned} \quad (0.14)$$

Constant H locus for Type 111 cubics

Let's start with the a cubic that has three real roots: $f(x) = xw(x+w)$. The coefficient matrix for this is

$$\begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \quad (0.15)$$

The coefficients of the Hessian are

$$\begin{aligned} \delta_1 &= AC - BB = -1 \\ \delta_2 &= AD - BC = -1 \\ \delta_3 &= BD - CC = -1 \end{aligned} \quad (0.16)$$

so the Hessian is $-2x^2 - 2xw - 2w^2$, which has no real roots. We combine equations (0.13) (0.14) and (0.16) to get the set of transformations that leave this Hessian intact:

$$\begin{bmatrix} t & u \\ s & v \end{bmatrix} = \begin{bmatrix} \alpha + \beta & -2\alpha \\ 2\alpha & -\alpha + \beta \end{bmatrix} \quad (0.17)$$

or

$$\begin{bmatrix} t & u \\ s & v \end{bmatrix} = \begin{bmatrix} \gamma - \varepsilon & -\gamma \\ -\varepsilon & -\gamma + \varepsilon \end{bmatrix}$$

The determinants of these transformation are $3\alpha^2 + \beta^2$ and $-(\varepsilon^2 - \varepsilon\gamma + \gamma^2)$ respectively. They are always positive so the transformations can never be singular. Now let's see what happens to the cubic under these transformations. I will do this explicitly for just the (α, β) transformation; the (γ, ε) transformation gives the same net result. Combining the general transformation equation (0.5) with equations (0.15) and (0.17) gives.

$$\begin{bmatrix} \tilde{A} \\ \tilde{B} \\ \tilde{C} \\ \tilde{D} \end{bmatrix} = \begin{bmatrix} 6\alpha^3 - 6\alpha\beta^2 \\ +3\alpha^3 - 9\alpha^2\beta - 3\alpha\beta^2 + \beta^3 \\ -3\alpha^3 - 9\alpha^2\beta + 3\alpha\beta^2 + \beta^3 \\ 6\alpha\beta^2 - 6\alpha^3 \end{bmatrix} \quad (0.18)$$

So what kind of path in $[ABCD]$ space does this trace out as (α, β) vary? Remarkably, it's a straight line.

We can see this by rewriting it as

$$\begin{bmatrix} \tilde{A} \\ \tilde{B} \\ \tilde{C} \\ \tilde{D} \end{bmatrix} = (-9\alpha^2\beta + \beta^3) \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} + (3\alpha^3 - 3\alpha\beta^2) \begin{bmatrix} 2 \\ 1 \\ -1 \\ -2 \end{bmatrix}$$

Even though the two parenthesized scalars on the right hand side do not vary linearly with (α, β) , the result must always lie on the straight line through the points represented by the two column vectors. In fact, if you note that the scalar factors can themselves be factored into

$$\begin{aligned} -9\alpha^2\beta + \beta^3 &= \beta(-3\alpha + \beta)(3\alpha + \beta) \\ 3\alpha^3 - 3\alpha\beta^2 &= \alpha(\alpha + \beta)(\alpha - \beta) \end{aligned}$$

you can see that equation (0.18) traces out this line three times as (α, β) go through all their values

Constant H locus for Type $1\bar{1}$ cubics

Now let's try this with a simple cubic that has only one real root: $f(x) = x^3 + w^3$. The coefficient vector is

$$\begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

The Hessian coefficients are

$$\begin{aligned} \delta_1 &= AC - BB = 0 \\ \delta_2 &= AD - BC = 1 \\ \delta_3 &= BD - CC = 0 \end{aligned}$$

The class of transformation matrices that leave the Hessian intact are

$$\begin{bmatrix} t & u \\ s & v \end{bmatrix} = \begin{bmatrix} -\alpha + \beta & 0 \\ 0 & \alpha + \beta \end{bmatrix}$$

or

$$\begin{bmatrix} t & u \\ s & v \end{bmatrix} = \begin{bmatrix} 0 & \gamma \\ \varepsilon & 0 \end{bmatrix}$$

The determinants of these matrices are $\beta^2 - \alpha^2$ and $\varepsilon\gamma$ respectively so the matrix will be singular at two points on the (α, β) or (γ, ε) excursion. We will expect something odd to happen there.

Putting this all together as in the previous section gives the set of all cubics that share this Hessian:

$$\begin{bmatrix} \tilde{A} \\ \tilde{B} \\ \tilde{C} \\ \tilde{D} \end{bmatrix} = \begin{bmatrix} (\beta - \alpha)^3 \\ 0 \\ 0 \\ (\alpha + \beta)^3 \end{bmatrix} \text{ or } \begin{bmatrix} \gamma^3 \\ 0 \\ 0 \\ \varepsilon^3 \end{bmatrix}$$

This again is a straight line. It lies completely in the $\Delta < 0$ region, except for the two cubics with coefficients $[1,0,0,0]^T$ and $[0,0,0,1]^T$ where it hits the triple-root curve. In other words, the loci of constant \mathbf{H} cubics are straight lines that connect pairs of points on the triple root line.

We can see this more directly by constructing a formula for the homogeneous blend of two arbitrary points on the triple root line that have parameter values (t_1, s_1) and (t_2, s_2)

$$\begin{bmatrix} \tilde{A} \\ \tilde{B} \\ \tilde{C} \\ \tilde{D} \end{bmatrix}_{Type1\bar{1}} = u \begin{bmatrix} t_1^3 \\ t_1^2 s_1 \\ t_1 s_1^2 \\ s_1^3 \end{bmatrix} + v \begin{bmatrix} t_2^3 \\ t_2^2 s_2 \\ t_2 s_2^2 \\ s_2^3 \end{bmatrix}$$

Now plug this formula into the definition of the \mathbf{H} matrix from equation(0.8). After the dust settles you end up with.

$$\mathbf{H} = uv(t_1 s_2 - s_1 t_2)^2 \begin{bmatrix} 2t_1 t_2 & t_1 s_2 + s_1 t_2 \\ t_1 s_2 + s_1 t_2 & 2s_1 s_2 \end{bmatrix} \quad (0.19)$$

In other words the Hessian along the line (swept out by varying u and v) is unchanged except for the scalar constant factors. To check on the determinant of \mathbf{H} we first calculate

$$\det \begin{bmatrix} 2t_1 t_2 & t_1 s_2 + s_1 t_2 \\ t_1 s_2 + s_1 t_2 & 2s_1 s_2 \end{bmatrix} = -(t_1 s_2 - s_1 t_2)^2$$

And note that for all 2x2 matrices we have $\det(\alpha \mathbf{M}) = \alpha^2 \det \mathbf{M}$. So we get

$$\det \mathbf{H} = -u^2 v^2 (t_1 s_2 - s_1 t_2)^6$$

This is always negative (except where u or v are zero.) indicating two real roots for the quadratic \mathbf{H} .

One last tidbit: if we solve for the roots of \mathbf{H} in equation (0.19) we get

$$[x \ w] = [-s_1 \ t_1], [-s_2 \ t_2]$$

This gives a nice geometric interpretation of the real roots of the Hessian of a single-root cubic: they are simply the cube roots of the two intersections of the constant-Hessian line with the triple-root curve.

The net picture

We now have a picture of $[A,B,C,D]$ space as being filled with lines of constant Hessian. Such a “tiling” of this space is called a *foliation*. All points $[A,B,C,D]$ lie on exactly one such line, except for the points on the triple-root curve. In the region for type 111 and type 21 these lines do not intersect and they fill space. In the region for type $1\bar{1}$ cubics they always intersect the triple root curve twice.

A New View

Figure 3 does indeed solve our problem by showing the surface $\Delta = 0$, but I find it a bit unsatisfying. It’s not at all clear what happens as the surface approaches infinity. We found algebraically that interesting things happen there (the triple-root line is tangent there) but we can’t see it visually. Let’s see if we can find a different view that is more edifying. This problem was (probably) first addressed by Zeeman [4] and recounted in Ramshaw [5]. The development here is slightly different and requires rather less mathematical sophistication.

We had good luck in examining the space of possible quadratics by writing the homogeneous parameter pair in polar coordinates.

How to Solve a Cubic Equation – Part 1

$$[x \ w] = [r \cos \xi \ r \sin \xi]$$

The same luck will hold when we do this for cubics. Plugging the polar coordinates into the cubic equation (0.1) gives.

$$\begin{aligned} Ax^3 + 3Bx^2w + 3Cwx^2 + Dw^3 = \\ r^3 (A \cos^3 \xi + 3B \cos^2 \xi \sin \xi + 3C \cos \xi \sin^2 \xi + D \sin^3 \xi) \end{aligned}$$

Then applying the triple angle formulas

$$\begin{aligned} \cos^3 \xi &= \frac{3}{4} \cos \xi + \frac{1}{4} \cos 3\xi \\ \cos^2 \xi \sin \xi &= \frac{1}{4} \sin \xi + \frac{1}{4} \sin 3\xi \\ \cos \xi \sin^2 \xi &= \frac{1}{4} \cos \xi - \frac{1}{4} \cos 3\xi \\ \sin^3 \xi &= \frac{3}{4} \sin \xi - \frac{1}{4} \sin 3\xi \end{aligned}$$

we get

$$\begin{aligned} Ax^3 + 3Bx^2w + 3Cwx^2 + Dw^3 = \\ r^3 \left(\begin{aligned} &+ \frac{3}{4}(A+C) \cos \xi + \frac{3}{4}(B+D) \sin \xi \\ &+ \frac{1}{4}(A-3C) \cos 3\xi + \frac{1}{4}(3B-D) \sin 3\xi \end{aligned} \right) \end{aligned}$$

So we define four new coordinates

$$\begin{aligned} E &= (A+C)/4 \\ F &= (B+D)/4 \\ G &= (A-3C)/4 \\ H &= (3B-D)/4 \end{aligned}$$

or in matrix form

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -3 & 0 \\ 0 & 3 & 0 & -1 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = 4 \begin{bmatrix} E \\ F \\ G \\ H \end{bmatrix} \quad (0.20)$$

(This coordinate system has some sign flips and constant factors different from Zeeman's system, but I'll stick to this one since it makes some other calculations neater.) The polynomial now looks like

$$\begin{aligned} Ax^3 + 3Bx^2w + 3Cwx^2 + Dw^3 \\ = r^3 (+3(E \cos \xi + F \sin \xi) + G \cos 3\xi + H \sin 3\xi) \end{aligned}$$

We now have the cubic polynomial represented as the sum of two sine curves of frequency ξ and 3ξ with phases and amplitudes a simple function of the original coefficients.

Writing the discriminant and Hessian in terms of the new coefficients $[E, F, G, H]$ does not, at first, lead to something simpler. But the new system does have the following interesting property. Let's look at transformations that are simple rotations in parameter space

$$\begin{bmatrix} t & u \\ s & v \end{bmatrix} = \begin{bmatrix} \cos \rho & -\sin \rho \\ \sin \rho & \cos \rho \end{bmatrix} \quad (0.21)$$

Combining equations (0.5), (0.20) and (0.21) will give the effect of a parameter rotation on the coefficients in $[E, F, G, H]$ space. After some more trig identities you get

$$\begin{bmatrix} \tilde{E} \\ \tilde{F} \\ \tilde{G} \\ \tilde{H} \end{bmatrix} = \begin{bmatrix} \cos \rho & \sin \rho & 0 & 0 \\ -\sin \rho & \cos \rho & 0 & 0 \\ 0 & 0 & \cos 3\rho & \sin 3\rho \\ 0 & 0 & -\sin 3\rho & \cos 3\rho \end{bmatrix} \begin{bmatrix} E \\ F \\ G \\ H \end{bmatrix}$$

In other words, as the parameter space rotates by the angle ρ the vector $[E, F]$ also rotates by ρ and the vector $[G, H]$ rotates by the angle 3ρ . Nifty.

Now let's see what happens when we draw the triple root lines and the double root surface in this coordinate system.

The Triple Root Line

Combine equation (0.10) and equation (0.20) to get the parametric form of the triple-root line in $[E, F, G, H]$ space. Then turn the homogeneous parameter (t, s) to an angular parameter by $(t, s) = (\cos \theta, \sin \theta)$. Finally apply the triple angle identities. The final result is

$$4 \begin{bmatrix} \tilde{E} \\ \tilde{F} \\ \tilde{G} \\ \tilde{H} \end{bmatrix}_{Type3} = \begin{bmatrix} t^3 + ts^2 \\ t^2s + s^3 \\ t^3 - 3ts^2 \\ 3t^2s - s^3 \end{bmatrix} = \begin{bmatrix} \cos \theta \\ \sin \theta \\ \cos 3\theta \\ \sin 3\theta \end{bmatrix}$$

We plot this line by picking one of E, F, G or H as the homogeneous coordinate to divide by to project into 3D. Using E will generate pretty much the same picture as using F , and using G will generate pretty much the same picture as using H . We will therefore show just two plots, one with the projection

$X = \frac{E}{G}, Y = \frac{F}{G}, Z = \frac{H}{G}$ (figure 4) and one with the projection $X = \frac{E}{E}, Y = \frac{F}{E}, Z = \frac{H}{E}$ (Figure 5). In figure 4 since we are dividing by $G = \cos 3\theta$ the curve will pass through infinity three times as theta progresses from 0 to 180. But each such transit is not tangent to the plane at infinity, so it's a bit easier to visualize.

The Double Root Surface

We generate a parametric equation for the $\Delta = 0$ surface by transforming equation (0.11) via equation (0.20) to get

$$4 \begin{bmatrix} \tilde{E} \\ \tilde{F} \\ \tilde{G} \\ \tilde{H} \end{bmatrix}_{Type21} = \begin{bmatrix} (3t^2 + s^2)u + (2ts)v \\ (2ts)u + (t^2 + 3s^2)v \\ (3t^2 - 3s^2)u - (6ts)v \\ (6ts)u + (3t^2 - 3s^2)v \end{bmatrix} \quad (0.22)$$

As before this is parameterized by the two homogeneous pairs (t, s) and (u, v) . I will now do a bit of a trick for plotting convenience. As before I'll do the angular substitution $(t, s) = (\cos \theta, \sin \theta)$ and application of (this time) some double angle identities, but I'll leave (u, v) alone. And I'll break equation (0.22) up into two smaller matrix equations. The result is

$$\begin{aligned} \begin{bmatrix} \tilde{E} \\ \tilde{F} \end{bmatrix} &= \begin{bmatrix} 2 + \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & 2 - \cos(2\theta) \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \\ \begin{bmatrix} \tilde{G} \\ \tilde{H} \end{bmatrix} &= 3 \begin{bmatrix} \cos(2\theta) & -\sin(2\theta) \\ \sin(2\theta) & \cos(2\theta) \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \end{aligned} \quad (0.23)$$

Now for the plotting convenience trick. Instead of using (u, v) as our second parameter we can use the linearly related (\tilde{G}, \tilde{H}) pair as the second parameter for figure 4, and the linearly related (\tilde{E}, \tilde{F}) pair as the second parameter for figure 5. In each case this has the nice effect of making slices of constant Z come from freezing the $Z = \tilde{H}/\tilde{G}$ parameter (figure 4) or freezing the $Z = \tilde{F}/\tilde{E}$ parameter (figure 5). We do this by first eliminating (u, v) from equation (0.23) by the appropriate matrix inversions and multiplications giving.

$$\begin{bmatrix} \tilde{E} \\ \tilde{F} \end{bmatrix} = \frac{1}{3} \begin{bmatrix} \cos(4\theta) + 2\cos(2\theta) & \sin(4\theta) + 2\sin(2\theta) \\ \sin(4\theta) - 2\sin(2\theta) & -\cos(4\theta) + 2\cos(2\theta) \end{bmatrix} \begin{bmatrix} \tilde{G} \\ \tilde{H} \end{bmatrix} \quad (0.24)$$

Then for figure 4 we have

$$\begin{aligned} X &= \frac{\tilde{E}}{\tilde{G}} = \frac{(\cos(4\theta) + 2\cos(2\theta))\tilde{G} + (\sin(4\theta) + 2\sin(2\theta))\tilde{H}}{3\tilde{G}} \\ Y &= \frac{\tilde{F}}{\tilde{G}} = \frac{(\sin(4\theta) - 2\sin(2\theta))\tilde{G} + (-\cos(4\theta) + 2\cos(2\theta))\tilde{H}}{3\tilde{G}} \\ Z &= \frac{\tilde{H}}{\tilde{G}} \end{aligned}$$

which gives X and Y as functions of (θ, Z)

$$\begin{aligned} X &= \frac{1}{3}(\cos(4\theta) + 2\cos(2\theta)) + \frac{1}{3}(\sin(4\theta) + 2\sin(2\theta))Z \\ Y &= \frac{1}{3}(\sin(4\theta) - 2\sin(2\theta)) + \frac{1}{3}(-\cos(4\theta) + 2\cos(2\theta))Z \end{aligned}$$

In particular the surface intersects the plane $Z=0$ at the curve

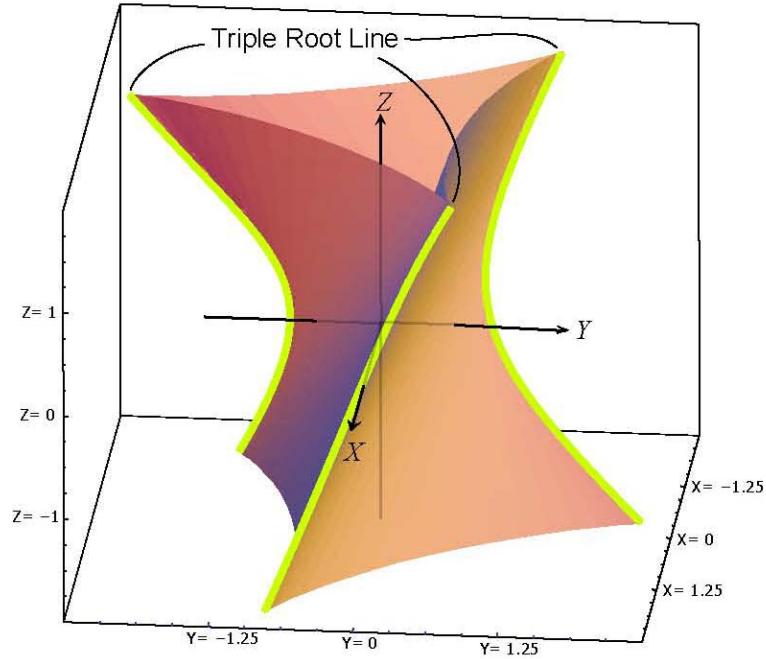
$$\begin{aligned} X &= \frac{1}{3}\cos(4\theta) + \frac{2}{3}\cos(-2\theta) \\ Y &= \frac{1}{3}\sin(4\theta) + \frac{2}{3}\sin(-2\theta) \end{aligned}$$

Note that I have moved the minus sign on the second terms inside the angle. In this form it shows an epicyclic generation of the dented triangle shaped curve (called an astroid) which you can see on the top edge of figure 4. Imagine a vector of length $1/3$ rotating by angle 4θ added to a vector of length $2/3$ rotating by angle -2θ . If we now do the trigonometric substitution $(\tilde{G}, \tilde{H}) = (\cos\phi, \sin\phi)$ and use some angle summation formulas we get

$$\begin{aligned} X &= \frac{\cos(4\theta - \phi) + 2\cos(2\theta - \phi)}{3\cos\phi} \\ Y &= \frac{\sin(4\theta - \phi) - 2\sin(2\theta - \phi)}{3\cos\phi} \\ Z &= \tan\phi \end{aligned}$$

In other words as ϕ varies away from zero the astroid curve simply rotates by minus that angle and magnifies by $1/\cos\phi$. (That's why the top edge of figure 4 has the same shape as the $Z=0$ section, just at a different angle.)

Figure 4. The discriminant surface plotted with $X=E/G$, $Y=F/G$, $Z=H/G$



For figure 5 we invert equation (0.24) and use

$$\begin{bmatrix} \tilde{G} \\ \tilde{H} \end{bmatrix} = \begin{bmatrix} -\cos(4\theta) + 2\cos(2\theta) & -\sin(4\theta) - 2\sin(2\theta) \\ -\sin(4\theta) + 2\sin(2\theta) & \cos(4\theta) + 2\cos(2\theta) \end{bmatrix} \begin{bmatrix} \tilde{E} \\ \tilde{F} \end{bmatrix}$$

so

$$X = \frac{\tilde{G}}{\tilde{E}} = \frac{(-\cos(4\theta) + 2\cos(2\theta))\tilde{E} + (-\sin(4\theta) - 2\sin(2\theta))\tilde{F}}{\tilde{E}}$$

$$Y = \frac{\tilde{H}}{\tilde{E}} = \frac{(-\sin(4\theta) + 2\sin(2\theta))\tilde{E} + (\cos(4\theta) + 2\cos(2\theta))\tilde{F}}{\tilde{E}}$$

$$Z = \frac{\tilde{F}}{\tilde{E}}$$

This again gives X and Y as functions of (θ, Z)

$$X = (-\cos(4\theta) + 2\cos(2\theta)) + (-\sin(4\theta) - 2\sin(2\theta))Z$$

$$Y = (-\sin(4\theta) + 2\sin(2\theta)) + (\cos(4\theta) + 2\cos(2\theta))Z$$

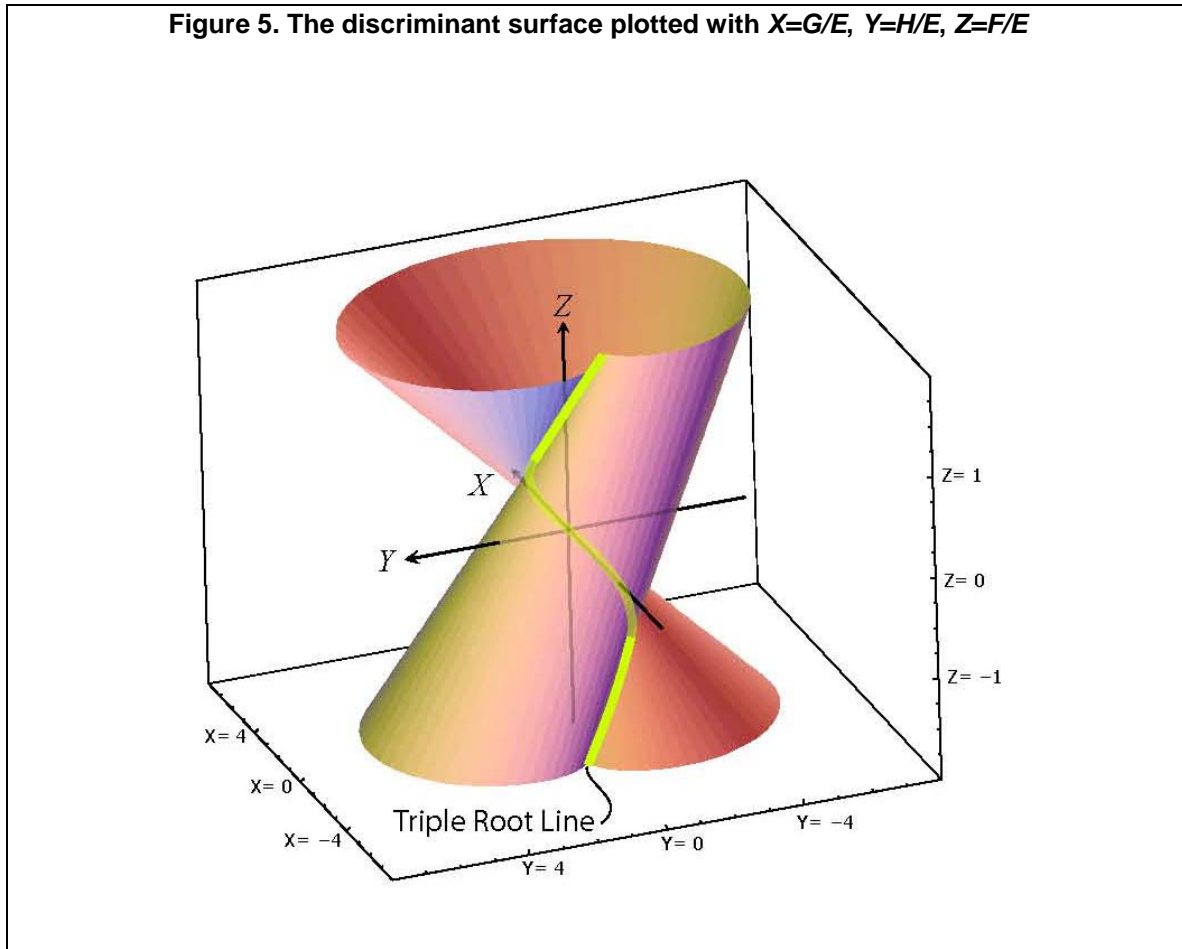
At $Z=0$ the curve is another epicycle:

$$X = -\cos(4\theta) + 2\cos(2\theta)$$

$$Y = -\sin(4\theta) + 2\sin(2\theta)$$

This generates the heart shaped curve (called a cardioid) that you see on the top face of figure 5. And as Z varies away from zero this curve also simply rotates and magnifies.

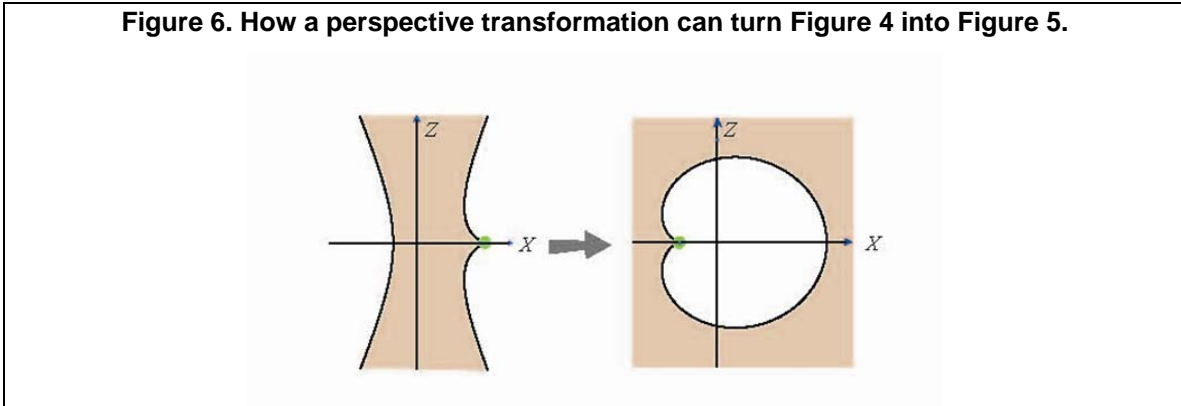
Figure 5. The discriminant surface plotted with $X=G/E$, $Y=H/E$, $Z=F/E$



Conclusion

We should now have a pretty good idea of what's going on in the space of all possible cubics. Remember that the plots shown in figures 3, 4, and 5 are all the same shape, just different perspective projections of it from 4D to 3D. The region inside the shape of figure 4 becomes the region outside the shape in figure 5, and vice versa. Note that the crease of the triple-root curve always points toward the type $1\bar{1}$ region. This allows lines through pairs of triple-root points to always stay in that region. What is especially odd, though, is the fact that the triple-root curve appears in three pieces in figure 5 but only in one piece in figures 3 and 4. But that's the sort of thing that happens when you turn tori inside out by passing them through a perspective transformation. You can get an idea of how this happens from Figure 6. The left part of the figure is a cross section through Figure 4 at $Y=0$. Now imagine a projective transform that moves the origin to the left and out to infinity. The left branch wraps around infinity and comes in from the right, turning the shape inside out and giving the right side of figure 6. Then a rotation about the X -axis gives the cross section of Figure 5.

Figure 6. How a perspective transformation can turn Figure 4 into Figure 5.



Next time we'll use some of these insights to solve the cubic.

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