CSE590B Lecture 7
Cubic Curves

Implicit and Parametric

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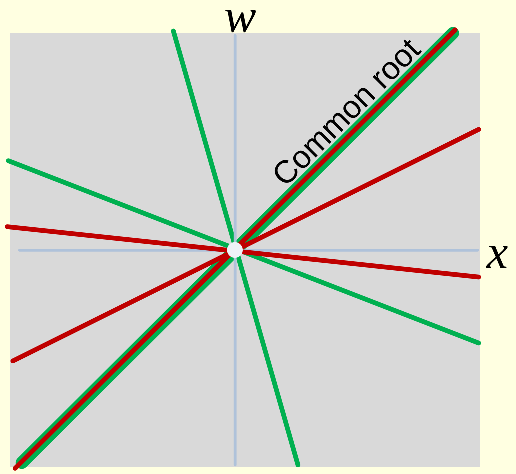
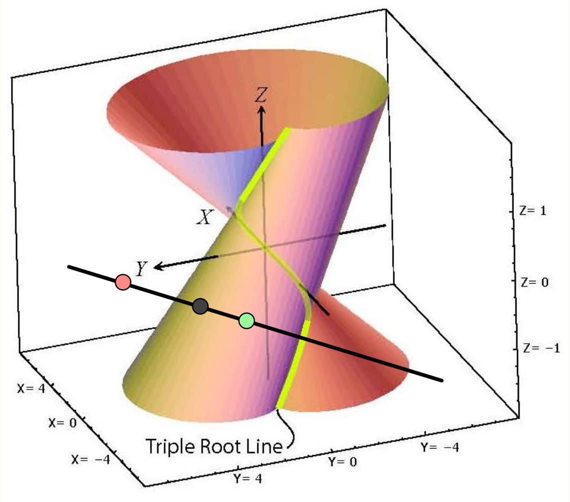
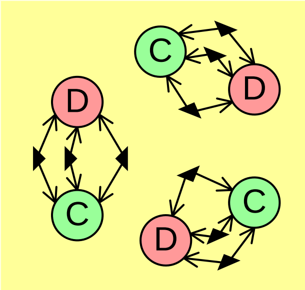
http://courses.cs.washington.edu/courses/cse590b/13au/

Resultant of Two Cubics

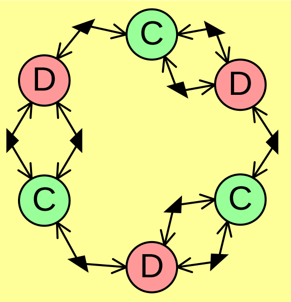




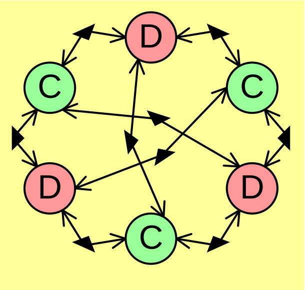
$216\,\rho =$

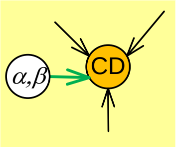


$+\ 9$

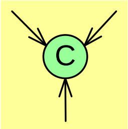


$+\ 9$

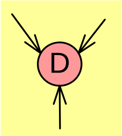




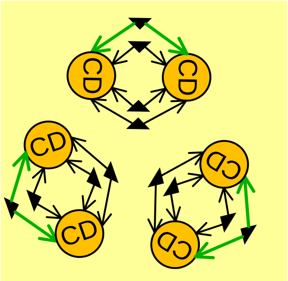
$= \alpha$



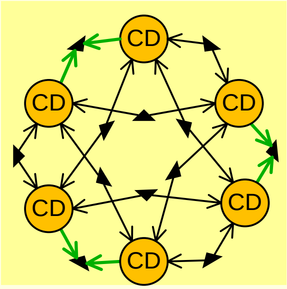
$+\beta$



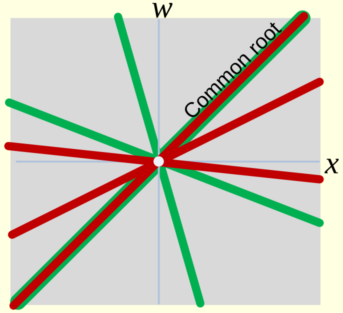
$\rho(\mathrm{C,D}) = -$

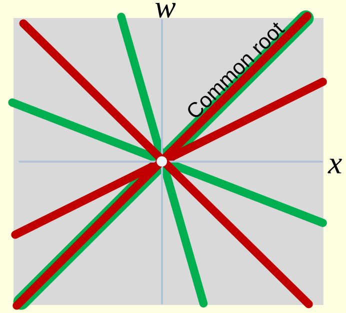


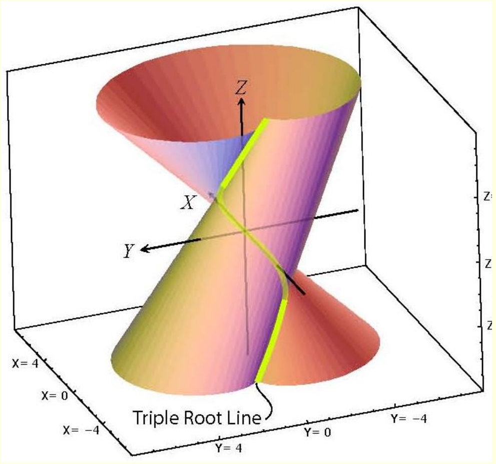
-36

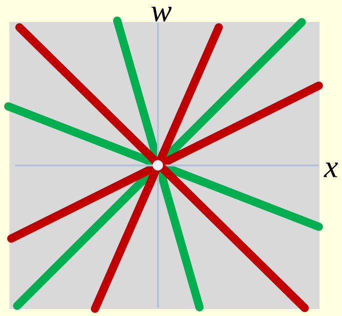


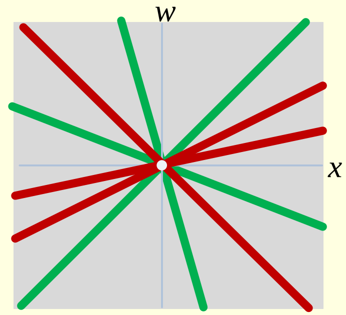
Other Relations Between Cubics

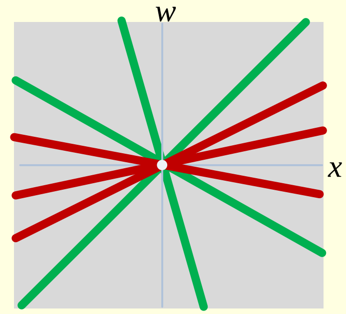


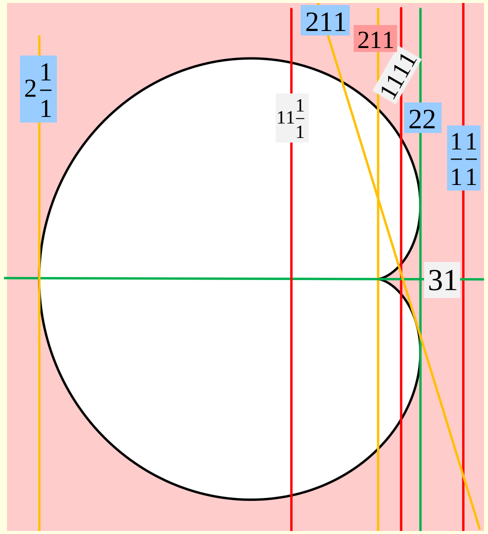


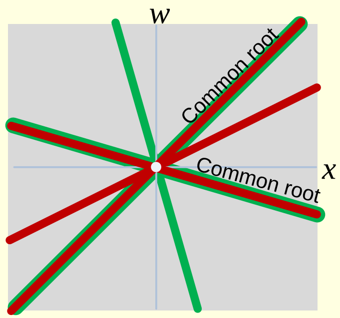


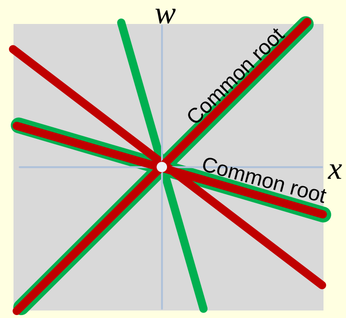








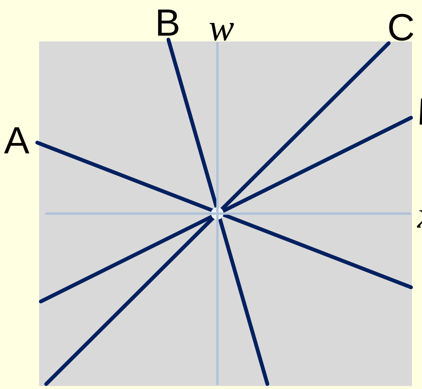


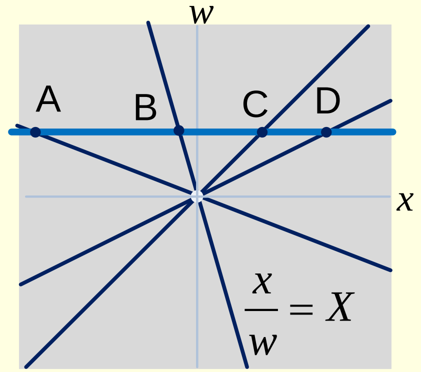


Etc…

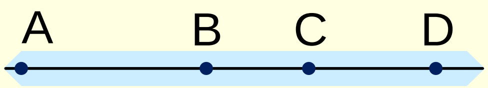
P^1 Space Visualizations

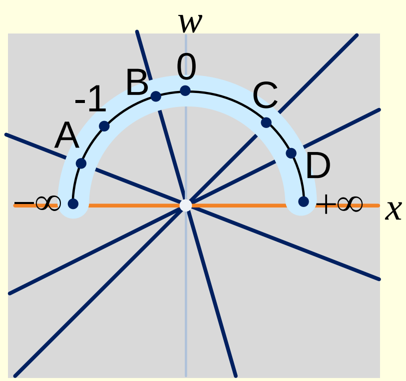
2D lines in [*x*,*w*]

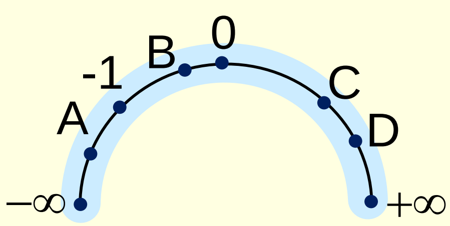




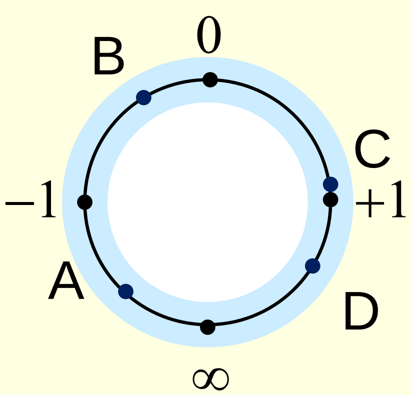
P^1





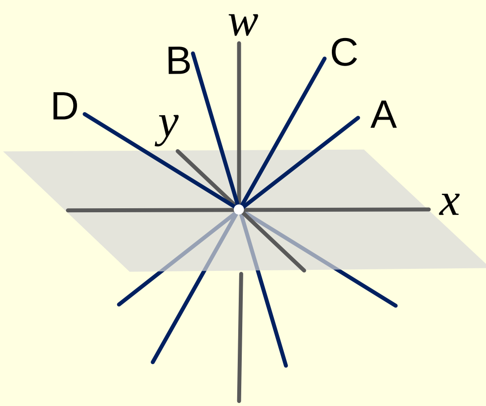


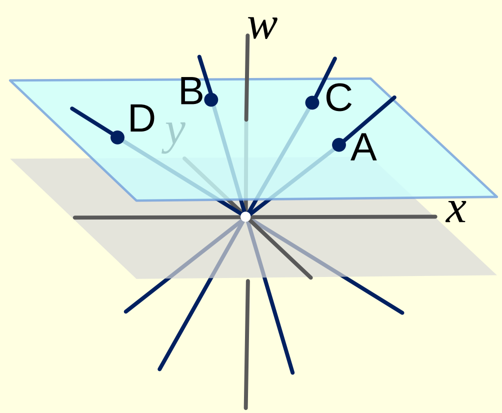
Topology

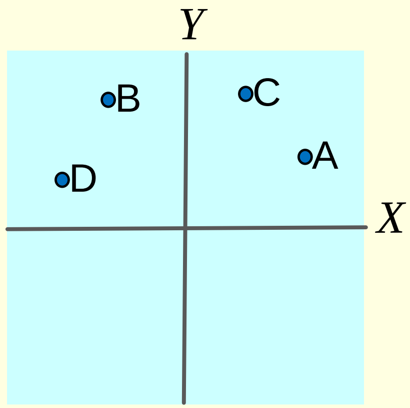


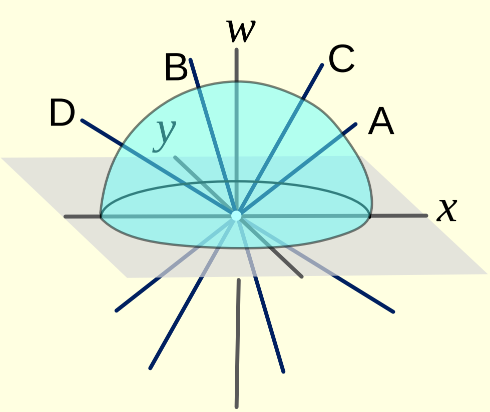
360° generates P^1 twice

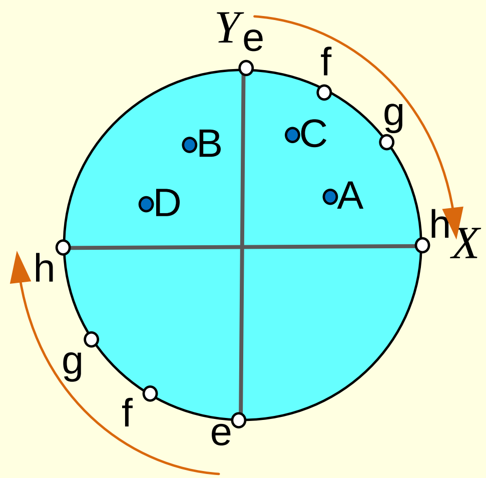
P^2 Space Visualizations



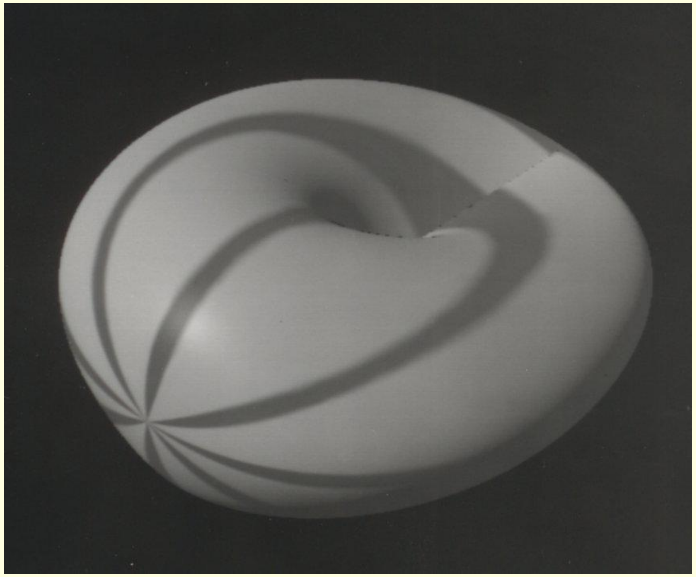


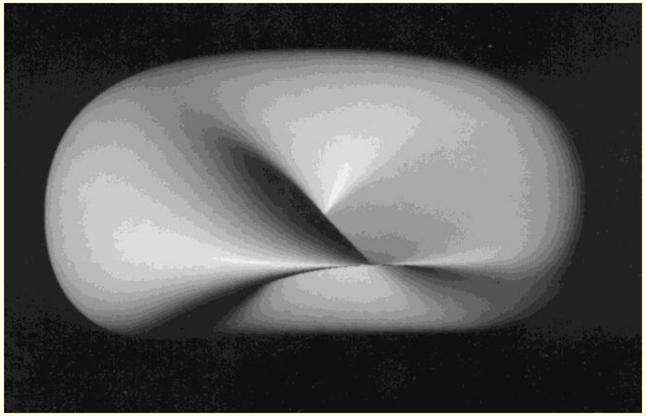


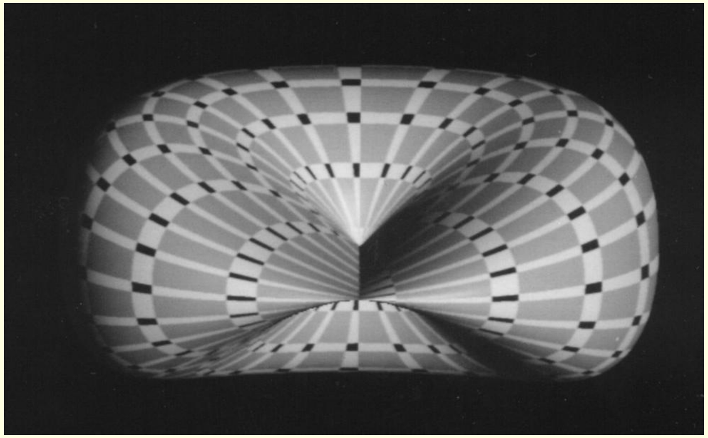




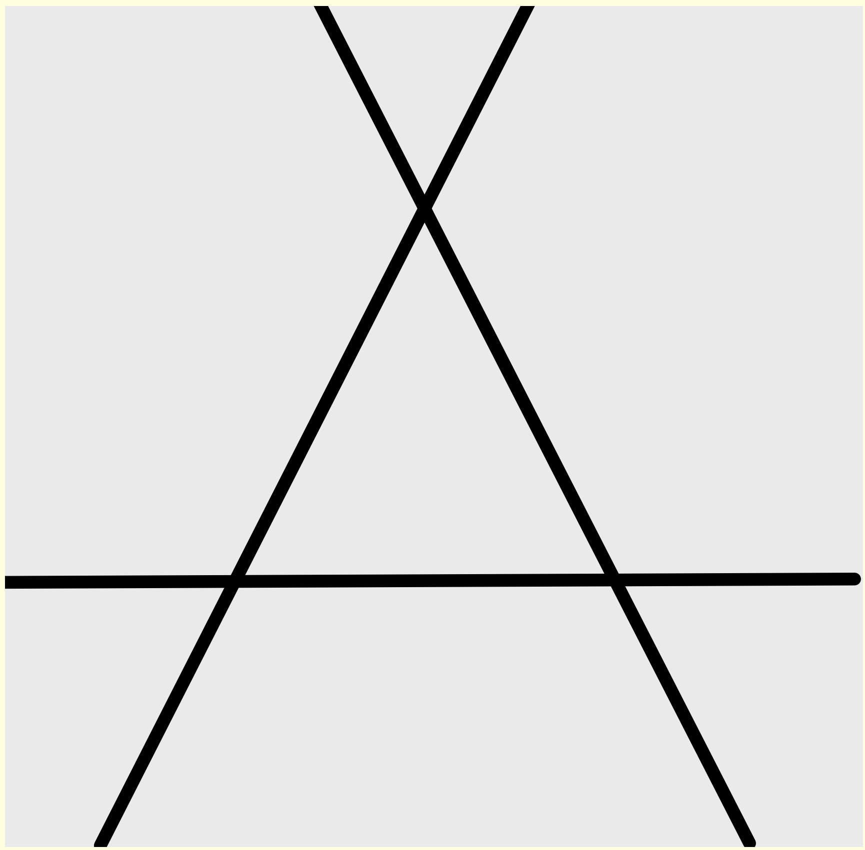
P^2 Space Visualizations



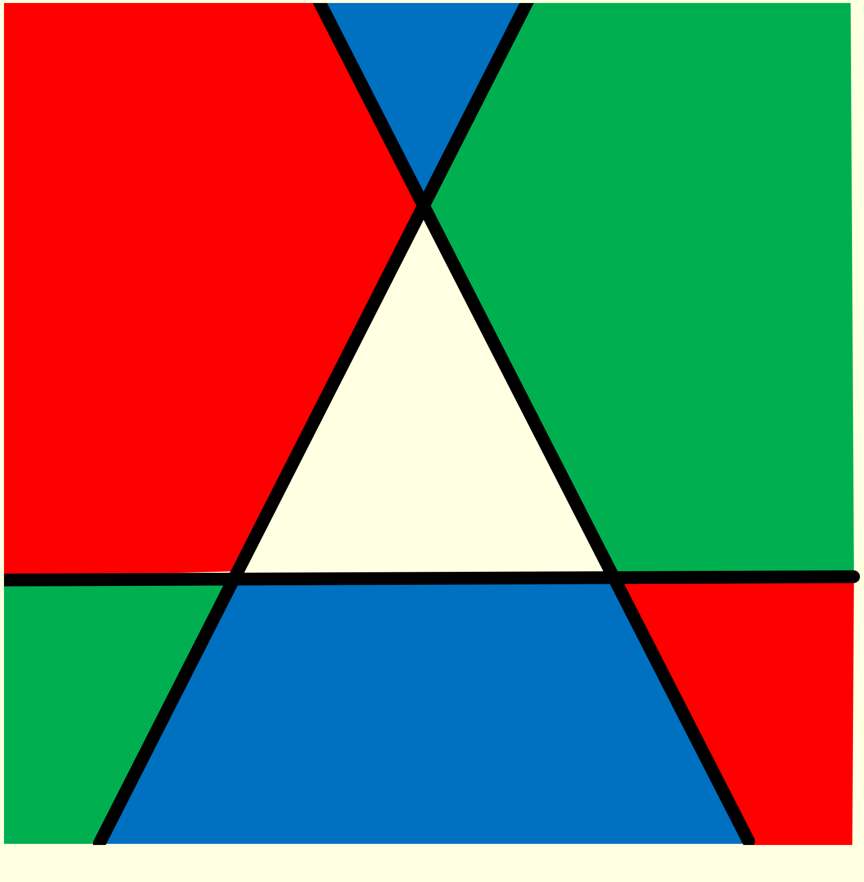




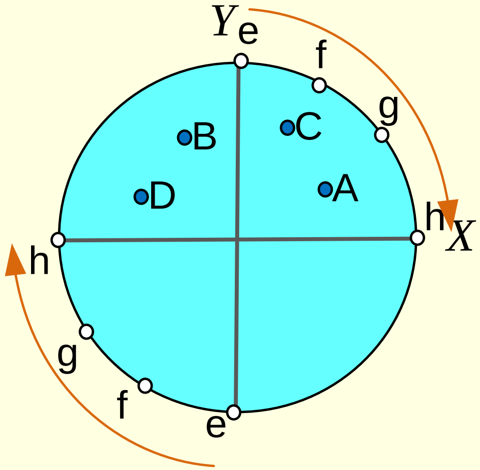
How Many Triangles?

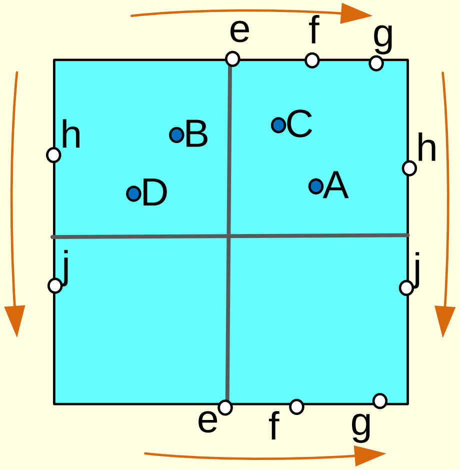


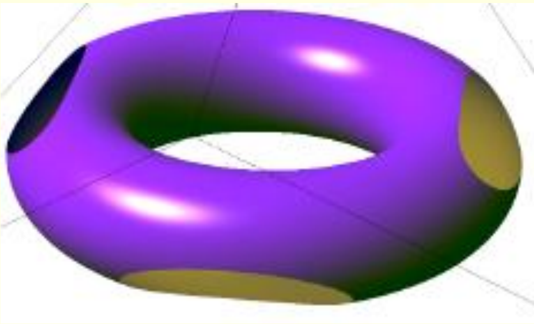
Four



Other Space Topologies



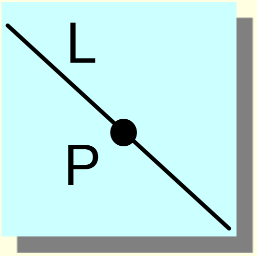


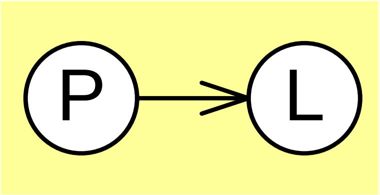


Point on a Line

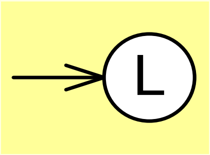
$$Ax + By + Cw = 0$$

$$\begin{bmatrix} x & y & w \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix} = 0$$





$= 0$

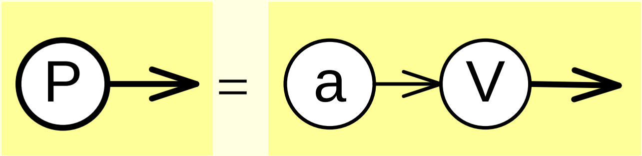


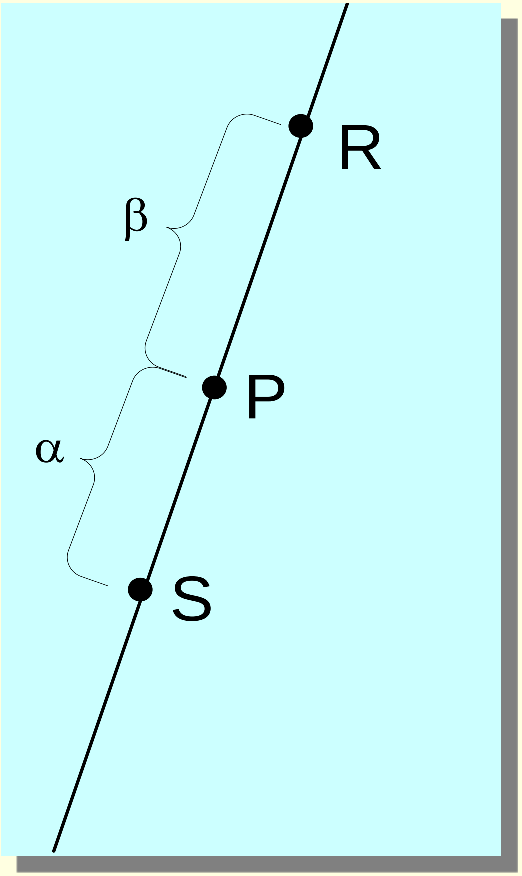
Parametric Line

$$\mathbf{P}(\alpha,\beta)=\alpha\mathbf{R}+\beta\mathbf{S}$$

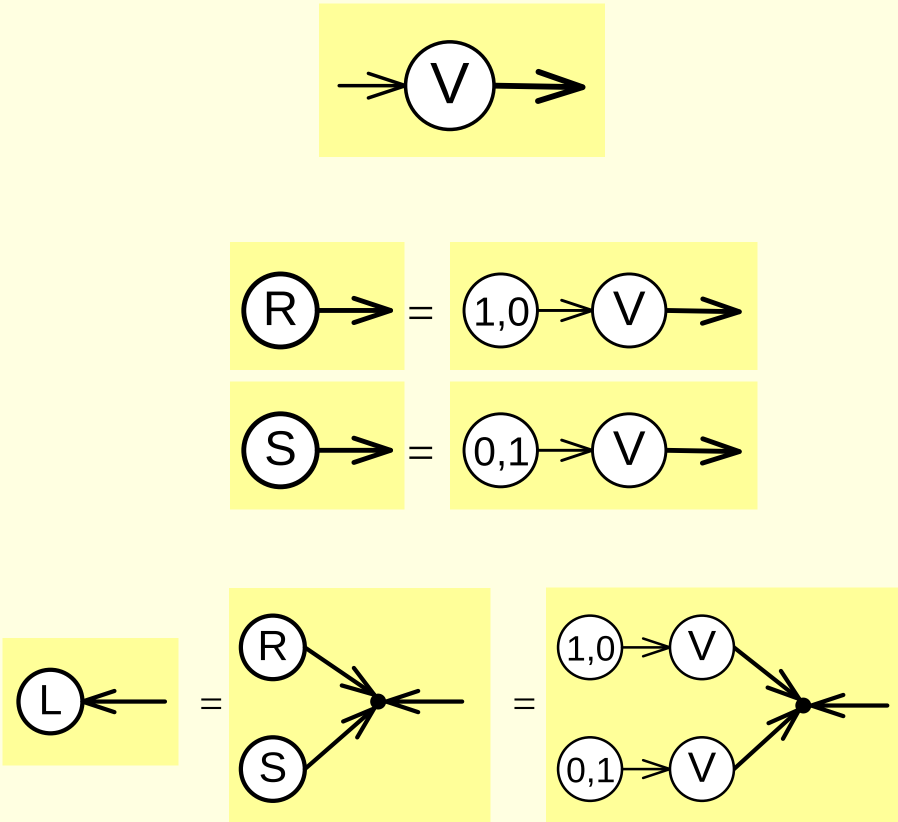
$$\mathbf{P}=\begin{bmatrix}\alpha & \beta\end{bmatrix}\begin{bmatrix}R^1 & R^2 & R^3\\ S^1 & S^2 & S^3\end{bmatrix}$$

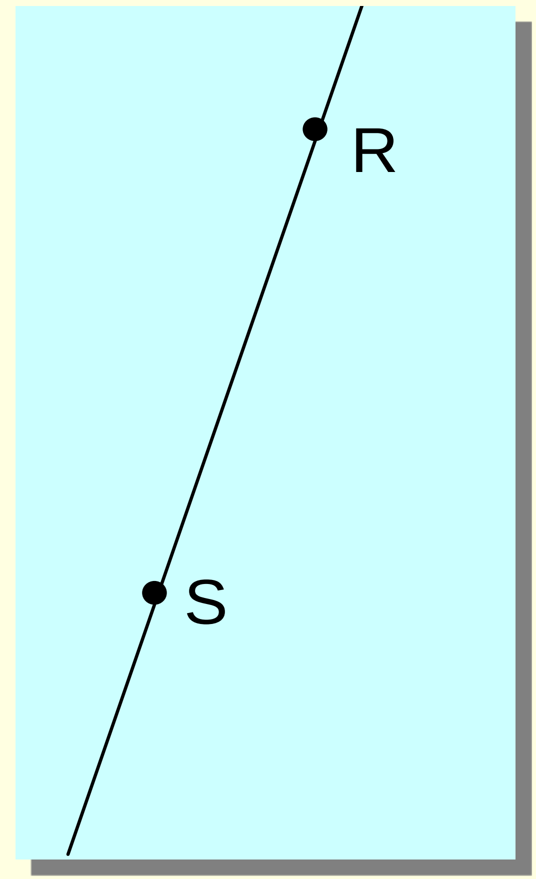
$$\mathbf{P}=\mathbf{aV}$$



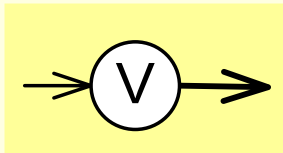


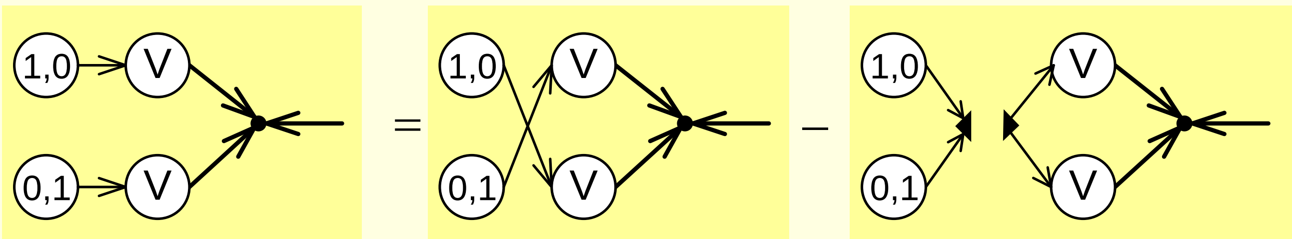
Implicitize Parametric Line

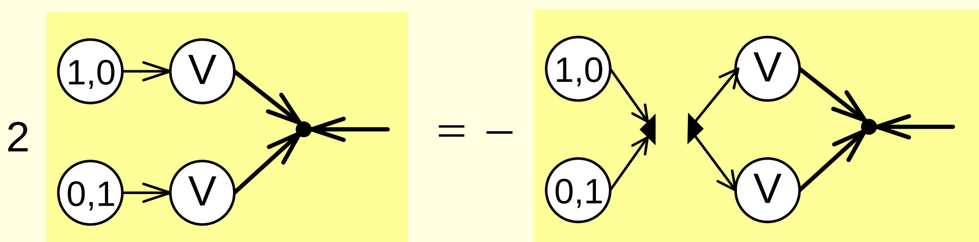


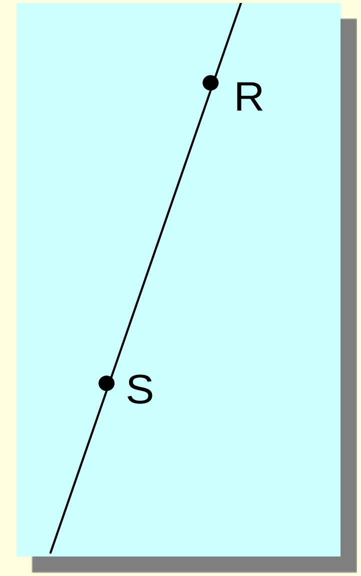


Implicitize Parametric Line



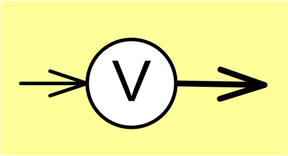




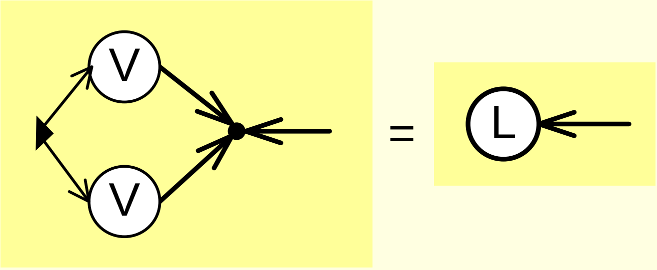


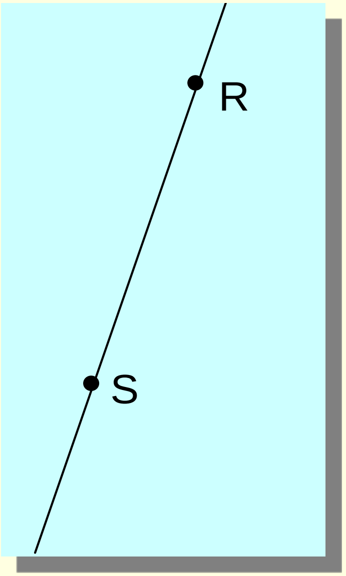
Implicitize Parametric Line

Parametric

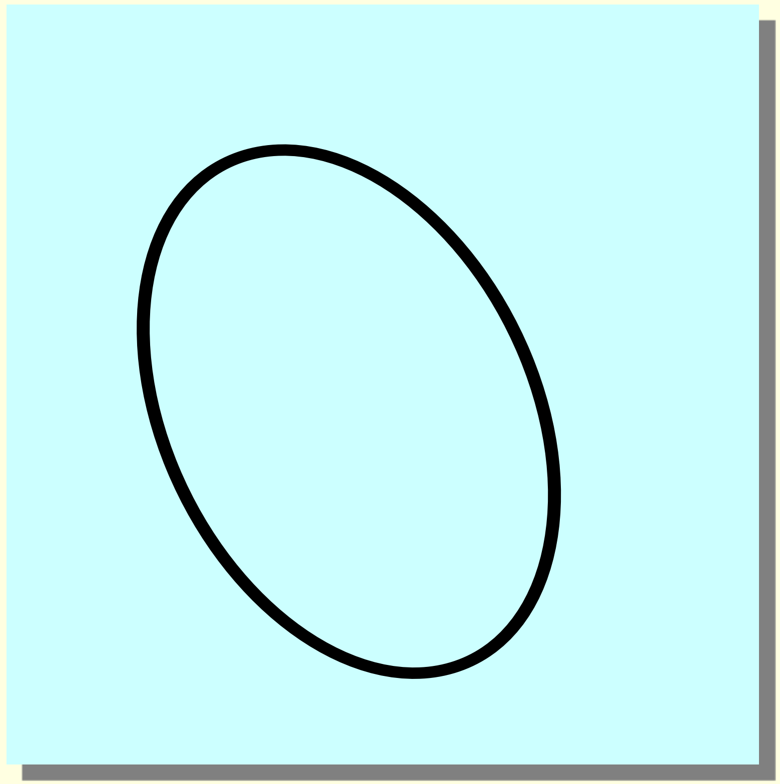


Implicit





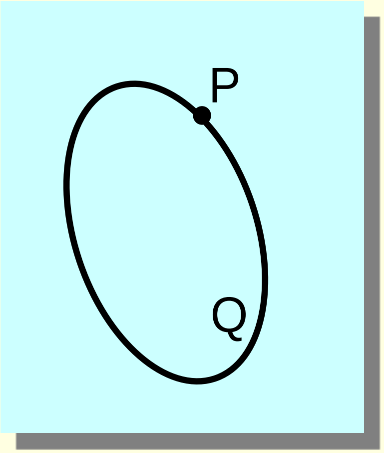
Quadratic Curve

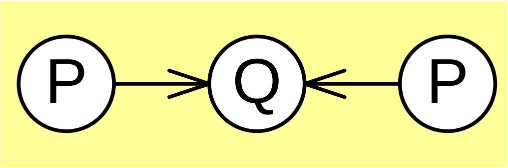


Point on a Quadratic Curve

$$Ax^2 + 2Bxy + 2Cxw$$
$$+Dy^2 + 2Eyw$$
$$+Fw^2 = 0$$

$$\begin{bmatrix} x & y & w \end{bmatrix} \begin{bmatrix} A & B & C \\ B & D & E \\ C & E & F \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix} = 0$$

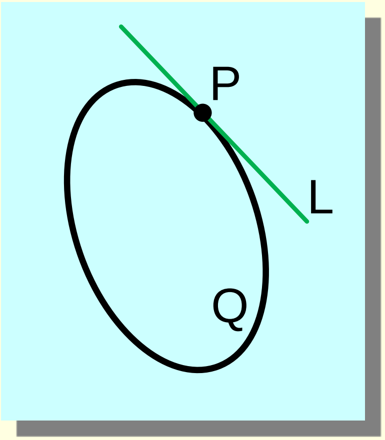




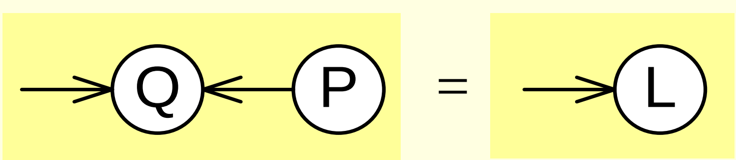
$= 0$

Tangent to Quadratic at P

$$\mathbf{Q}(x,y,w) = Ax^2 + 2Bxy + 2Cxw$$
$$+Dy^2 + 2Eyw$$
$$+Fw^2$$



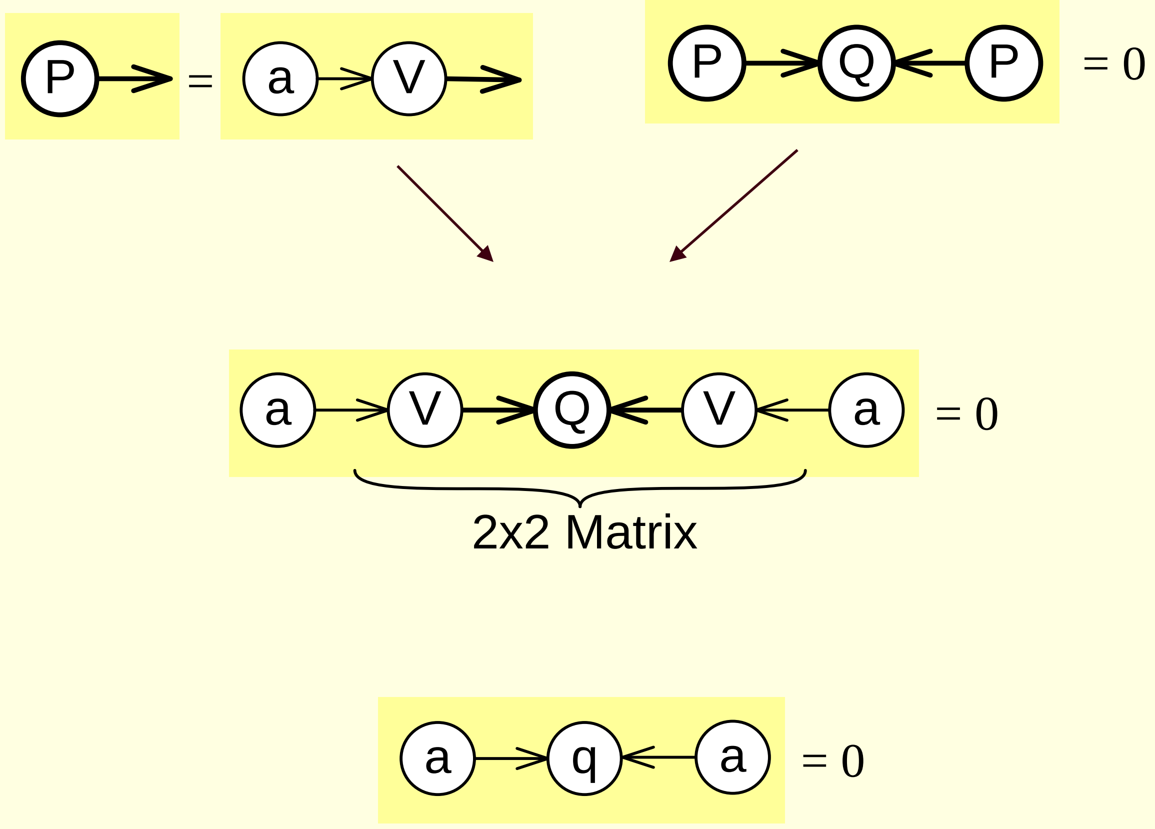
$$\begin{bmatrix} \mathbf{Q}_x \\ \mathbf{Q}_y \\ \mathbf{Q}_w \end{bmatrix} = 2\begin{bmatrix} Ax+By+Cw \\ Bx+Dy+Ew \\ Cx+Ey+Fw \end{bmatrix} = 2\begin{bmatrix} A & B & C \\ C & D & E \\ B & C & F \end{bmatrix}\begin{bmatrix} x \\ y \\ w \end{bmatrix}$$

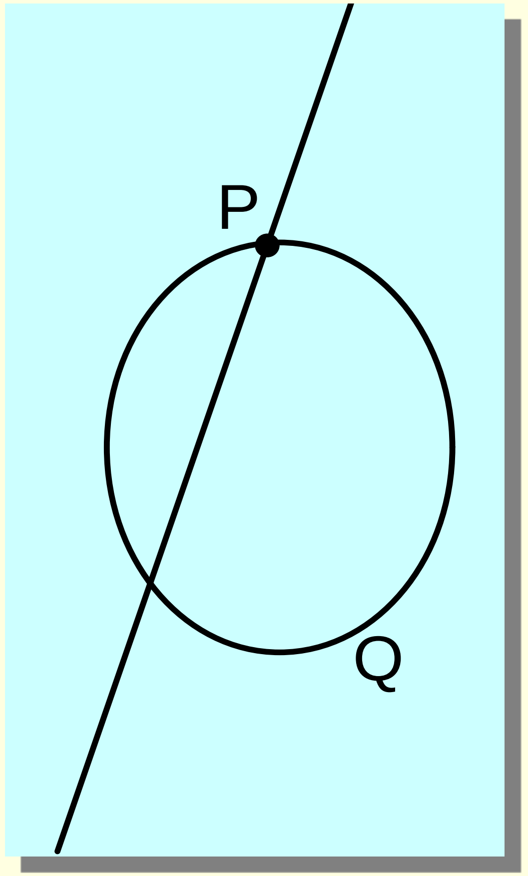


Intersect Line And Quadratic

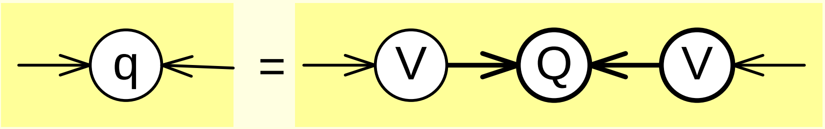
Parametric line

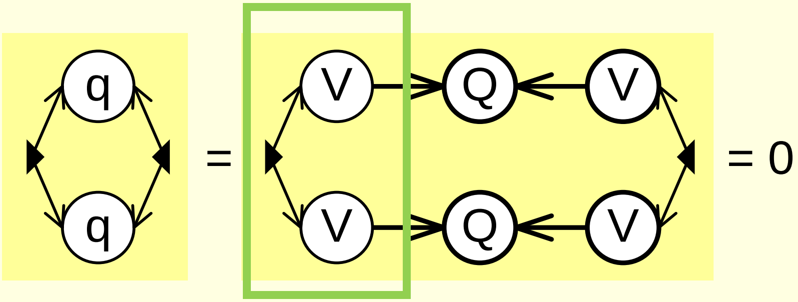
Implicit Quadratic

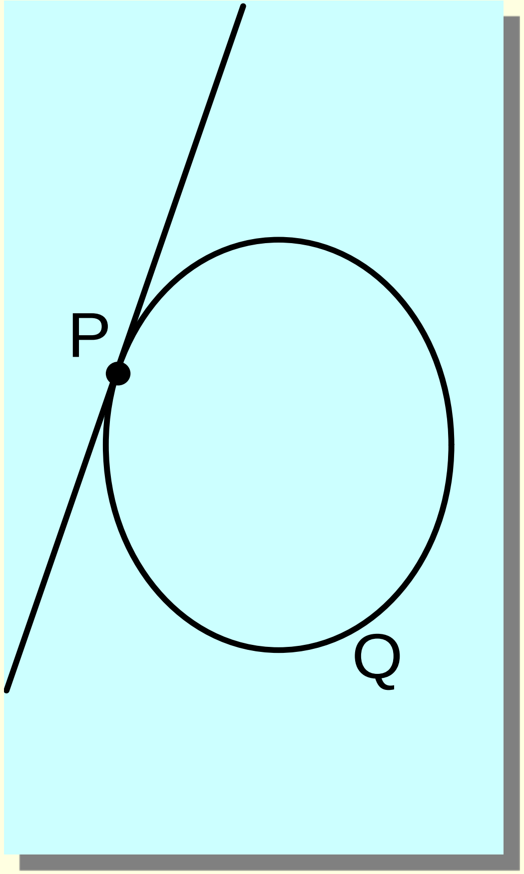




Double Root Means Tangent

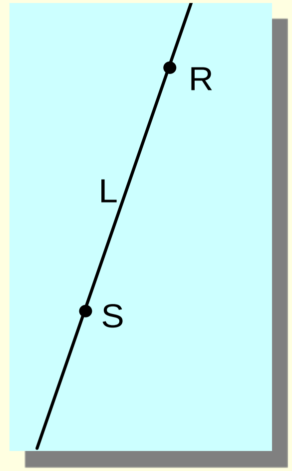




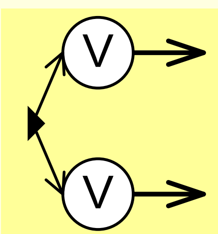


Reinterpret Diagram Fragment

$$\mathbf{V} = \begin{bmatrix} R^1 & R^2 & R^3 \\ S^1 & S^2 & S^3 \end{bmatrix}$$

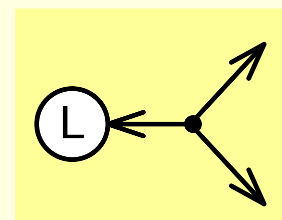


$$\mathbf{L} = \begin{bmatrix} L_1 \\ L_2 \\ L_3 \end{bmatrix}$$



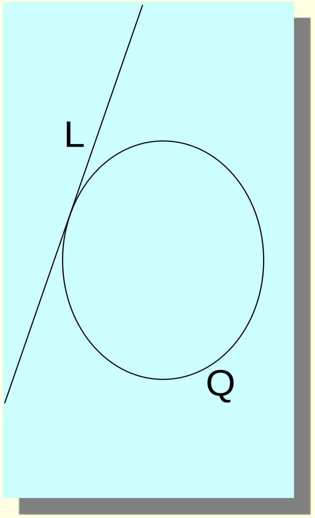
$$= \begin{bmatrix} R^1 & S^1 \\ R^2 & S^2 \\ R^3 & S^3 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} R^1 & R^2 & R^3 \\ S^1 & S^2 & S^3 \end{bmatrix}$$

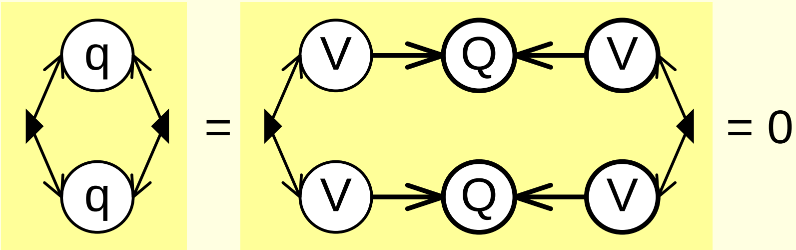
$$= \begin{bmatrix} 0 & R^1S^2 - R^2S^1 & R^1S^3 - R^3S^1 \\ R^2S^1 - R^1S^2 & 0 & R^2S^3 - R^3S^2 \\ R^3S^1 - R^1S^3 & R^3S^2 - R^2S^3 & 0 \end{bmatrix}$$

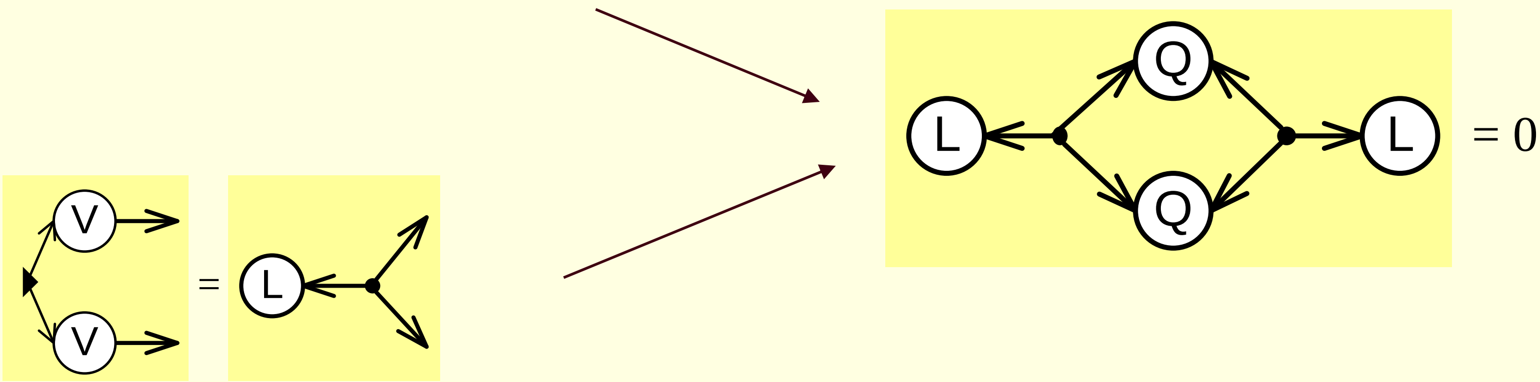


$$= \begin{bmatrix} 0 & L_3 & -L_2 \\ -L_3 & 0 & L_1 \\ L_2 & -L_1 & 0 \end{bmatrix}$$

Line Tangent To Quadric



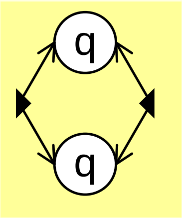




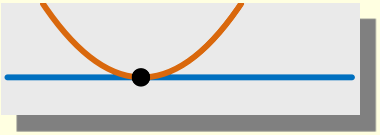
Klebsch Transfer Principle

Klebsch Transfer Principle

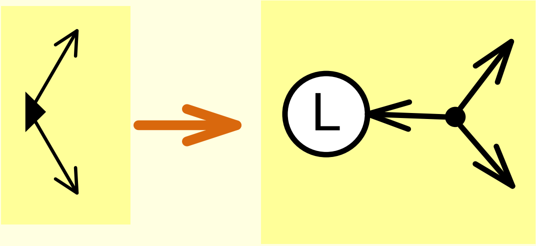
Condition for a double root in 2D (P^1)



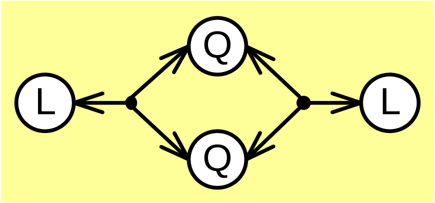
= 0



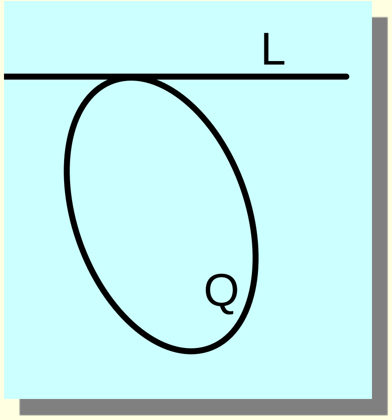
Make substitution



Condition for tangency in 3D (P^2)



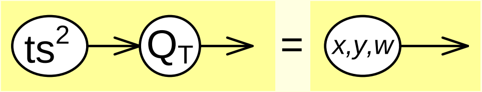
= 0



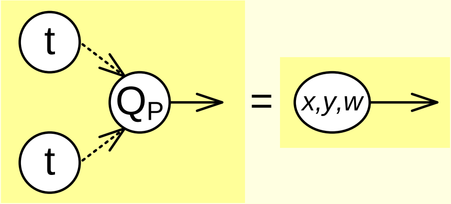
Parametric Quadratic curves

$$x(s,t) = A_x t^2 + 2B_x ts + C_x s^2$$
$$y(s,t) = A_y t^2 + 2B_y ts + C_y s^2$$
$$w(s,t) = A_w t^2 + 2B_w ts + C_w s^2$$

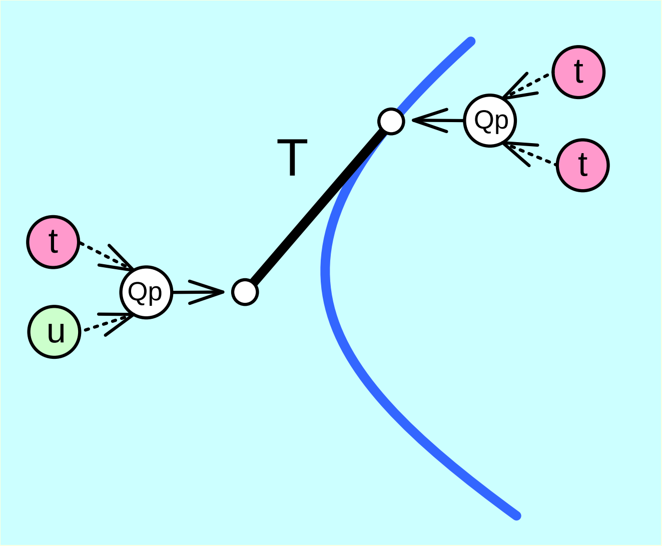
$$\begin{bmatrix} t^2 & 2ts & s^2 \end{bmatrix} \begin{bmatrix} A_x & A_y & A_w \\ B_x & B_y & B_w \\ C_x & C_y & C_w \end{bmatrix} = \begin{bmatrix} x & y & w \end{bmatrix}$$

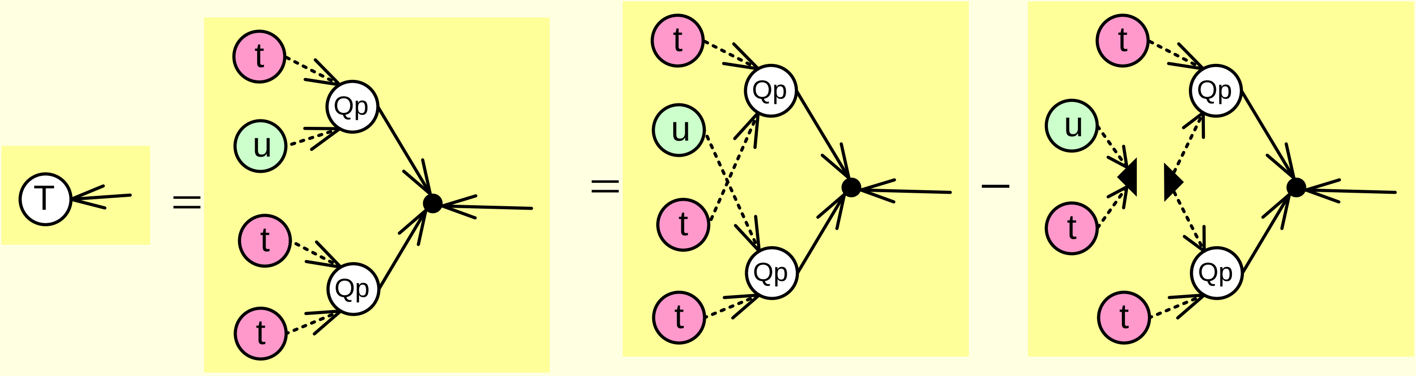


$$\begin{bmatrix} t & s \end{bmatrix} \begin{bmatrix} \begin{bmatrix} A_x & B_x \\ B_x & C_x \end{bmatrix} & \begin{bmatrix} A_y & B_y \\ B_y & C_y \end{bmatrix} & \begin{bmatrix} A_w & B_w \\ B_w & C_w \end{bmatrix} \end{bmatrix} \begin{bmatrix} t \\ s \end{bmatrix} = \begin{bmatrix} x & y & w \end{bmatrix}$$

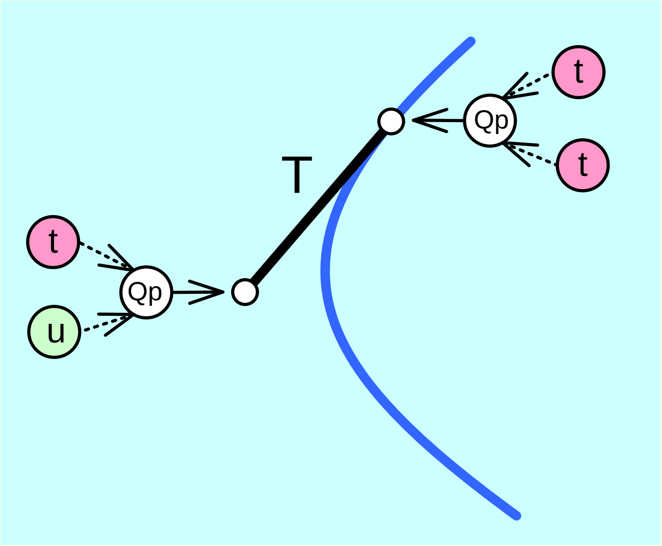


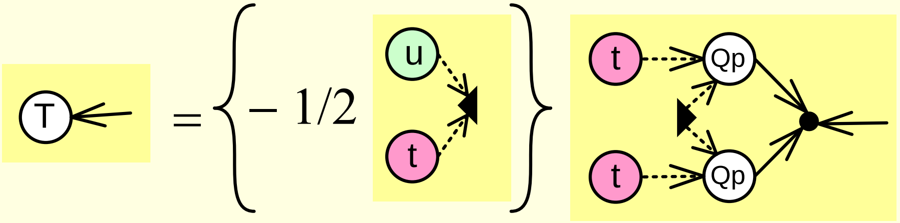
Points and Tangents



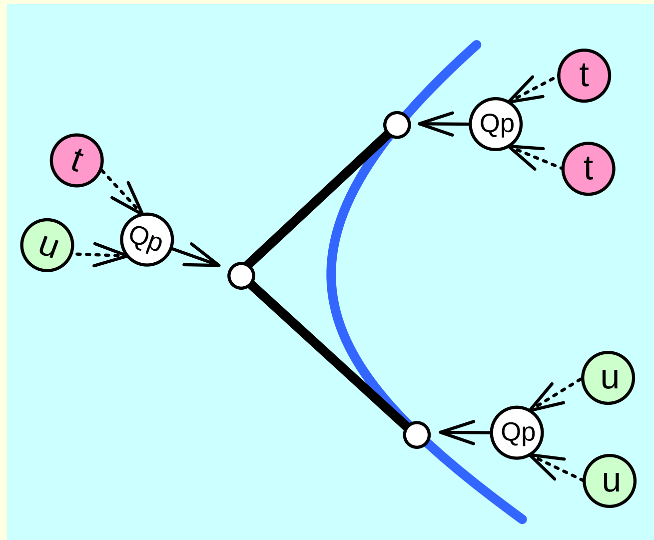


Points and Tangents

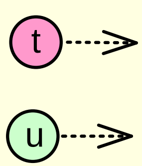




Points and Tangents



Plus:
Homogeneous
Scales of Points

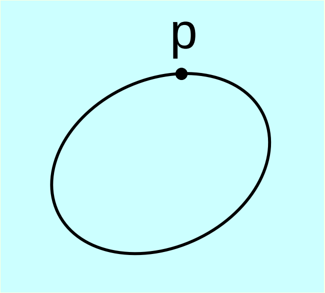
If 

$= \begin{bmatrix} 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1 & 1 \end{bmatrix}$

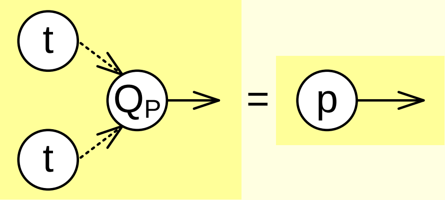
Bezier Control Points

Implicitizing A Parametric Quadratic Curve



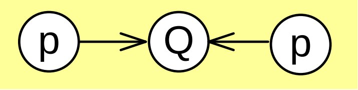
Given

$$\begin{bmatrix} t & s \end{bmatrix} \begin{bmatrix} \begin{bmatrix} A_x & B_x \\ B_x & C_x \end{bmatrix} & \begin{bmatrix} A_y & B_y \\ B_y & C_y \end{bmatrix} & \begin{bmatrix} A_w & B_w \\ B_w & C_w \end{bmatrix} \end{bmatrix} \begin{bmatrix} t \\ s \end{bmatrix} = \begin{bmatrix} x & y & w \end{bmatrix}$$



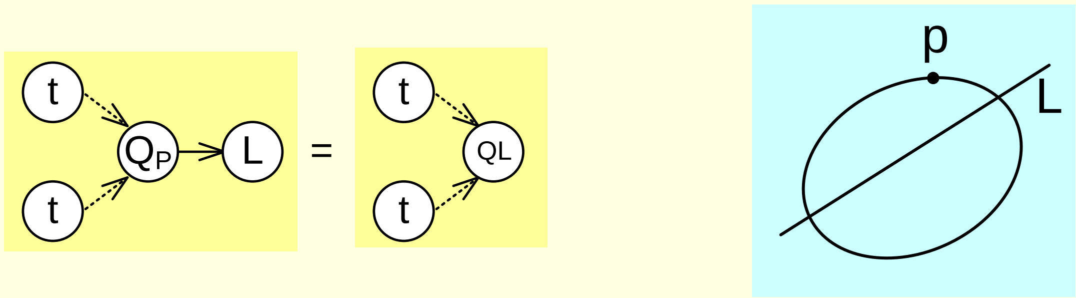
Want

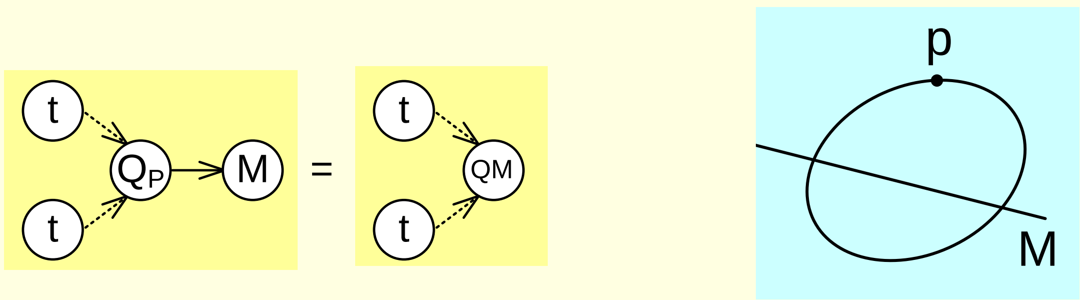
$$\begin{bmatrix} x & y & w \end{bmatrix} \begin{bmatrix} A & B & C \\ B & D & E \\ C & E & F \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix} = 0$$

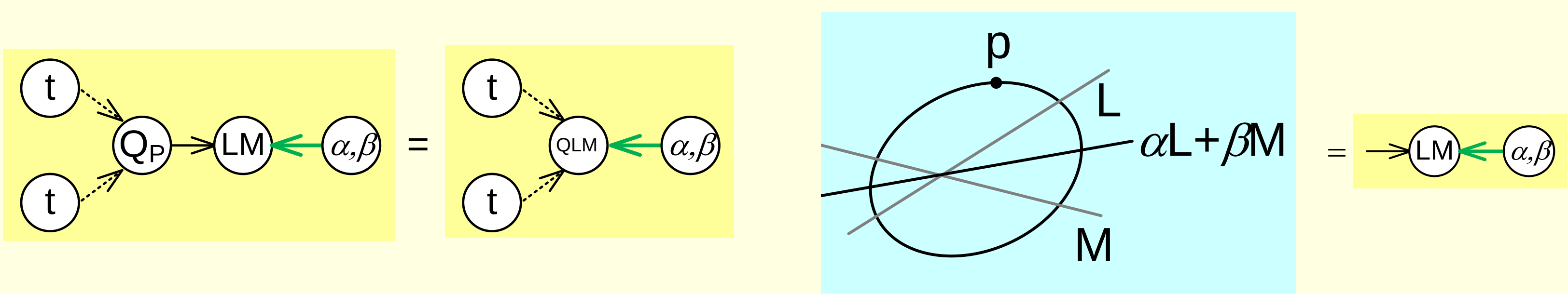


$= 0$

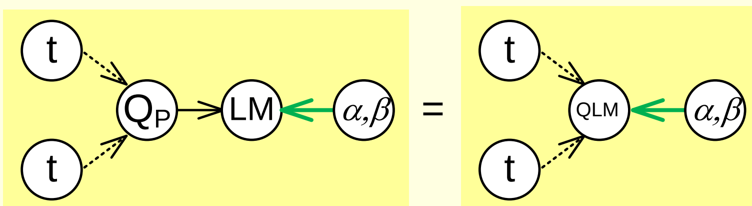
Implicitizing A Parametric Quadratic Curve

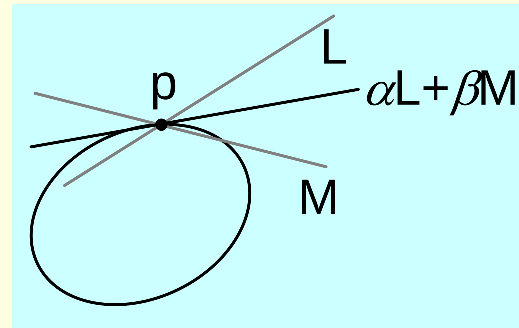




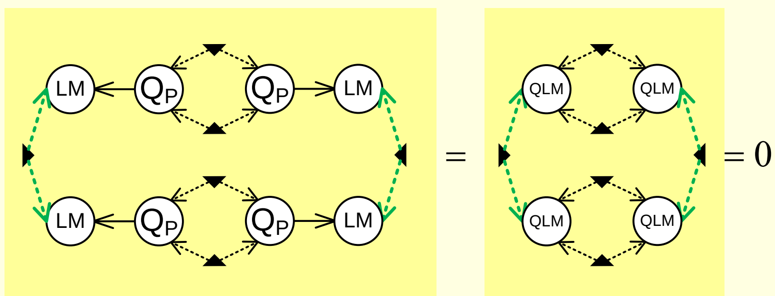


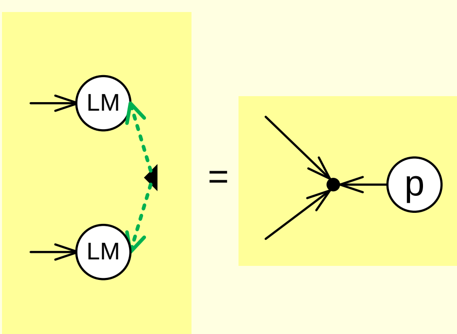
Implicitizing A Parametric Quadratic Curve

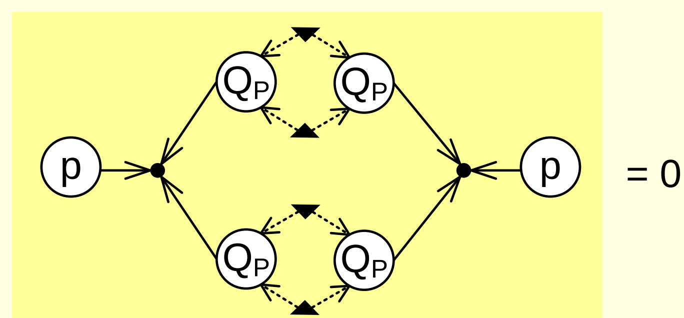




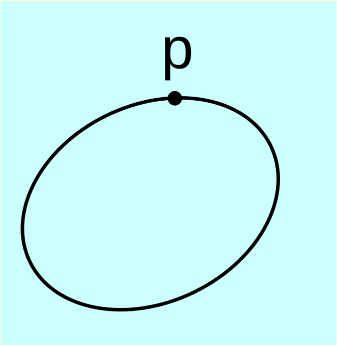
Use resultant:



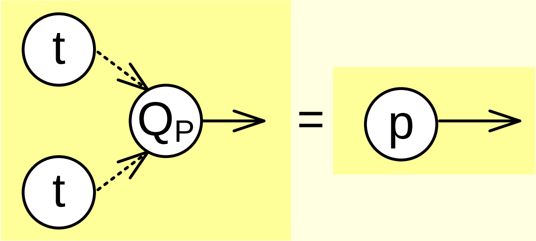




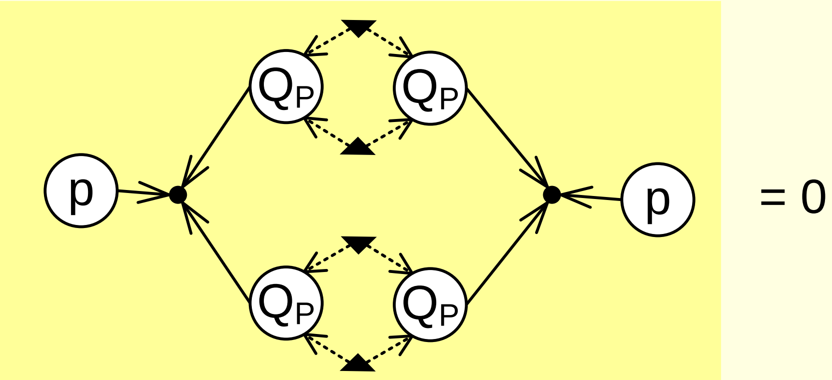
Implicitizing A Parametric Quadratic Curve



Given the parametric form:

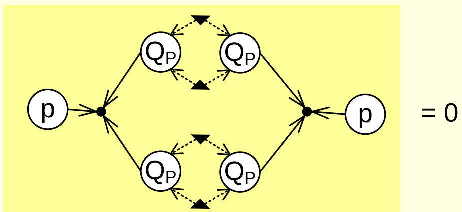


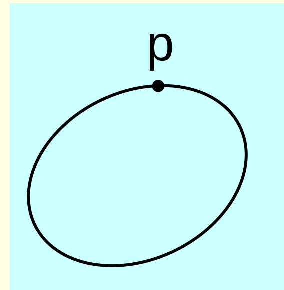
The implicit form is:

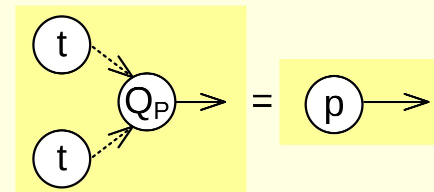


Line Tangency to Param. Quadratic Curve

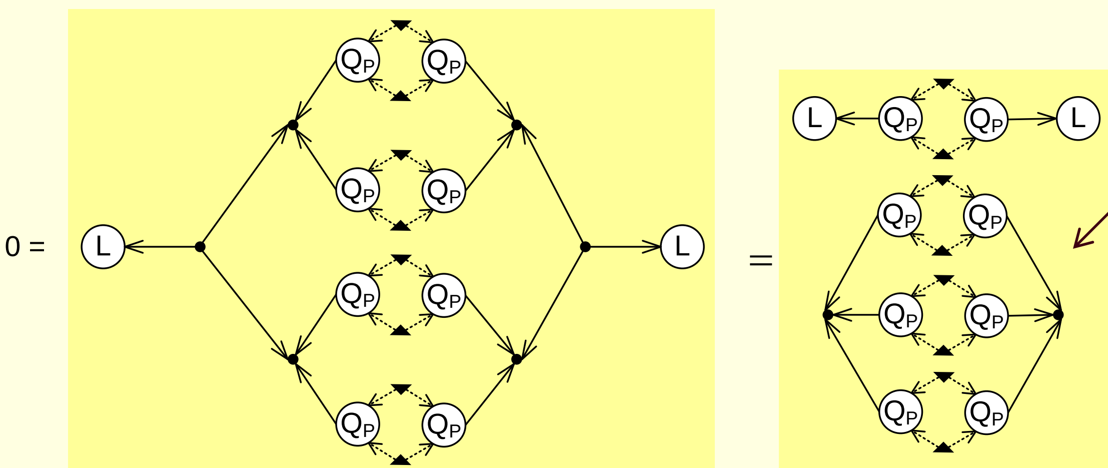
In terms of the implicitized parametric form:





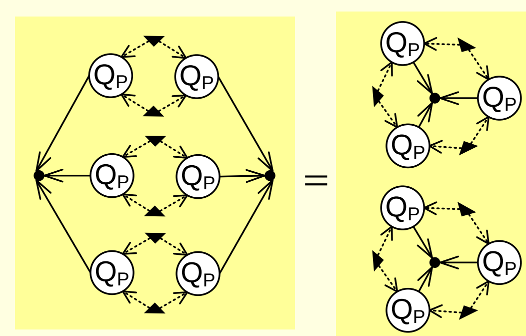


The line tangency condition is:



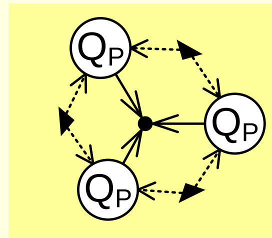
Sign?

Sign is positive

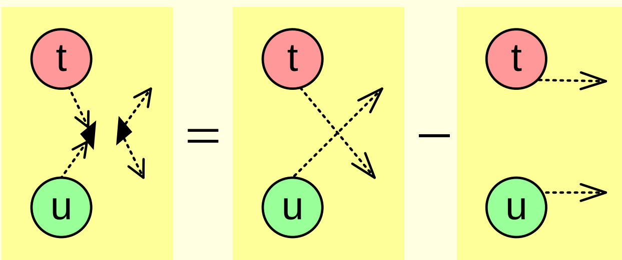


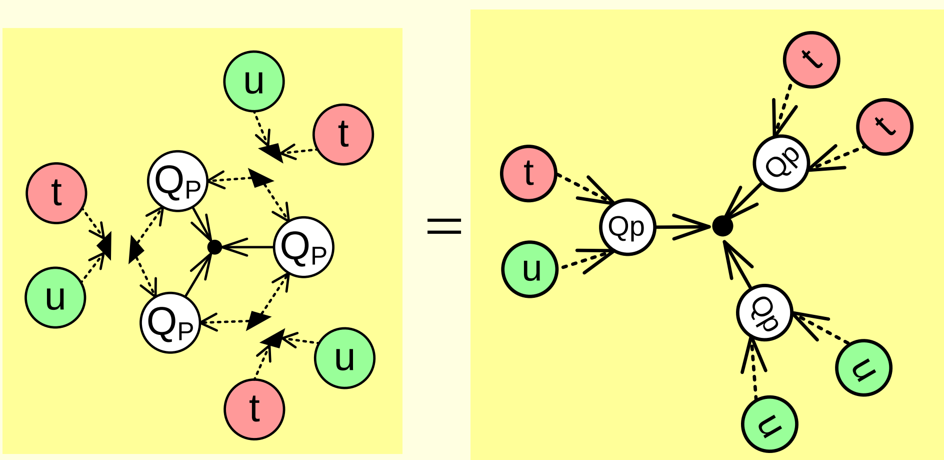
Singularity in a Param. Quadratic Curve

What does

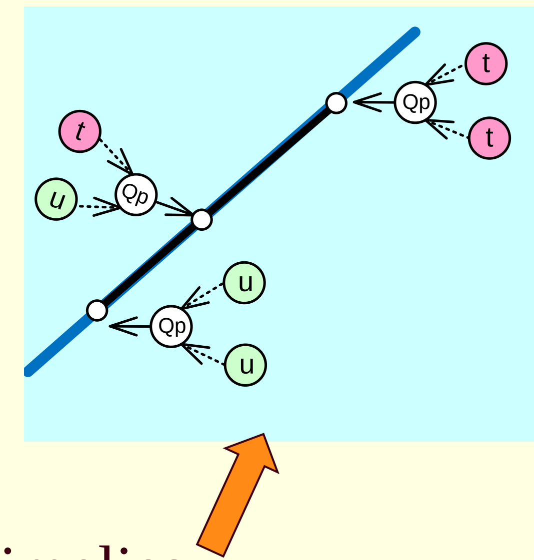


= 0 mean?



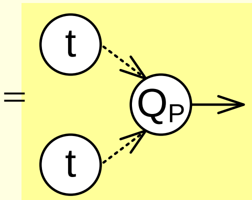


= 0 implies

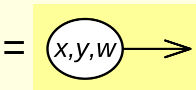


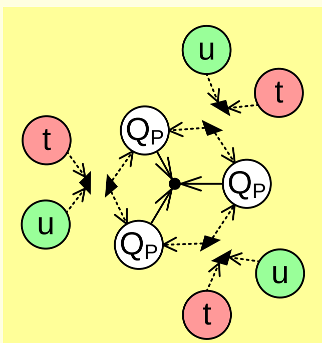
Forms of Singularity Condition

$$\begin{bmatrix} t_0 & t_1 \end{bmatrix} \left[\begin{bmatrix} A_x & B_x \\ B_x & C_x \end{bmatrix} \quad \begin{bmatrix} A_y & B_y \\ B_y & C_y \end{bmatrix} \quad \begin{bmatrix} A_w & B_w \\ B_w & C_w \end{bmatrix} \right] \begin{bmatrix} t_0 \\ t_1 \end{bmatrix} =$$



=





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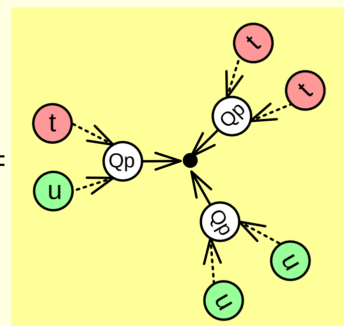
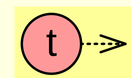
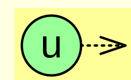
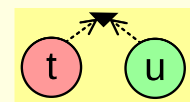




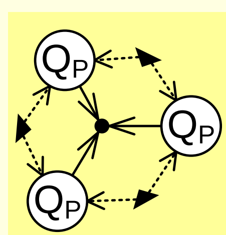
$= \begin{bmatrix} 1 & 0 \end{bmatrix}$



$= \begin{bmatrix} 0 & 1 \end{bmatrix}$



$= 1$



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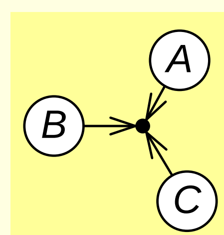
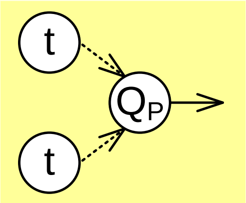


Forms of Singularity Condition

$$[t \quad s]\left[\begin{bmatrix}A_x & B_x\\ B_x & C_x\end{bmatrix}\ \begin{bmatrix}A_y & B_y\\ B_y & C_y\end{bmatrix}\ \begin{bmatrix}A_w & B_w\\ B_w & C_w\end{bmatrix}\right]\begin{bmatrix}t\\ s\end{bmatrix} =$$



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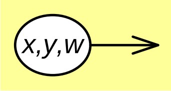
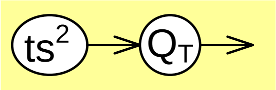


$$[t^2 \quad 2ts \quad s^2]\begin{bmatrix}A_x & A_y & A_w\\ B_x & B_y & B_w\\ C_x & C_y & C_w\end{bmatrix} =$$



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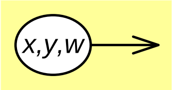

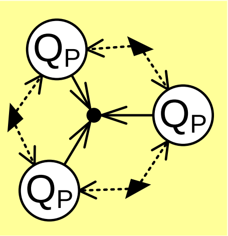

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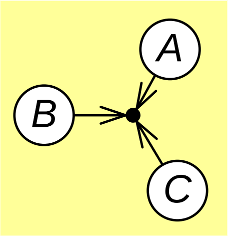

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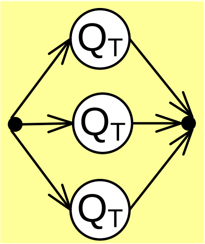

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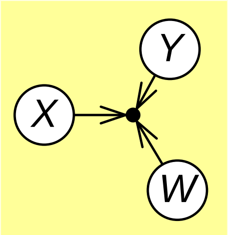
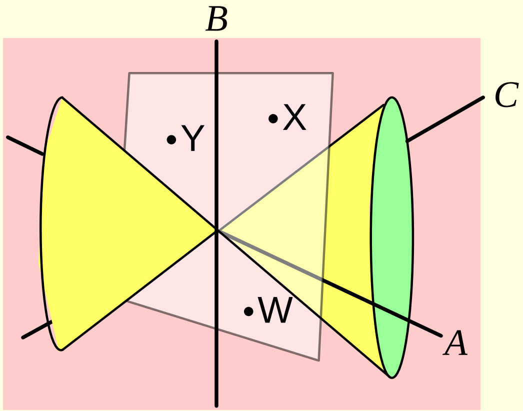
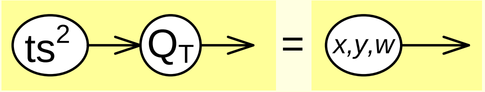


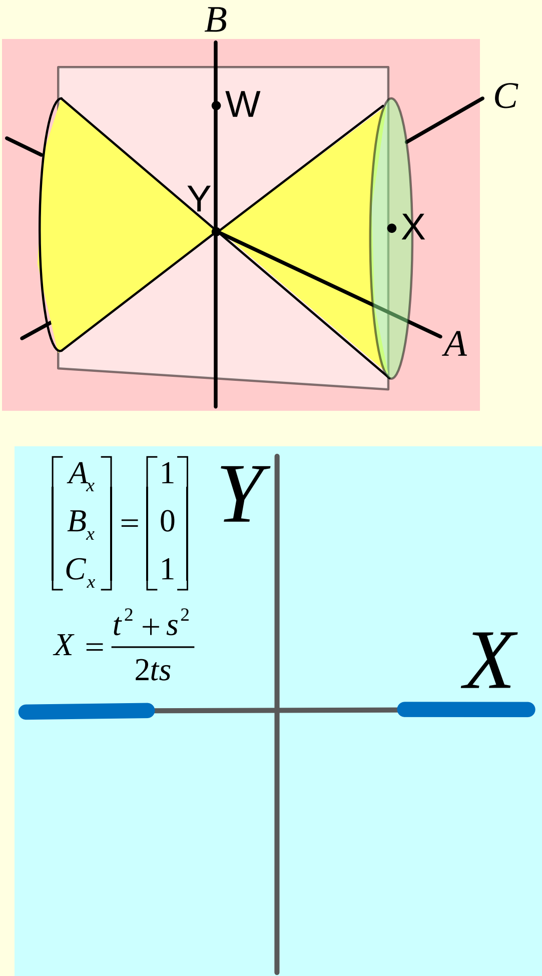


x,y,w coplanar in coefficient space

Sub-Forms of Singularity

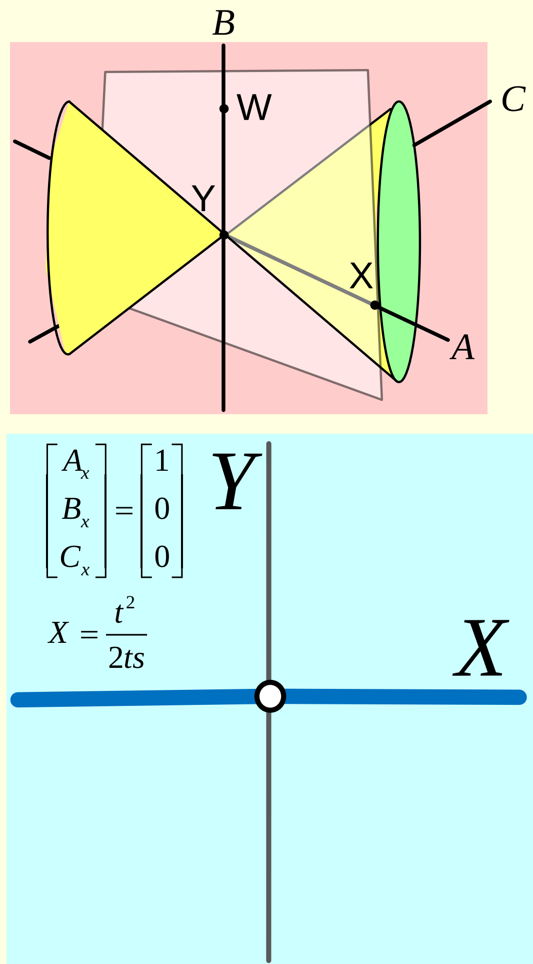
$$\begin{bmatrix} t^2 & 2ts & s^2 \end{bmatrix} \begin{bmatrix} A_x & 0 & 0 \\ B_x & 0 & 1 \\ C_x & 0 & 0 \end{bmatrix} =$$





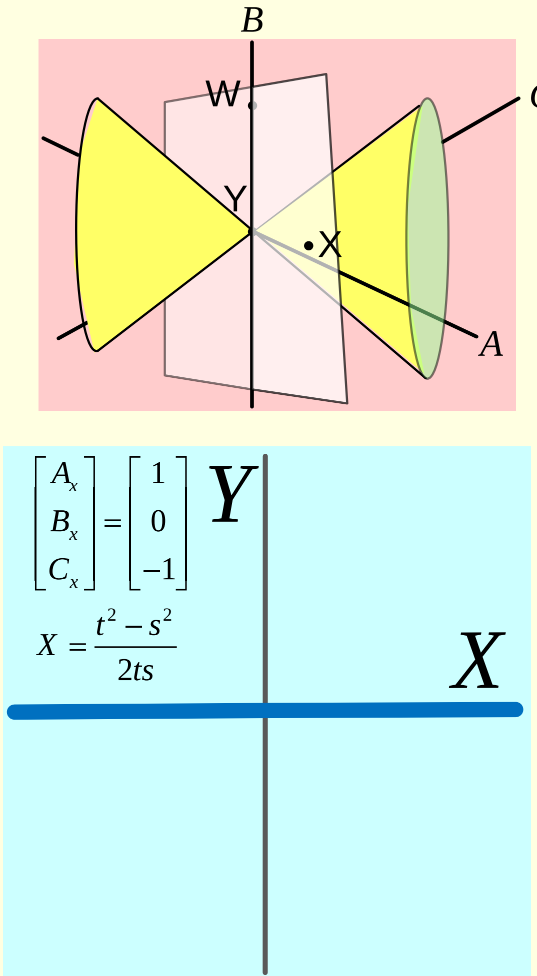
$$\begin{bmatrix} A_x \\ B_x \\ C_x \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$X = \frac{t^2 + s^2}{2ts}$$



$$\begin{bmatrix} A_x \\ B_x \\ C_x \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$X = \frac{t^2}{2ts}$$

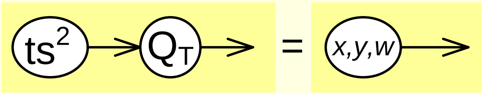


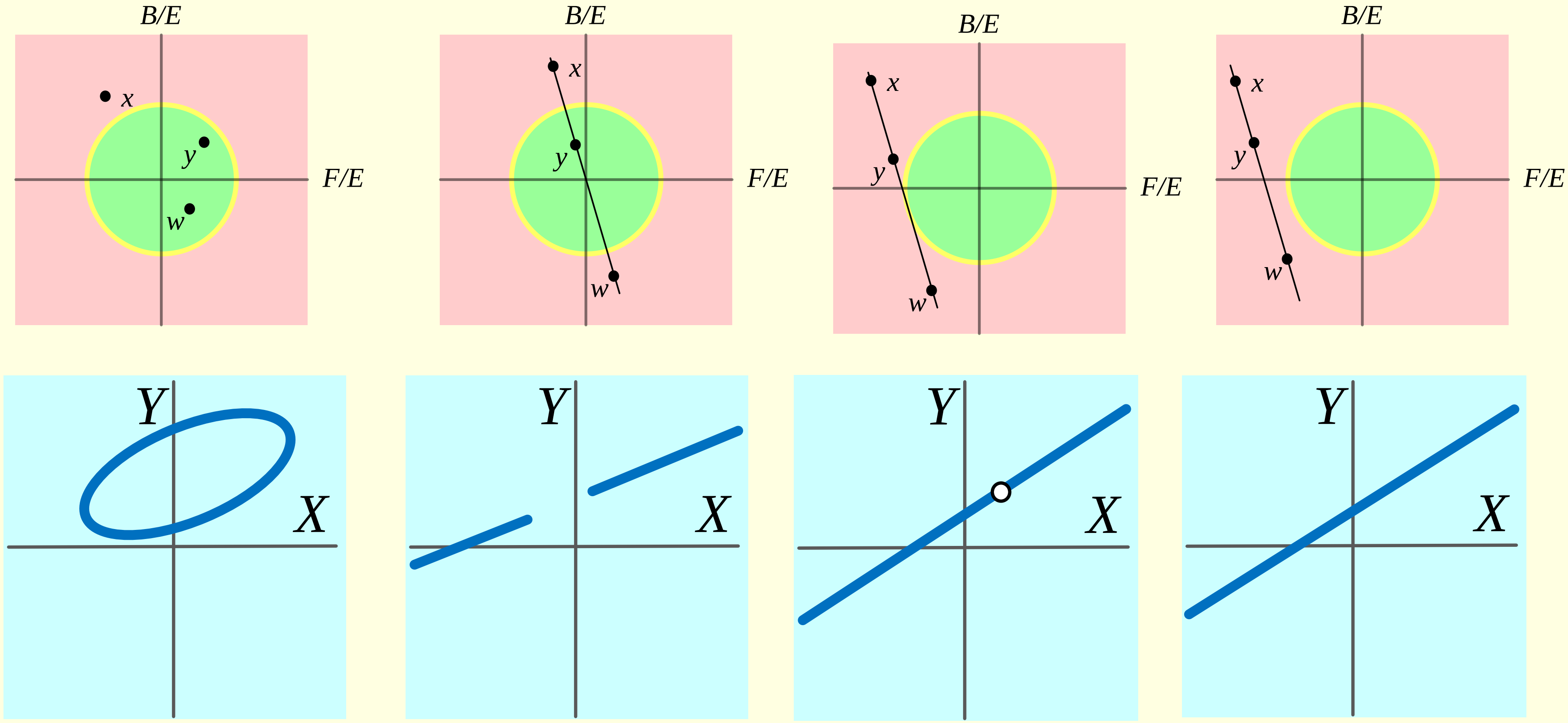
$$\begin{bmatrix} A_x \\ B_x \\ C_x \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$X = \frac{t^2 - s^2}{2ts}$$

Types of Quadratic

$$\begin{bmatrix} t^2 & 2ts & s^2 \end{bmatrix} \begin{bmatrix} A_x & A_y & A_w \\ B_x & B_y & B_w \\ C_x & C_y & C_w \end{bmatrix} =$$

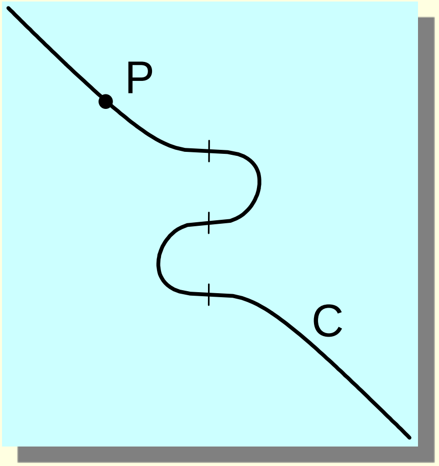


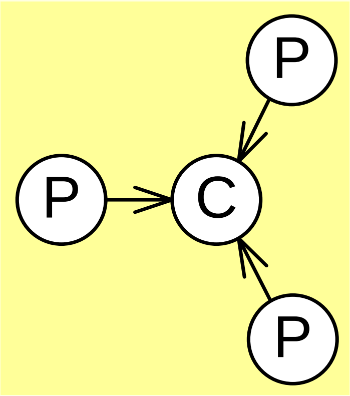


Cubic Curve in P2

$$Ax^3 + 3Bx^2y + 3Cxy^2 + Dy^3$$
$$+3Ex^2w + 6Fxyw + 3Gyw^2$$
$$+2Hxw^2 + 3Jyw^2$$
$$+Kw^3 = 0$$

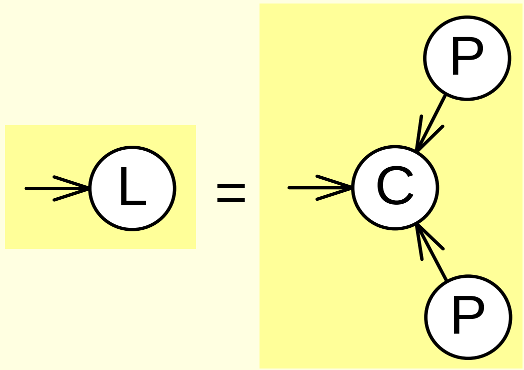
$$\left\{ \begin{bmatrix} x & y & w \end{bmatrix} \left[\begin{bmatrix} A & B & E \\ B & C & F \\ E & F & H \end{bmatrix} \begin{bmatrix} B & C & F \\ C & D & G \\ F & G & J \end{bmatrix} \begin{bmatrix} E & F & H \\ F & G & J \\ H & J & K \end{bmatrix} \right] \begin{bmatrix} x \\ y \\ w \end{bmatrix} \right\} \begin{bmatrix} x \\ y \\ w \end{bmatrix} = 0$$

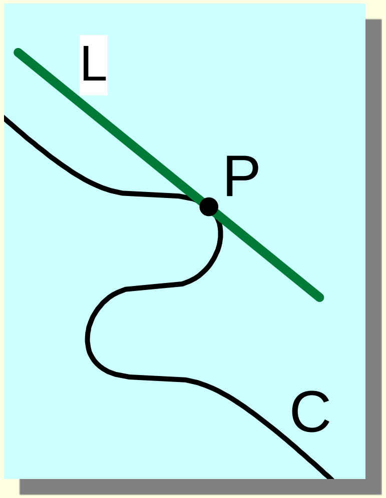




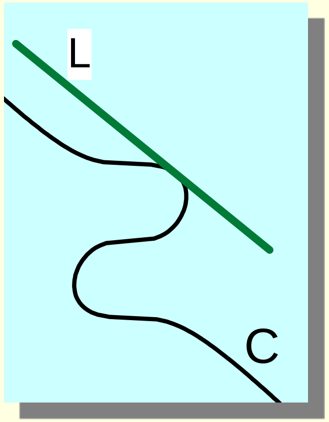
$$= 0$$

Cubic Curve Tangent at P

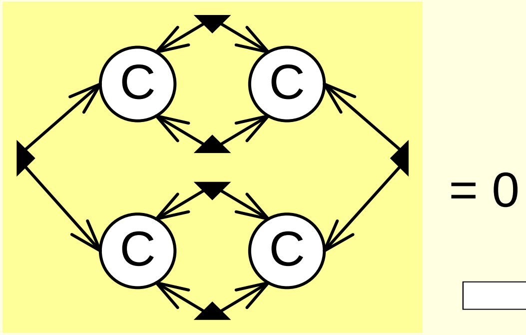


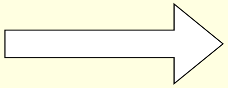


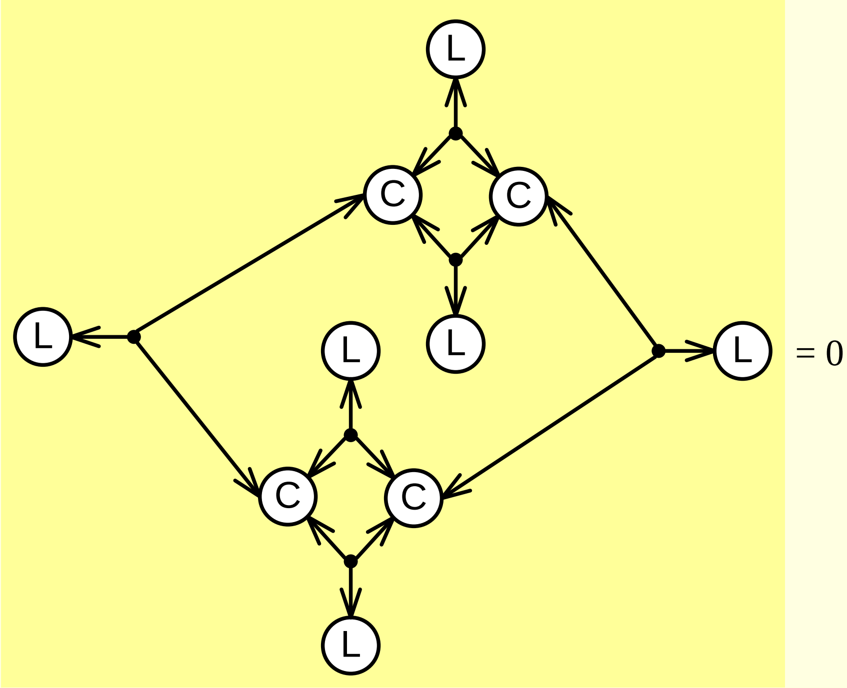
Is Line L Tangent to Cubic



Clebsch Translation Principle



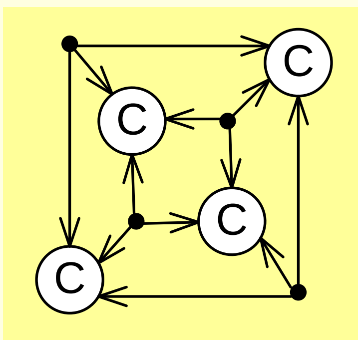




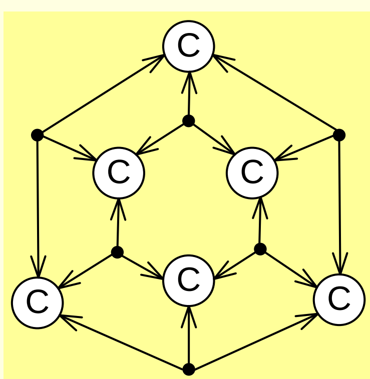
Wierstrass Standard Form

$$Y^2 = X^3 + cX + d$$

$$x^3 + cxw^2 + dw^3 - y^2w = 0$$



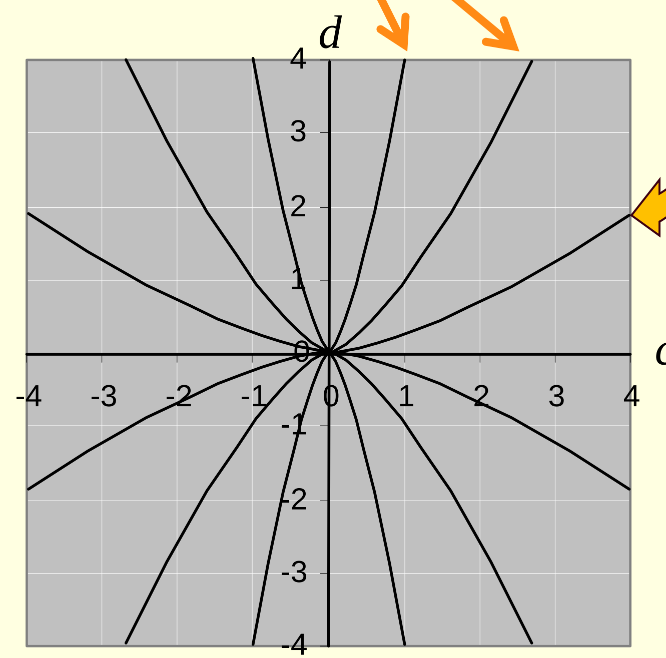
$$= I_4 = -\frac{8}{9}c$$



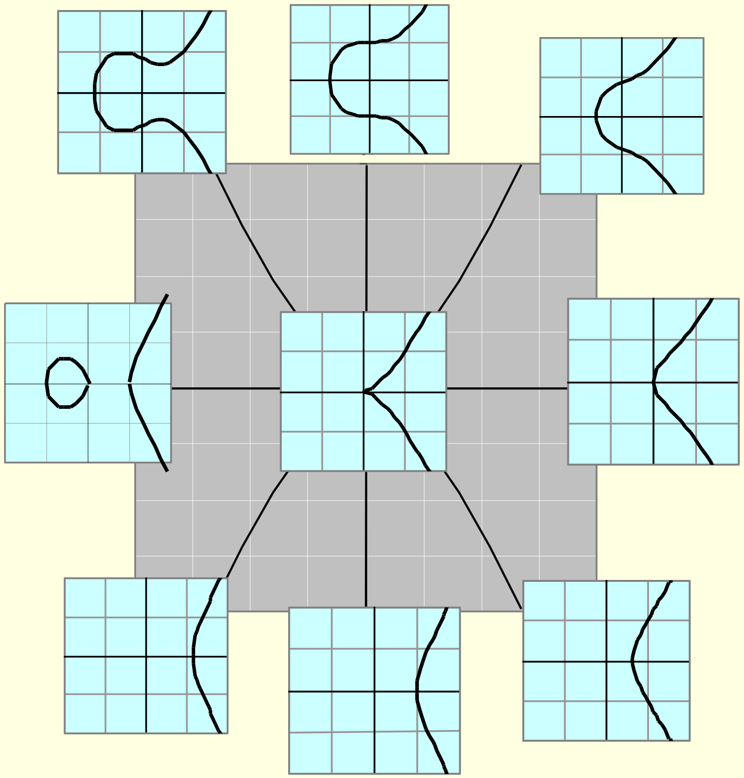
$$= I_6 = \frac{8}{9}d$$

c,d parameters along lines generate congruent curves

Different lines generate different curves

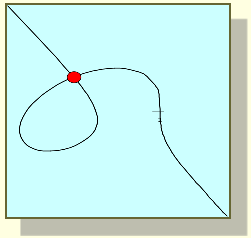


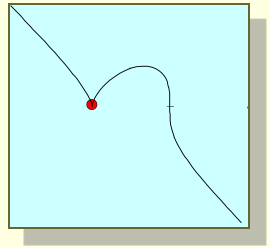
Cubic Continuum

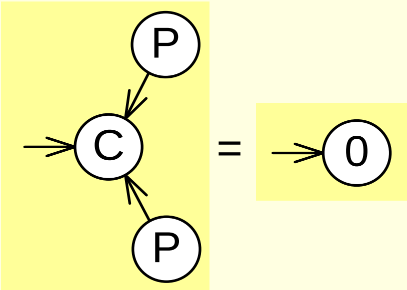


$$Y = \pm\sqrt{X^3 + cX + d}$$

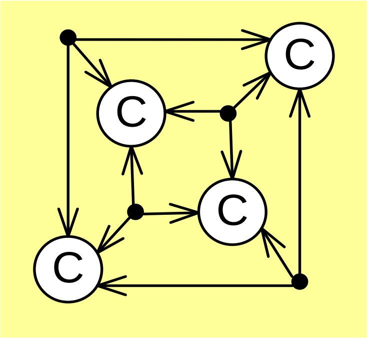
Nodes on a Cubic



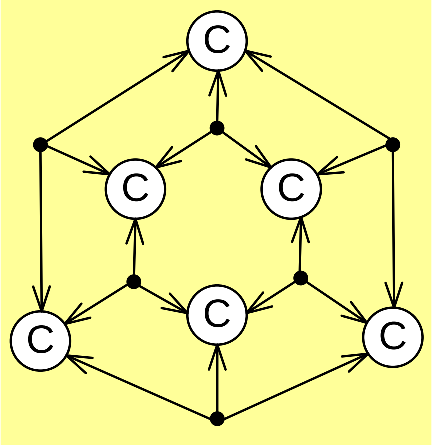




$I_4 =$



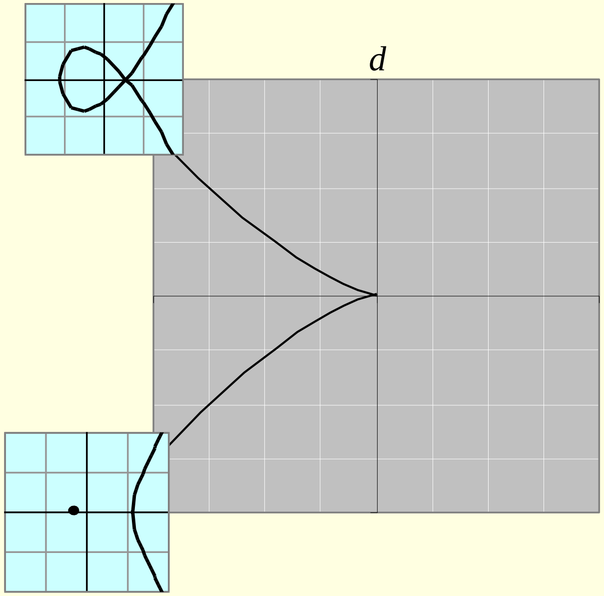
$I_6 =$



$$\Delta = I_4^{\ 3} - 6 I_6^{\ 2}$$

Loop and Serpentine

$$Y^2 = X^3 - 3X + 2 = (X-1)^2(X+2)$$

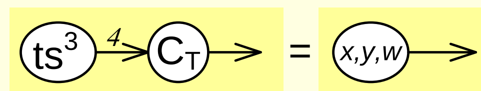


$$\Delta = I_4{}^3 - 6I_6{}^2$$
$$= -\left(\frac{8}{3}\right)^3 \left(\left(\frac{c}{3}\right)^3 + \left(\frac{d}{2}\right)^2\right)$$

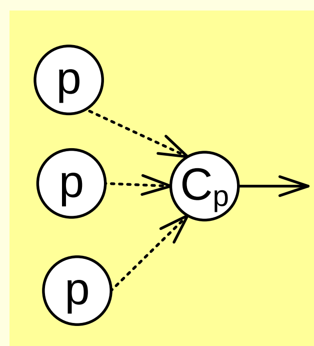
$$Y^2 = X^3 - 3X - 2 = (X+1)^2(X-2)$$

Parametric Cubic Curve

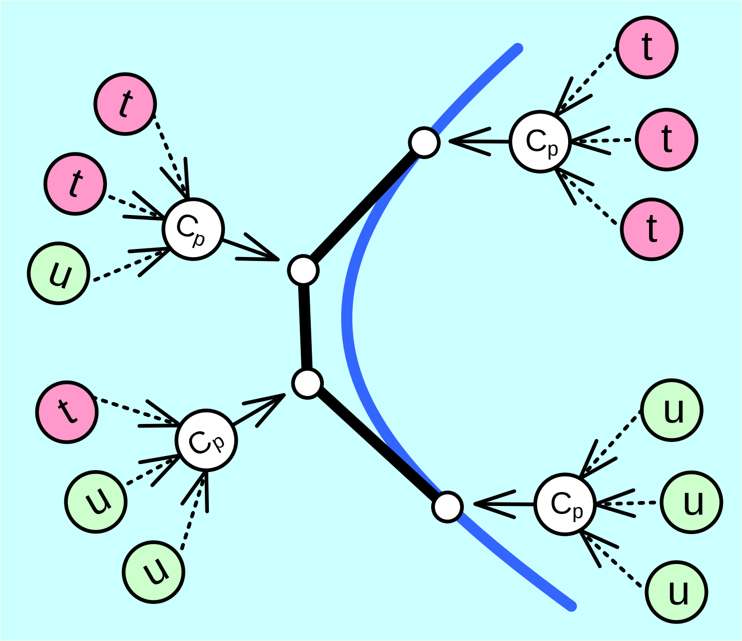
$$\begin{bmatrix} t^3 & 3t^2s & 3ts^2 & s^3 \end{bmatrix} \begin{bmatrix} A_x & A_y & A_w \\ B_x & B_y & B_w \\ C_x & C_y & C_w \\ D_x & D_y & D_w \end{bmatrix} = \begin{bmatrix} x & y & w \end{bmatrix}$$



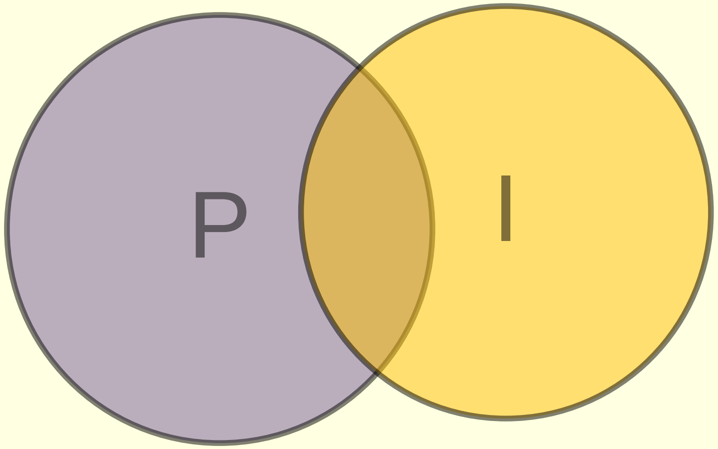
$$\begin{bmatrix} t & s \end{bmatrix} \begin{bmatrix} \begin{bmatrix} \begin{bmatrix} A_x & B_x \\ B_x & C_x \end{bmatrix} & \begin{bmatrix} B_x & C_x \\ C_x & D_x \end{bmatrix} \end{bmatrix} & \begin{bmatrix} \begin{bmatrix} A_y & B_y \\ B_y & C_y \end{bmatrix} & \begin{bmatrix} B_y & C_y \\ C_y & D_y \end{bmatrix} \end{bmatrix} & \begin{bmatrix} \begin{bmatrix} A_w & B_w \\ B_w & C_w \end{bmatrix} & \begin{bmatrix} B_w & C_w \\ C_w & D_w \end{bmatrix} \end{bmatrix} \end{bmatrix} \begin{bmatrix} t \\ s \end{bmatrix} \begin{bmatrix} t \\ s \end{bmatrix} = \begin{bmatrix} x & y & w \end{bmatrix}$$



Parametric Cubic Curve

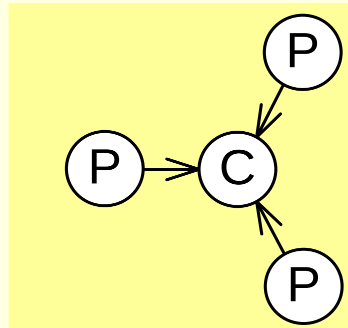


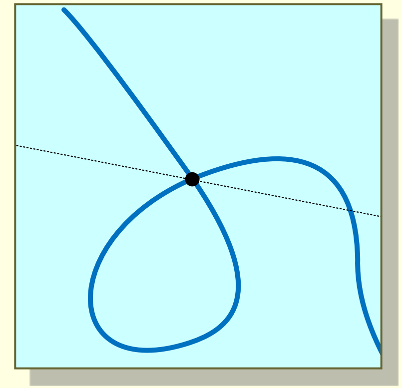
Parametric vs Implicit



Parametrizable Cubics

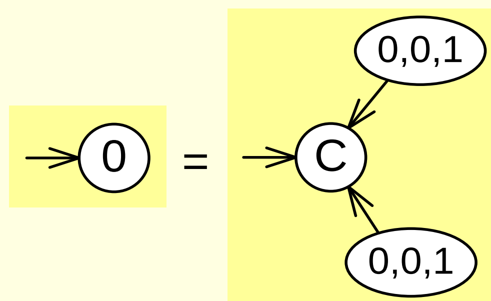
$$\begin{pmatrix} Ax^3 + 3Bx^2y + 3Cxy^2 + Dy^3 \\ +3Ex^2w + 6Fxyw + 3Gy^2w \\ +2Hxw^2 + 3Jyw^2 \\ +Kw^3 \end{pmatrix} =$$





Node at origin

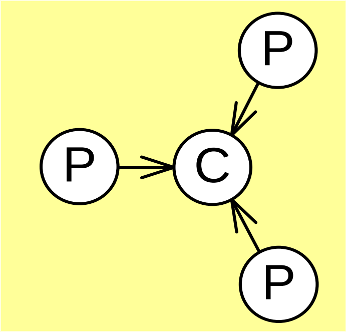
$H = J = K = 0$

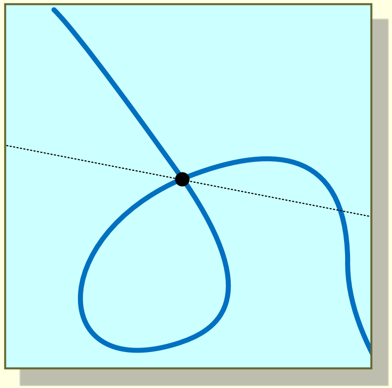


$$\frac{Ax^3 + 3Bx^2y + 3Cxy^2 + Dy^3}{-3\left(Ex^2 + 2Fxy + Gy^2\right)} = w$$

Parametrizable Cubics

$$\begin{pmatrix} Ax^3 + 3Bx^2y + 3Cxy^2 + Dy^3 \\ +3Ex^2w + 6Fxyw + 3Gy^2w \\ +3Hxw^2 + 3Jyw^2 \\ +Kw^3 \end{pmatrix} =$$





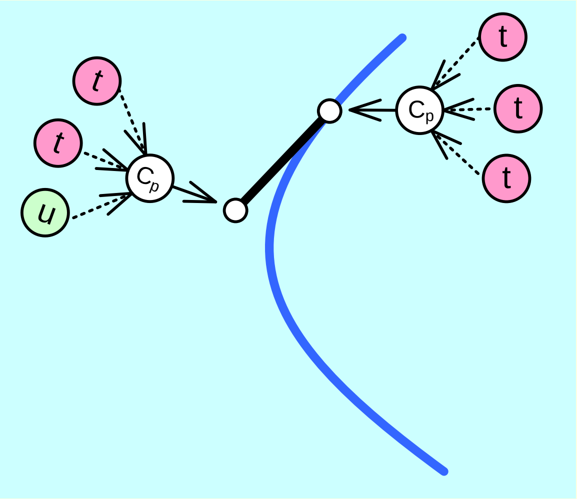
$$\begin{bmatrix} x & y & w \end{bmatrix} = \begin{bmatrix} x & y & \dfrac{Ax^3 + 3Bx^2y + 3Cxy^2 + Dy^3}{-3\left(Ex^2 + 2Fxy + Gy^2\right)} \end{bmatrix}$$

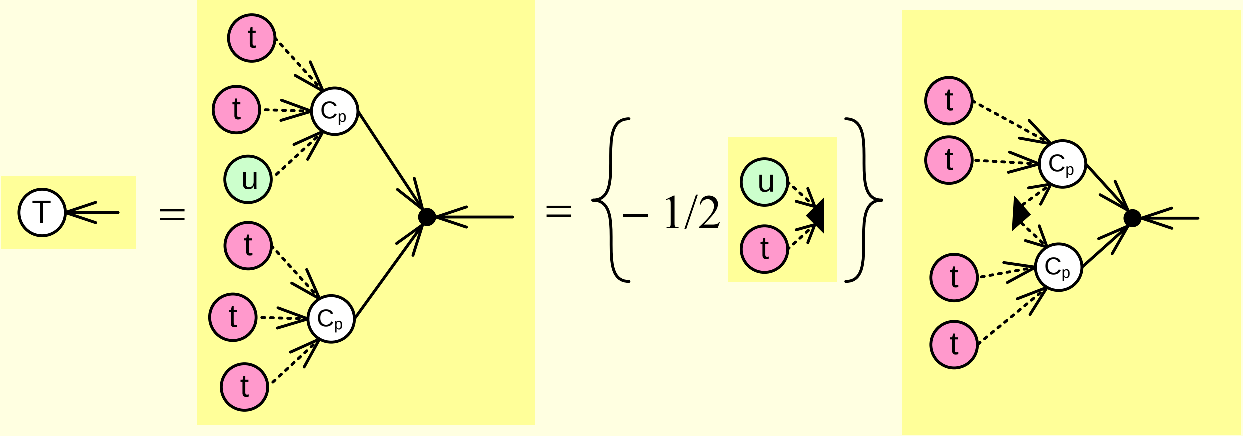
$$\begin{bmatrix} \tilde{x} & \tilde{y} & \tilde{w} \end{bmatrix} =$$

$$\begin{bmatrix} -3Ex^3 - 6Fx^2y - 3Gxy^2, & -3Ex^2y - 6Fxy^2 - 3Gy^3, & Ax^3 + 3Bx^2y + 3Cxy^2 + Dy^3 \end{bmatrix}$$

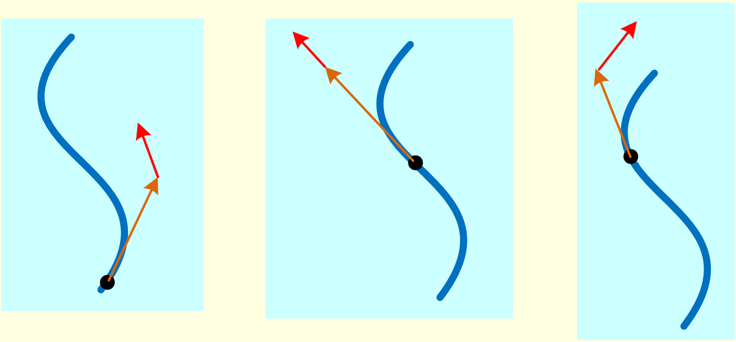
Parametrizable only if has Node

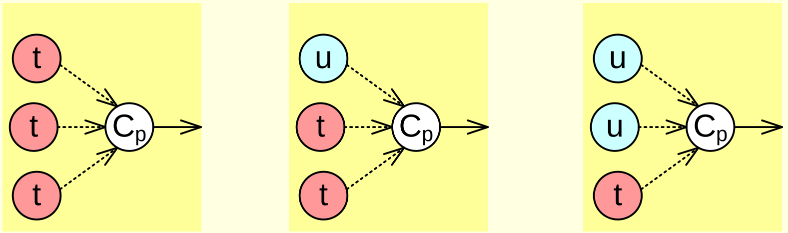
Tangent to Parametric Cubic Curve





Inflection point

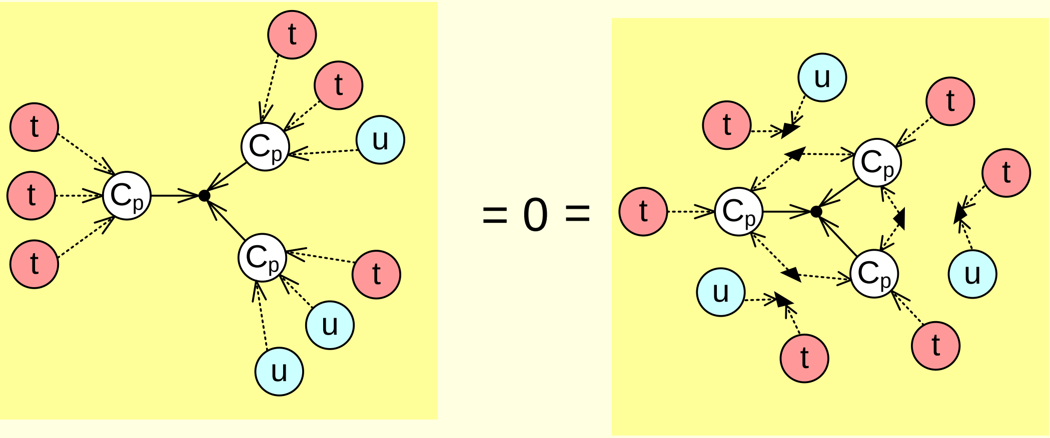


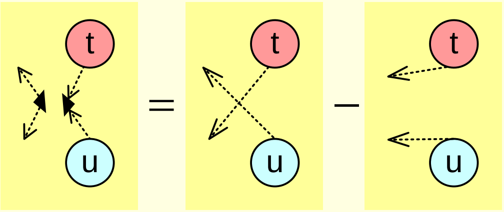


position

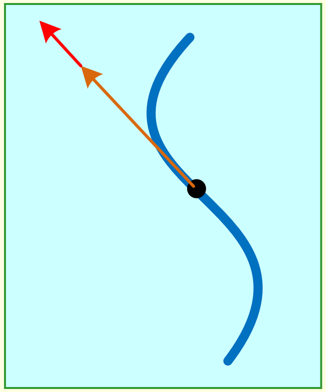
First derivative

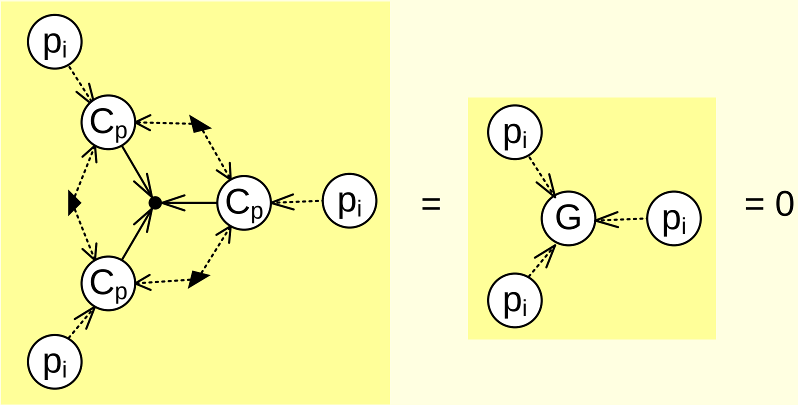
Second derivative

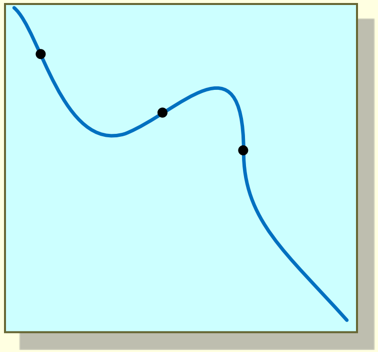




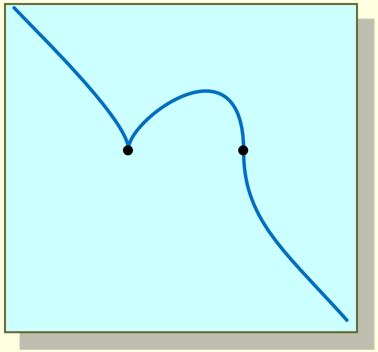
Inflection point



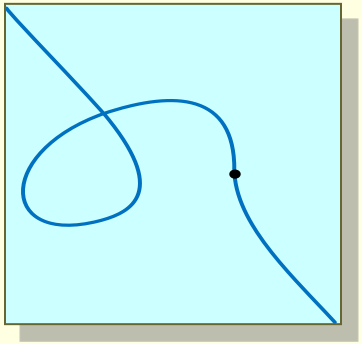




G is Type 111



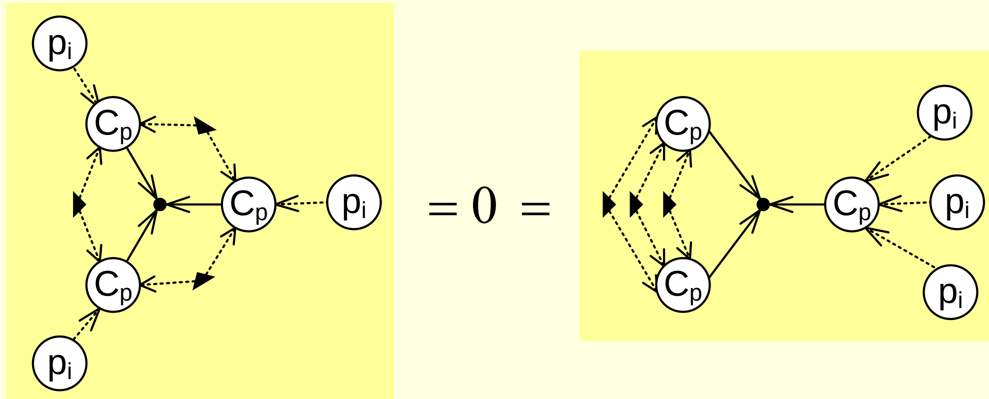
Type 12

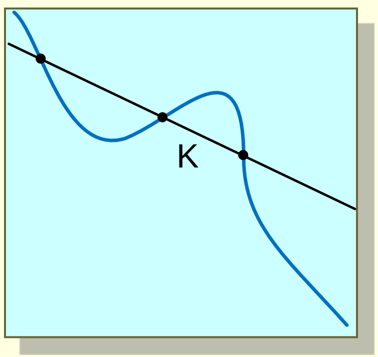


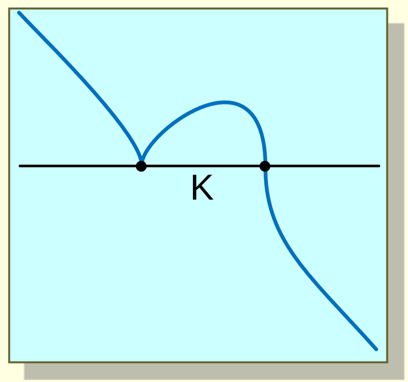
Type $1\frac{1}{1}$

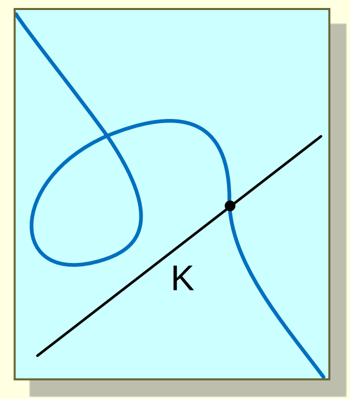
Type 3?

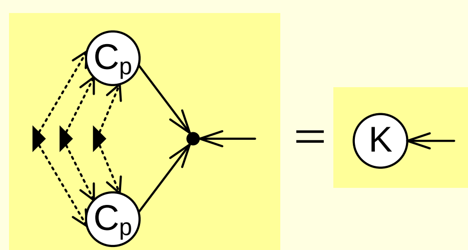
Inflection points are colinear



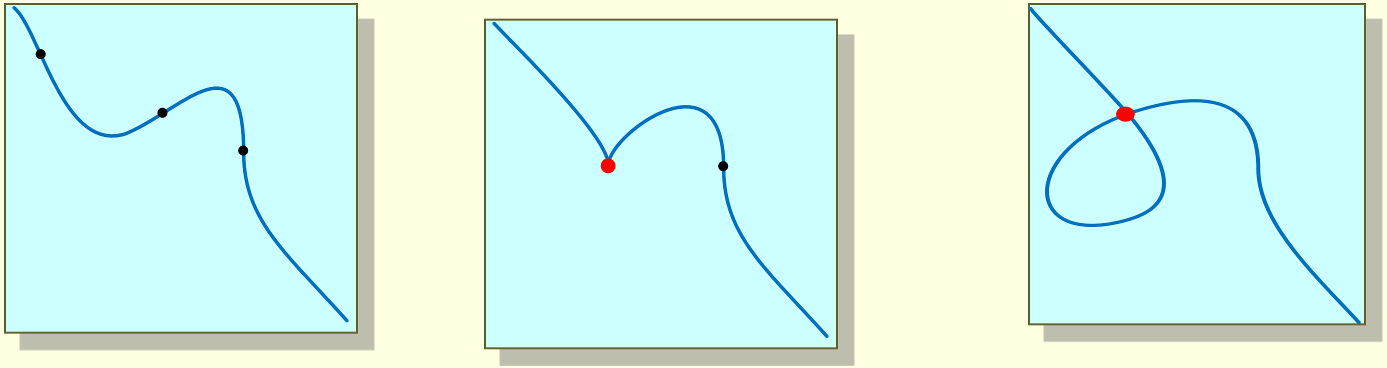


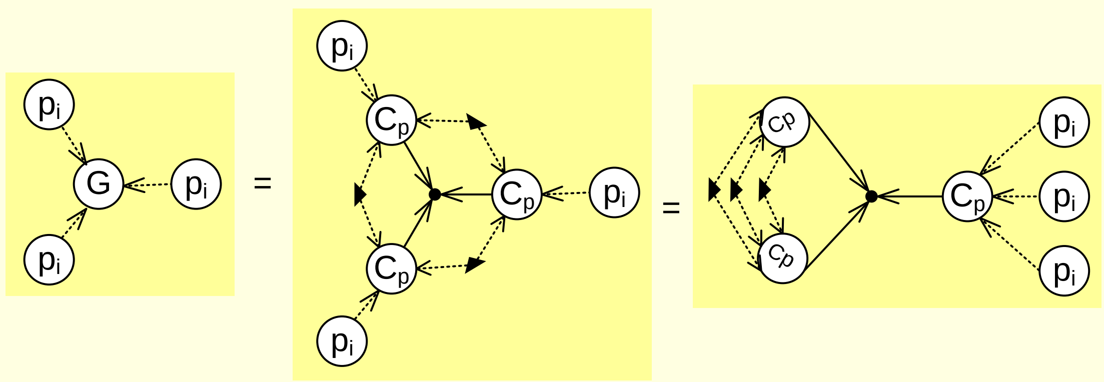




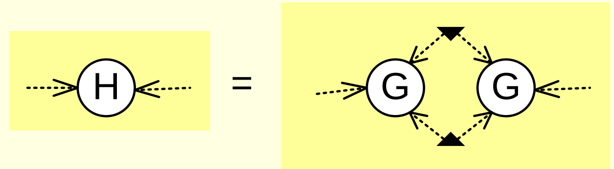


Double Points

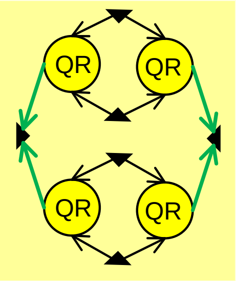
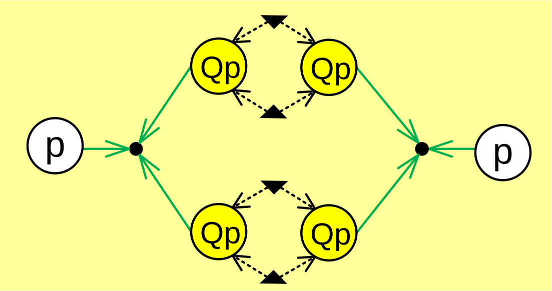
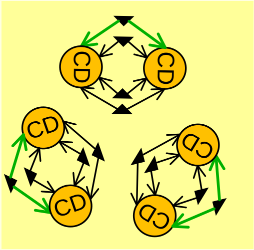
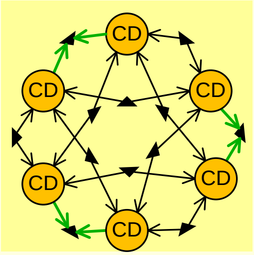
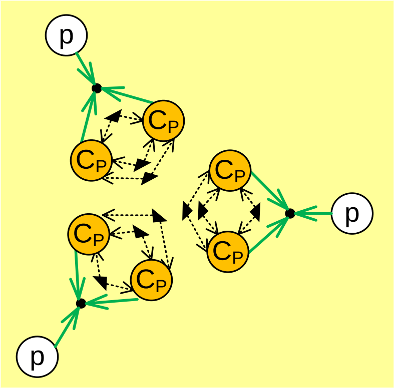
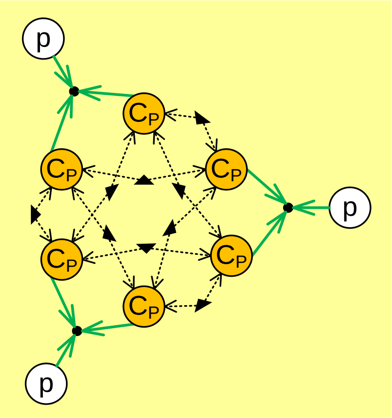




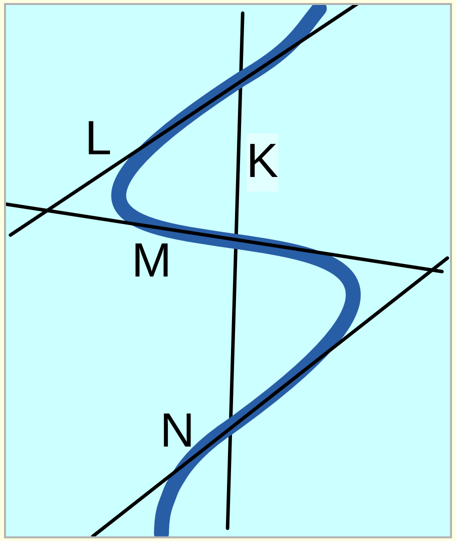
Parameters at double point are roots of Hessian(G)

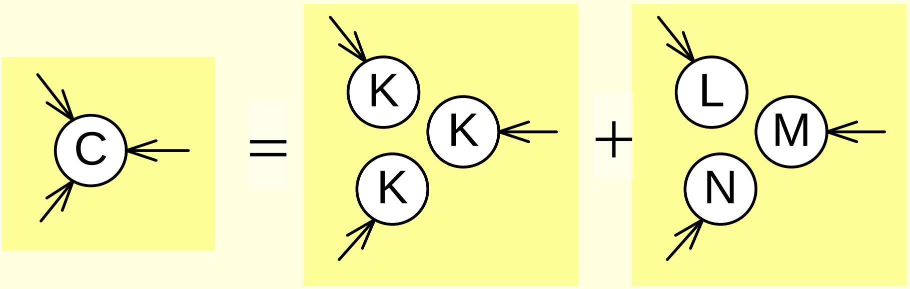


Implicitization of Parametric Quad,Cubic

	P^1 resultant	P^2 implicitization
Quadratic		
Cubic	 +36 	 +36 

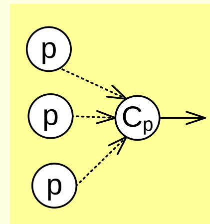
Analyzing Cubic Curves

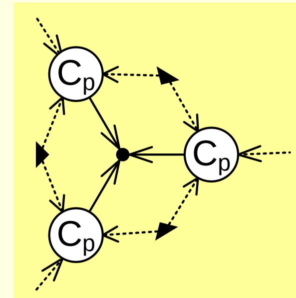




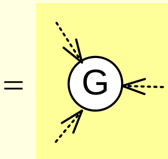
The Other Form of G

$$[t \quad s]\left[\begin{bmatrix}\begin{bmatrix}A_x & B_x\\ B_x & C_x\end{bmatrix} & \begin{bmatrix}B_x & C_x\\ C_x & D_x\end{bmatrix}\end{bmatrix} \quad \begin{bmatrix}\begin{bmatrix}A_y & B_y\\ B_y & C_y\end{bmatrix} & \begin{bmatrix}B_y & C_y\\ C_y & D_y\end{bmatrix}\end{bmatrix} \quad \begin{bmatrix}\begin{bmatrix}A_w & B_w\\ B_w & C_w\end{bmatrix} & \begin{bmatrix}B_w & C_w\\ C_w & D_w\end{bmatrix}\end{bmatrix}\right]\begin{bmatrix}t\\ s\end{bmatrix}\begin{bmatrix}t\\ s\end{bmatrix} =$$



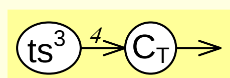


$=$



$$= \begin{bmatrix}\begin{bmatrix}A & B\\ B & C\end{bmatrix} & \begin{bmatrix}B & C\\ C & D\end{bmatrix}\end{bmatrix}$$

$$\begin{bmatrix}t^3 & 3t^2s & 3ts^2 & s^3\end{bmatrix}\begin{bmatrix}A_x & A_y & A_w\\ B_x & B_y & B_w\\ C_x & C_y & C_w\\ D_x & D_y & D_w\end{bmatrix} =$$



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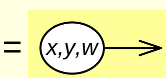

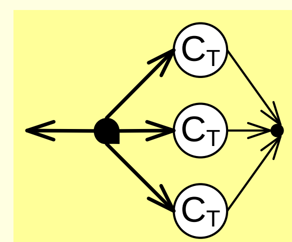

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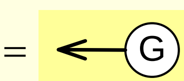

$$= \begin{bmatrix}A\\ B\\ C\\ D\end{bmatrix}$$

$$A = \det\begin{bmatrix}B_x & B_y & B_w\\ C_x & C_y & C_w\\ D_x & D_y & D_w\end{bmatrix},\ B = -\det\begin{bmatrix}A_x & A_y & A_w\\ C_x & C_y & C_w\\ D_x & D_y & D_w\end{bmatrix},\ C = \det\begin{bmatrix}A_x & A_y & A_w\\ B_x & B_y & B_w\\ D_x & D_y & D_w\end{bmatrix},\ D = -\det\begin{bmatrix}A_x & A_y & A_w\\ B_x & B_y & B_w\\ C_x & C_y & C_w\end{bmatrix}$$

Forms and Singularities of Curve

$$\begin{bmatrix} t^3 & 3t^2s & 3ts^2 & s^3 \end{bmatrix} \begin{bmatrix} A_x & A_y & A_w \\ B_x & B_y & B_w \\ C_x & C_y & C_w \\ D_x & D_y & D_w \end{bmatrix} = \begin{bmatrix} x & y & w \end{bmatrix}$$

Look at plane in coefficient space containing x,y,w cubics