CSE590B Lecture 3
More about P^1

Resultants, Division, Syzygies

and Transformations

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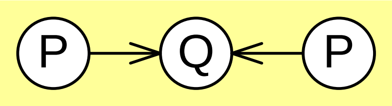
http://courses.cs.washington.edu/courses/cse590b/13au/

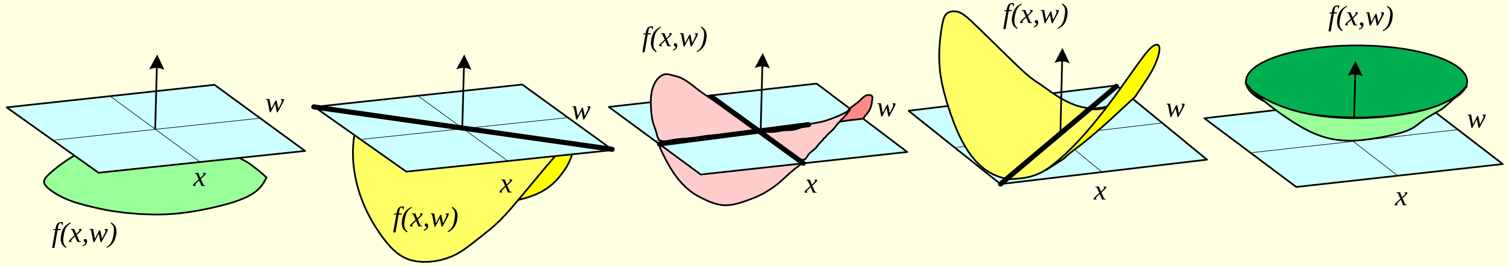
Previously On

CSE590b

Quadratic Polynomial

$$Q(x,w) = Ax^2 + 2Bxw + Cw^2 =$$

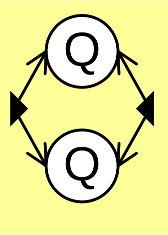


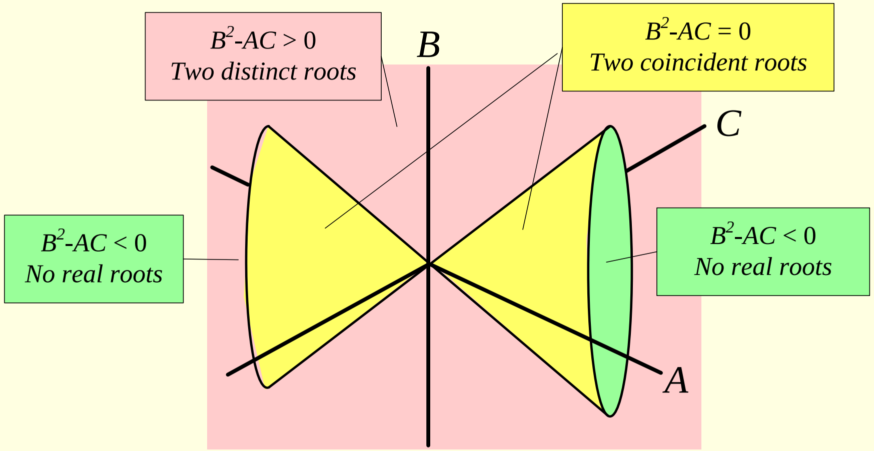


$\frac{1}{1}-$ 2- 11 2+ $\frac{1}{1}+$

$$\operatorname{discr}(Q) = B^2 - AC$$

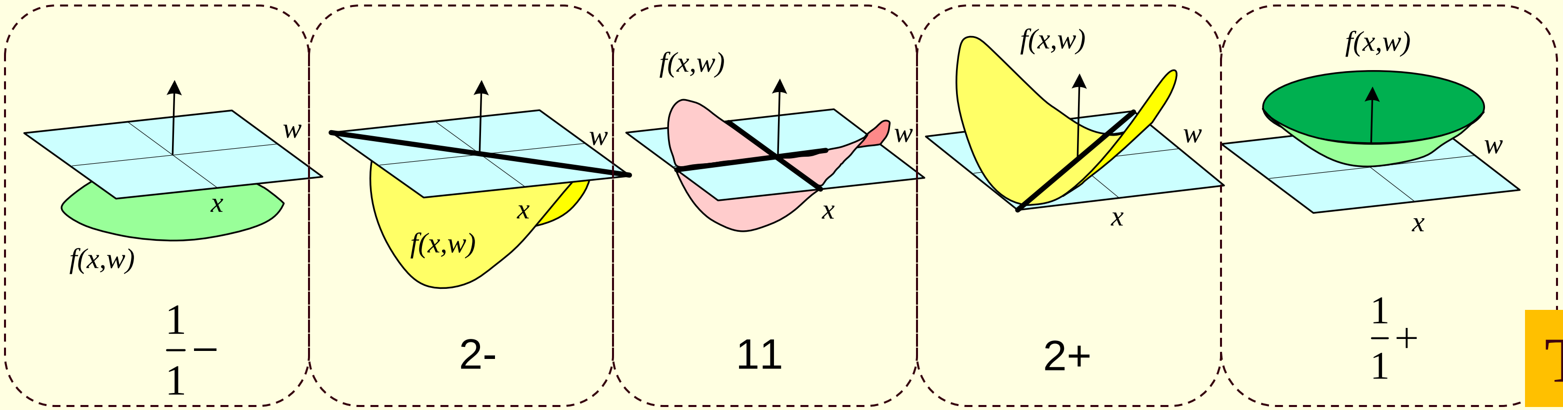
$$= -\det\begin{bmatrix} A & B \\ B & C \end{bmatrix} = 1/2$$





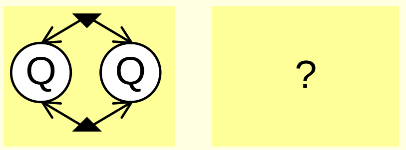
Topics

Equivalence Classes



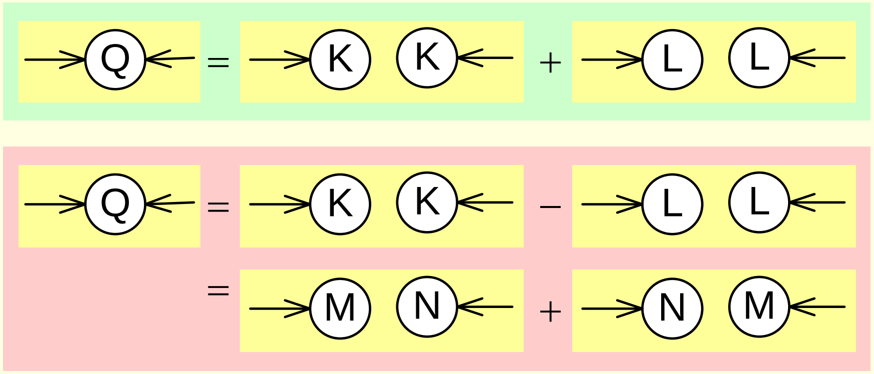
To test

Invariant Diagrams



Plug in

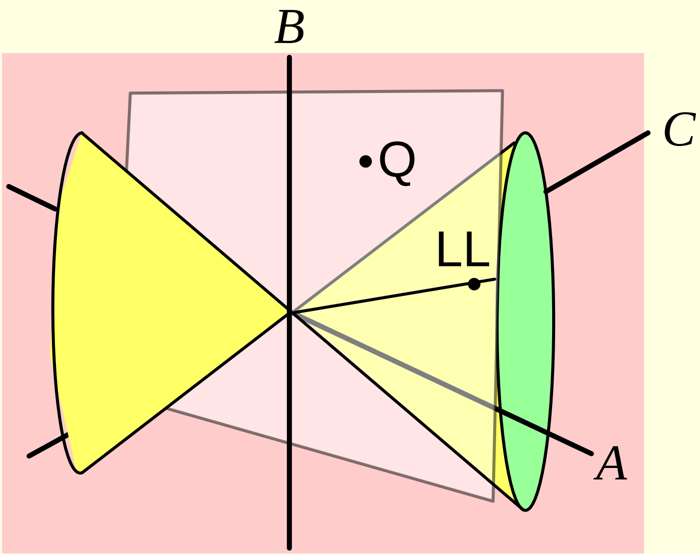
Internal Structure



And Now

Locus of Q's containing L as factor

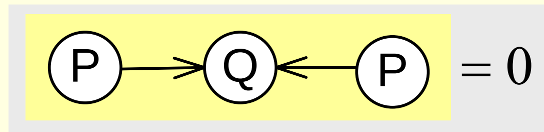
$$Q = KL$$
$$= (K_1 x + K_2 w)(L_1 x + L_2 w)$$
$$= K_1 L_1 x^2 + (K_1 L_2 + K_2 L_1) xw + K_2 L_2 w^2$$



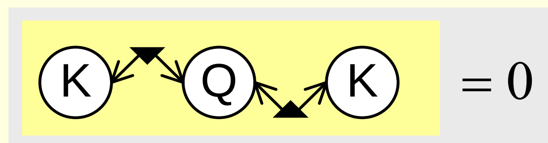
Plane tangent to cone along line LL

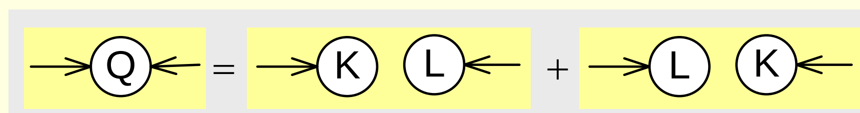
Root vs Factor

Root

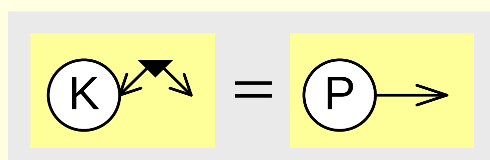


Factor

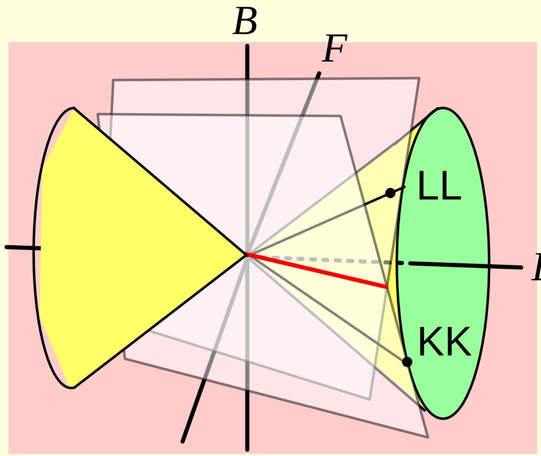


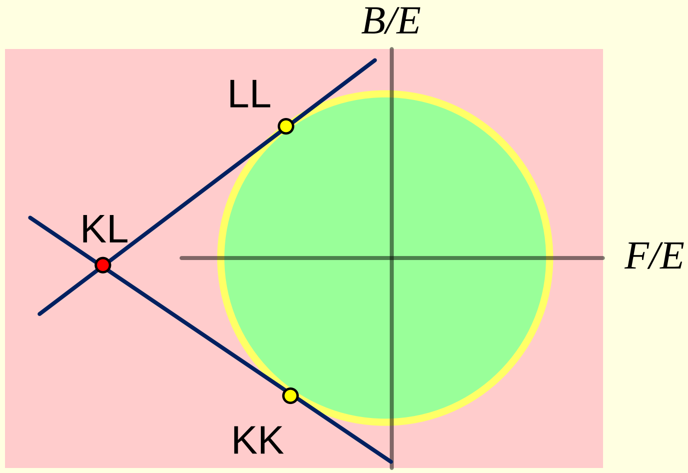


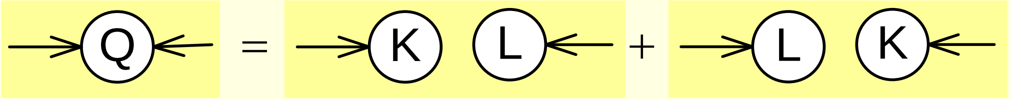
Relation between



Intersection of 2 loci



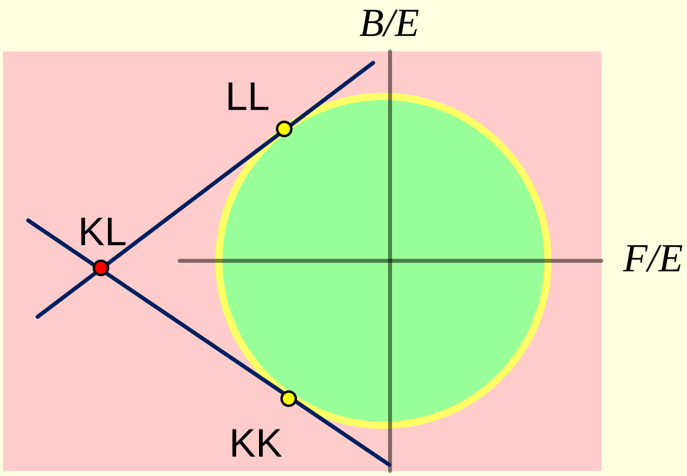


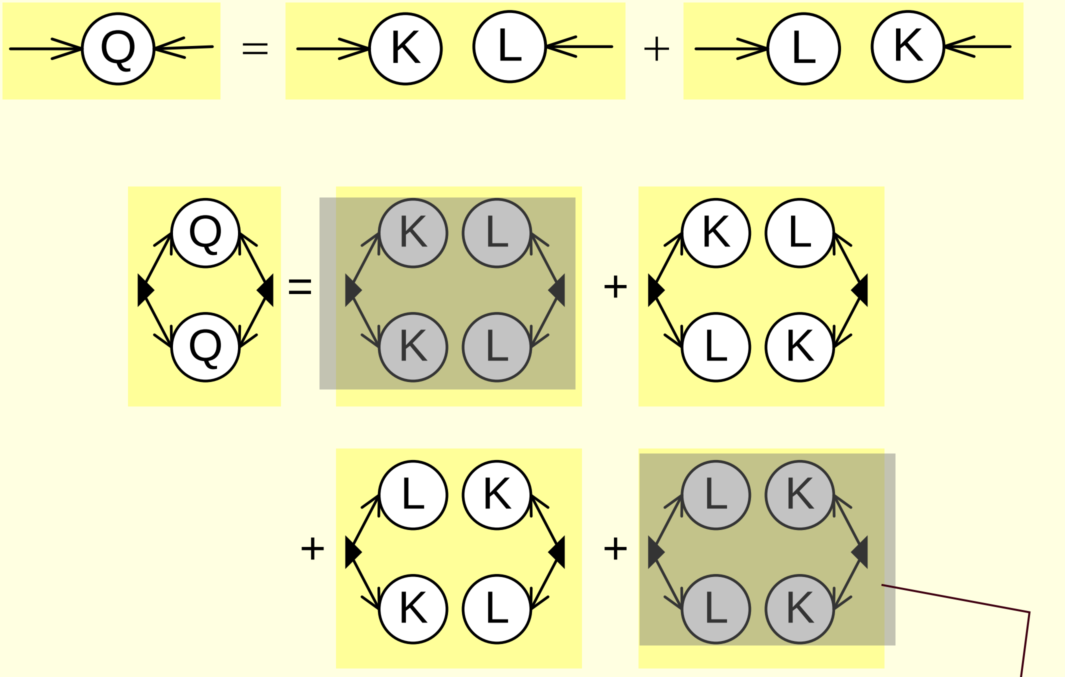


Given Q, find K,L

→Draw tangents from Q to cone

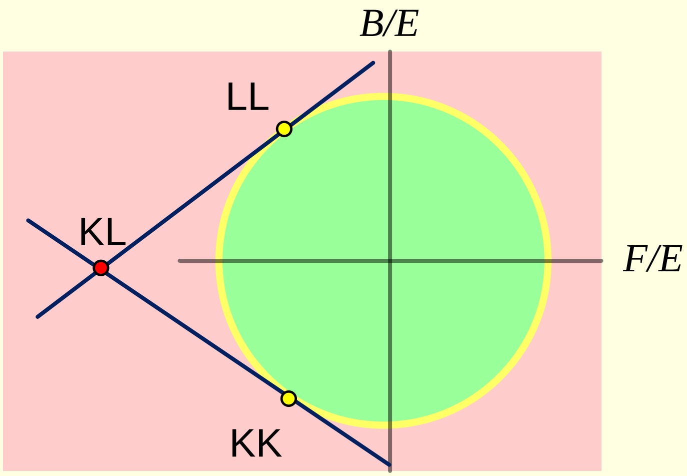
Another Invariant Test

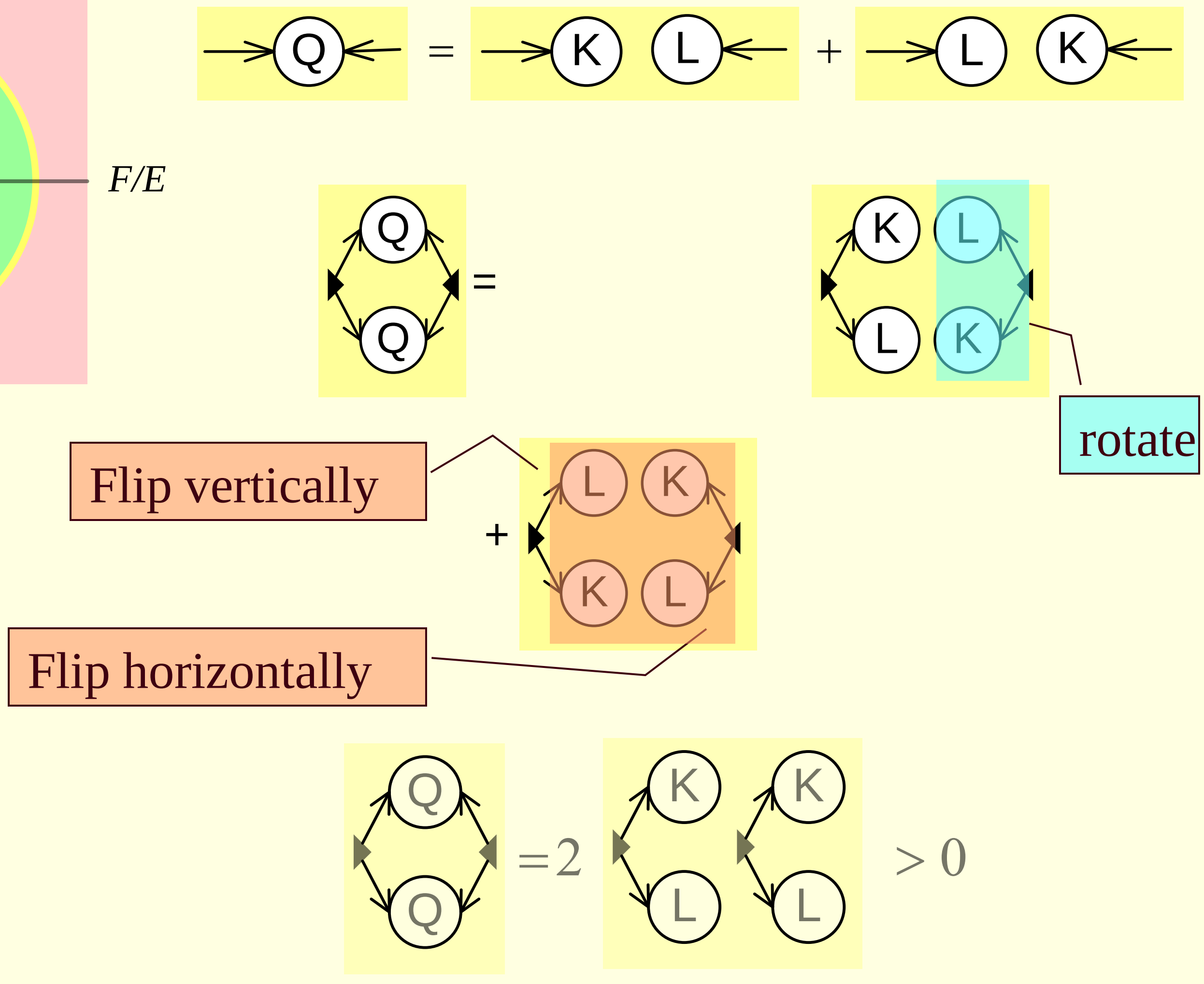




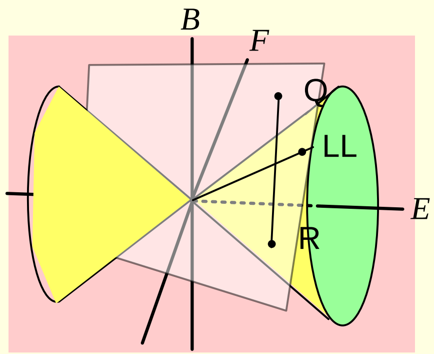
Grayed out indicates "identically zero"

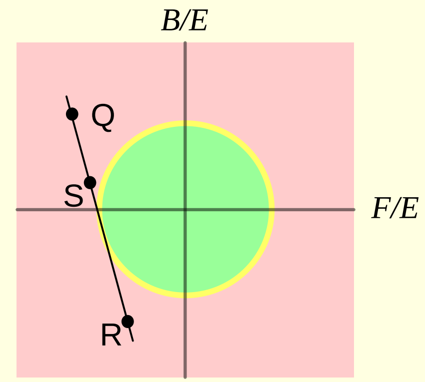
Another Invariant Test



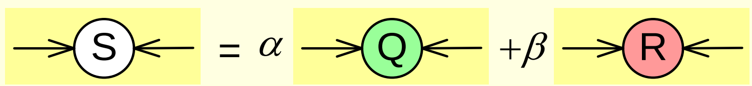


Do Two Quadratics share a root?



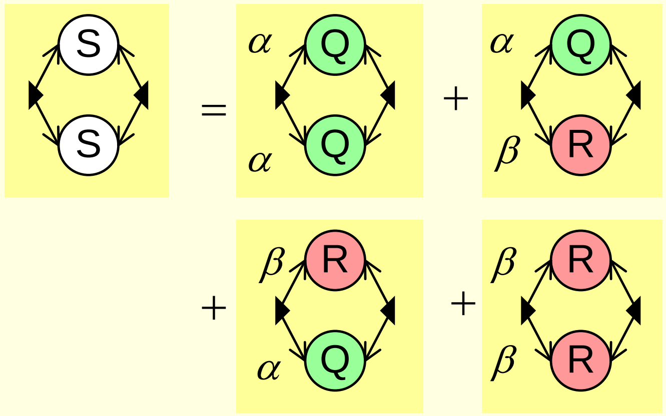


Is line from Q to R

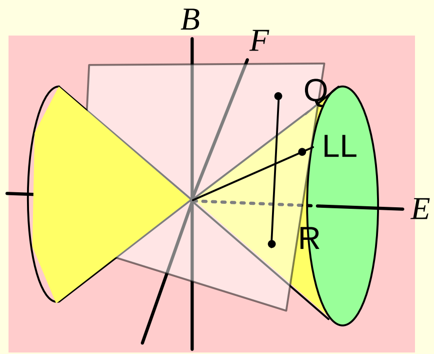


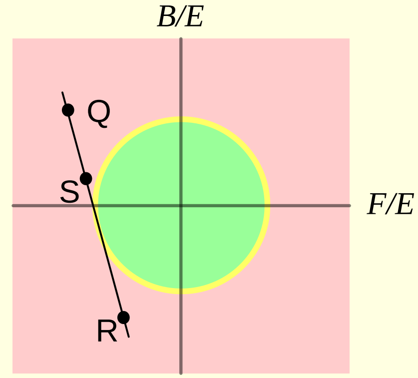
tangent to cone?

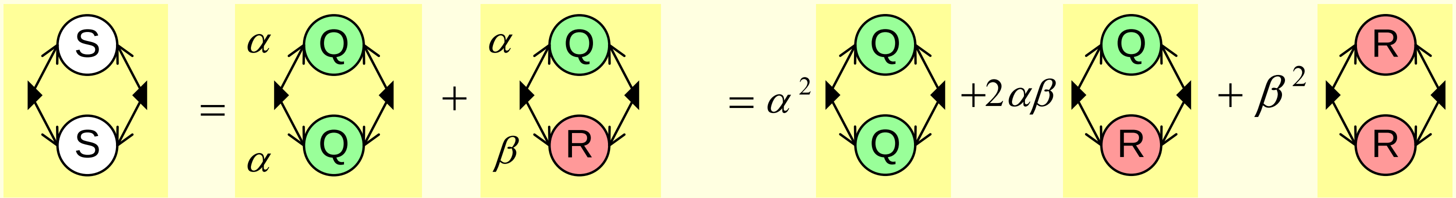
det (S(α,β)) has a double root

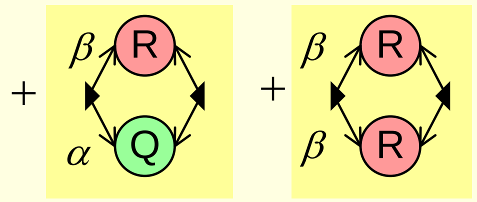


Do Two Quadratics share a root?

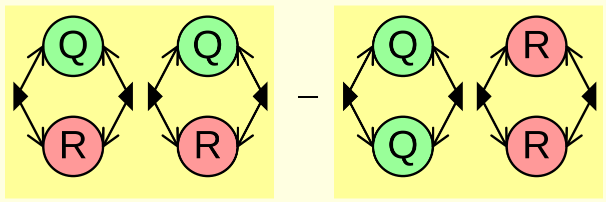




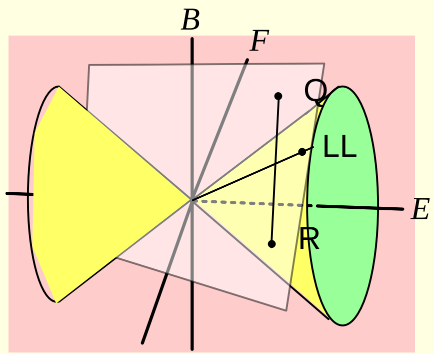


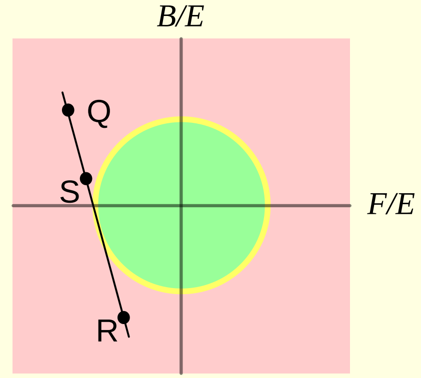


Discriminant:

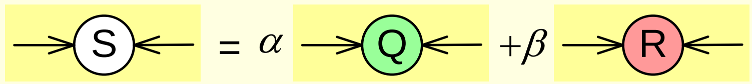


Do Two Quadratics share a root?





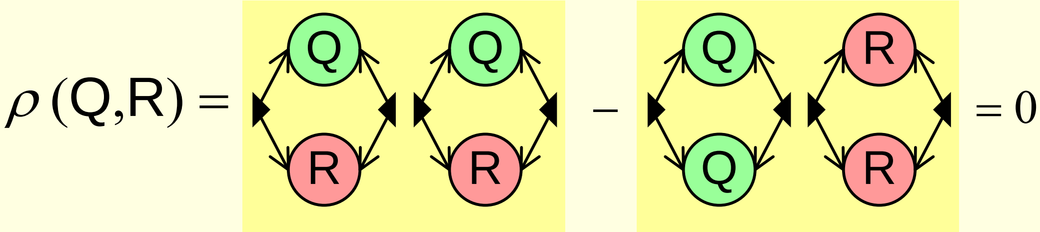
Is line from Q to R



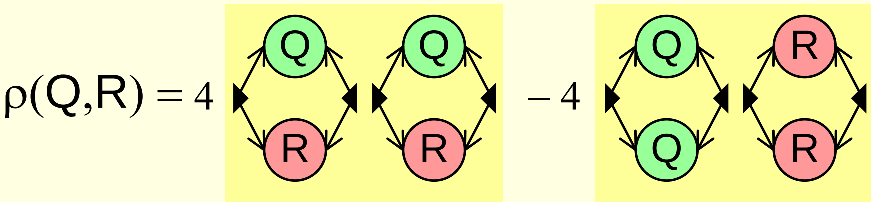
tangent to cone?

det S(α,β) has a double root

“Resultant”

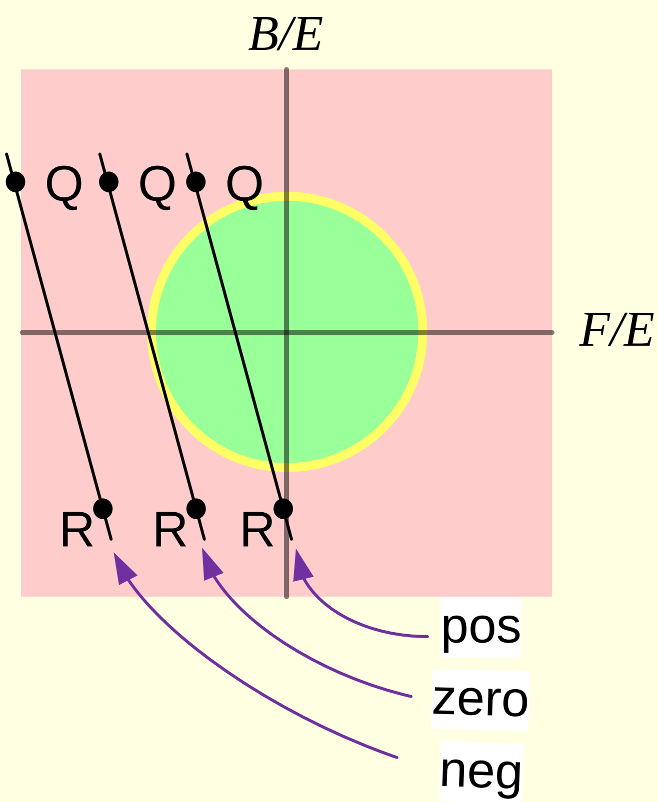


Resultant

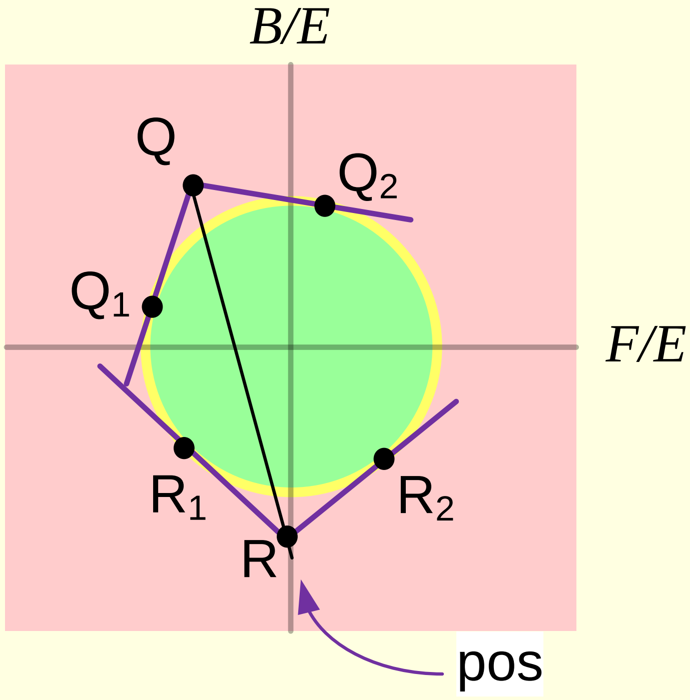


$$\mathbf{Q} = \begin{bmatrix} A_Q & B_Q \\ B_Q & C_Q \end{bmatrix}, \mathbf{R} = \begin{bmatrix} A_R & B_R \\ B_R & C_R \end{bmatrix}$$

$$\rho\left(\mathbf{Q},\mathbf{R}\right) = \det \underbrace{\begin{bmatrix} A_Q & \frac{1}{2}B_Q & C_Q & 0 \\ 0 & A_Q & \frac{1}{2}B_Q & C_Q \\ A_R & \frac{1}{2}B_R & C_R & 0 \\ 0 & A_R & \frac{1}{2}B_R & C_R \end{bmatrix}}_{\text{Sylvester Matrix}}$$

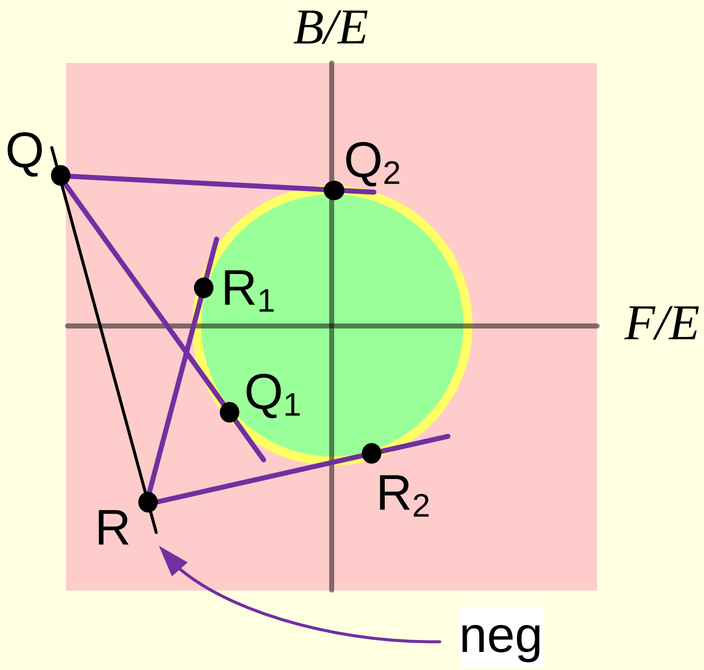


Resultant and root interleaving



$$\rho\left(Q,R\right)>0$$

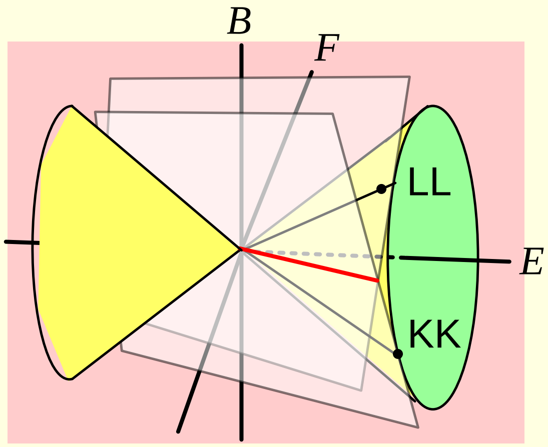
Roots disjoint

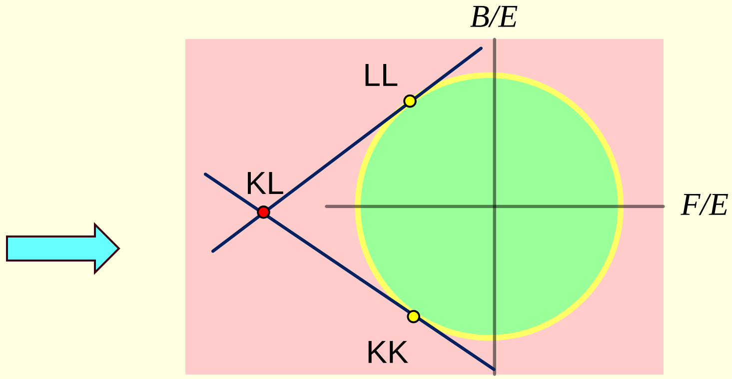


$$\rho\left(Q,R\right)<0$$

Roots interleaved

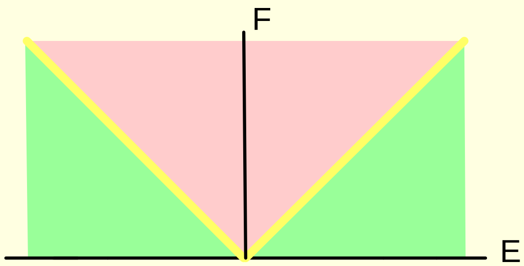
Two possible mappings 3D->2D





☺ Preserves lines

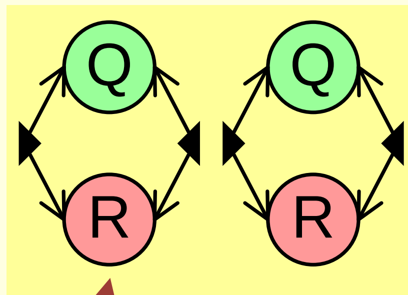
☹ Maps + and – cones together

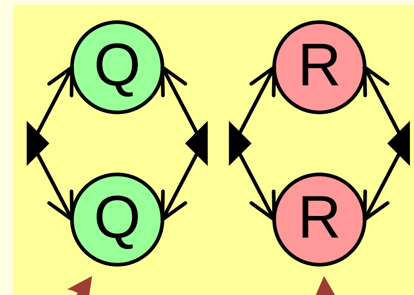


☺ + and – cones distinct

☹ Lines not preserved

Resultant

$\rho(Q,R) = 4$ 

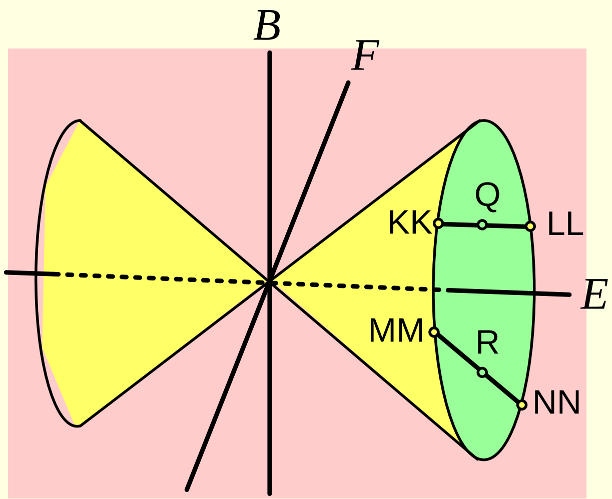
$-\,4$ 

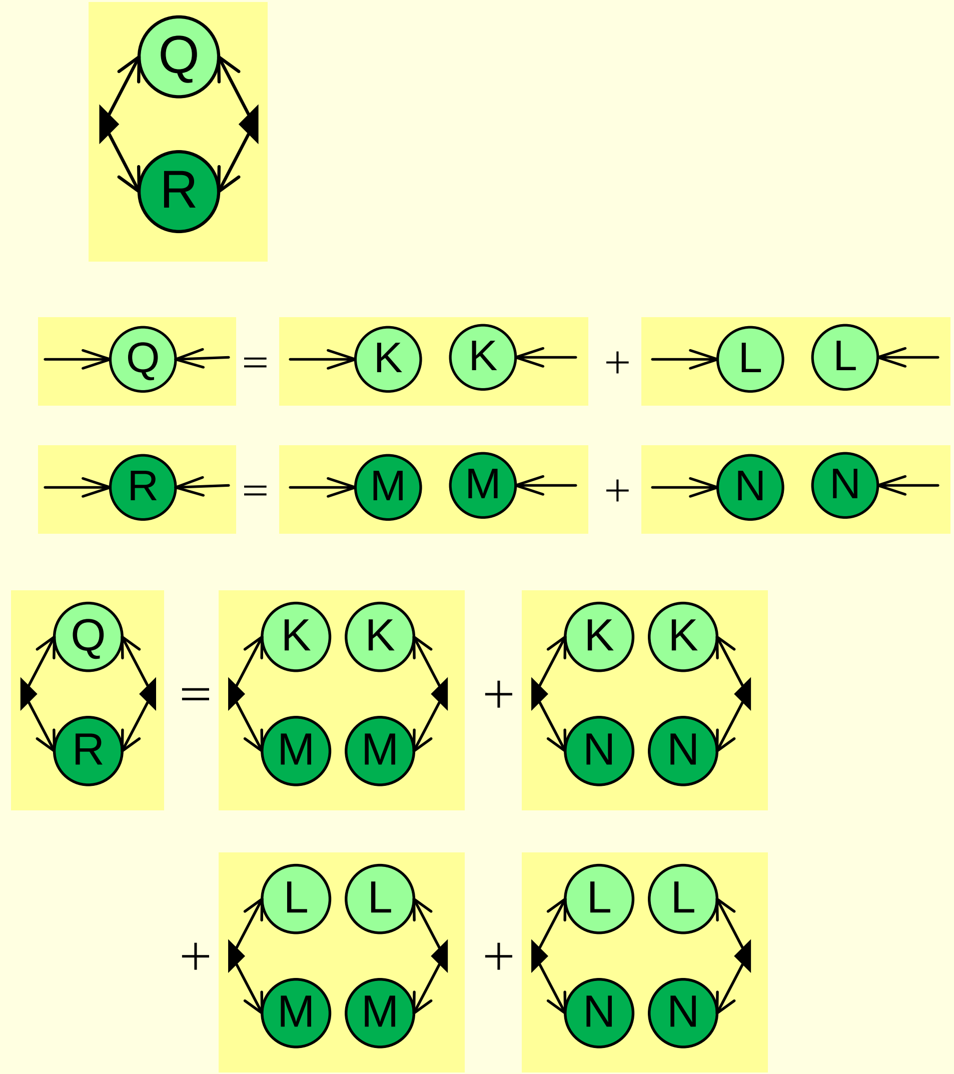
functional determinant

$-2\ \det Q$

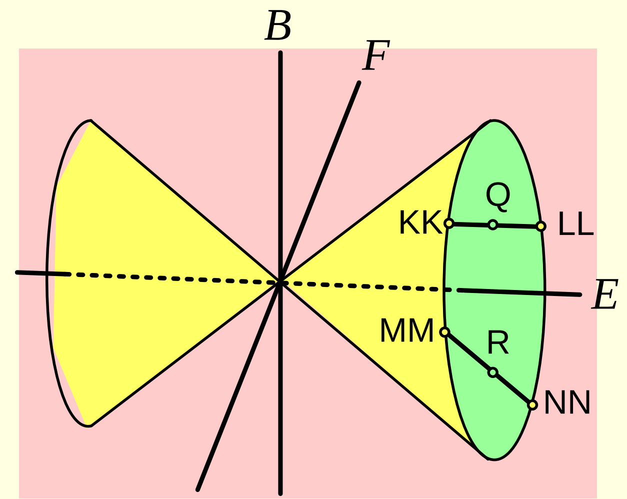
$-2\ \det R$

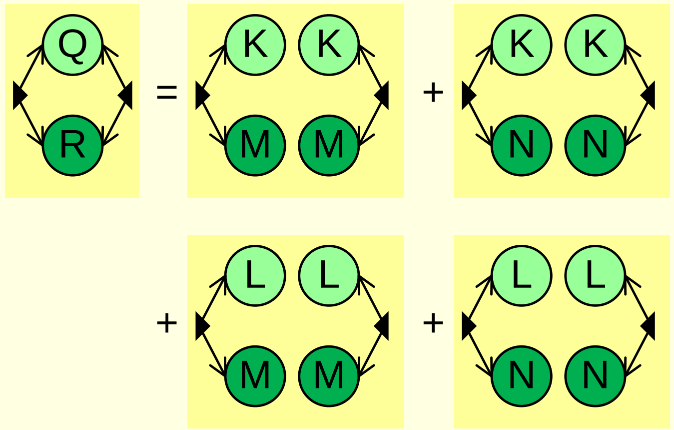
Functional Determinant

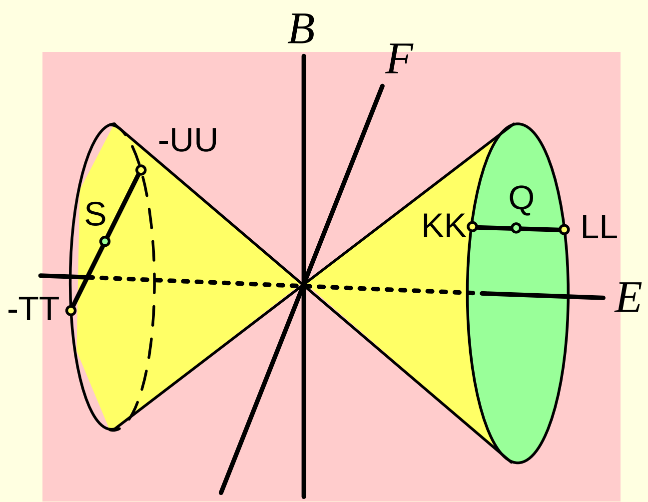


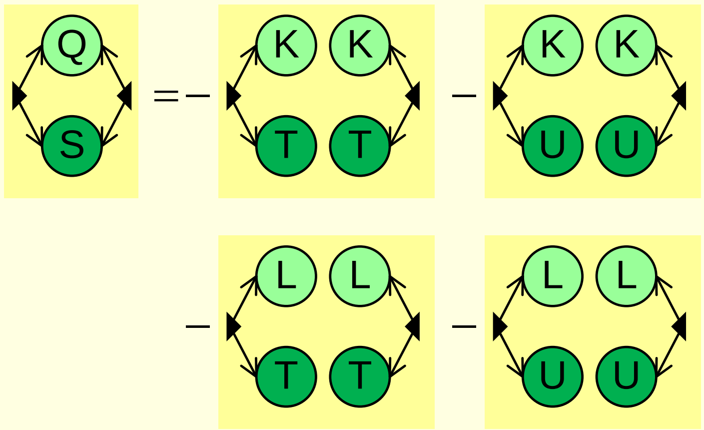


Functional Determinant

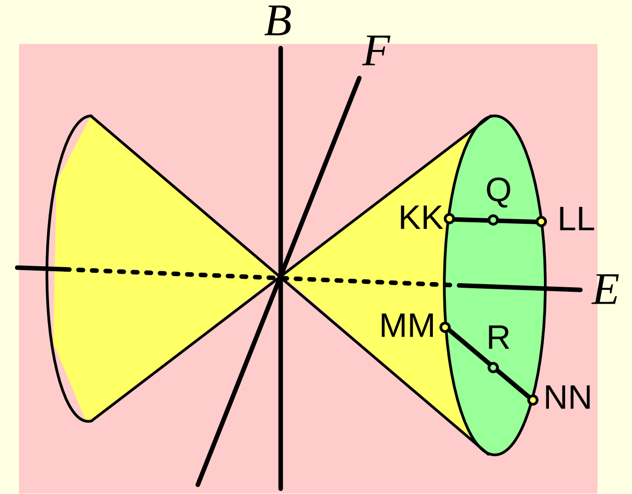


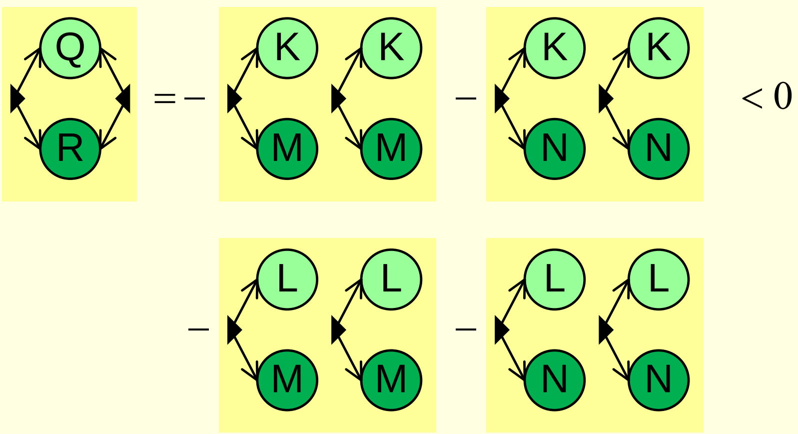


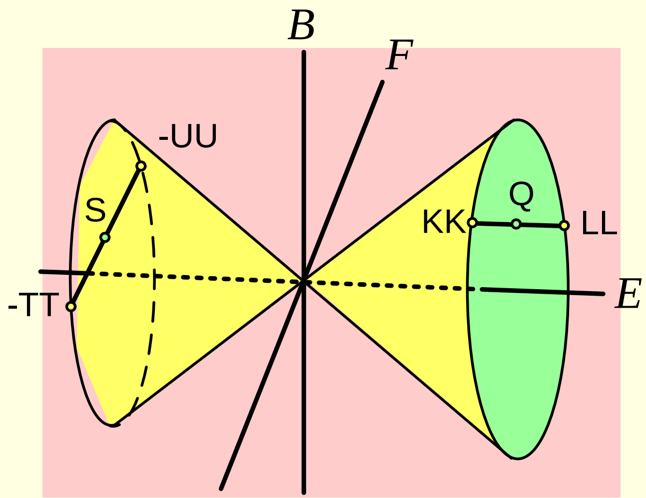


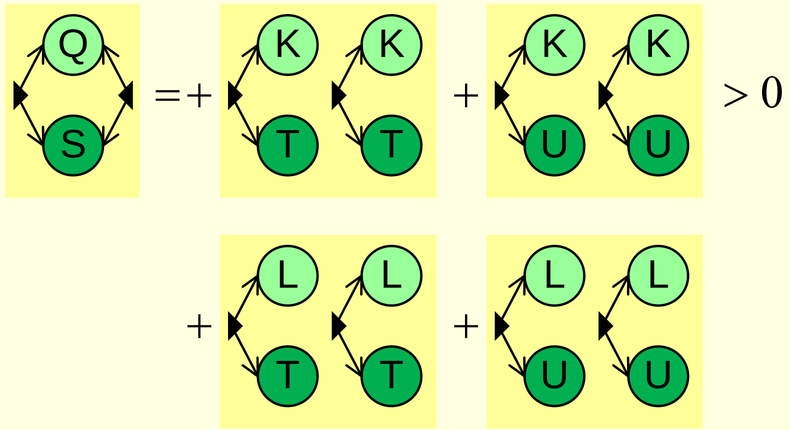


Functional Determinant

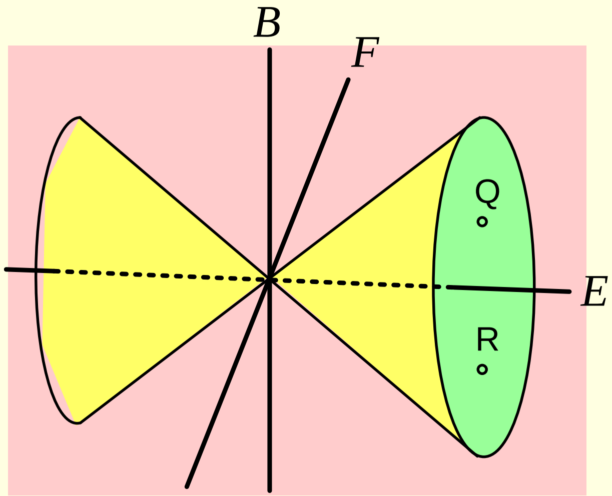




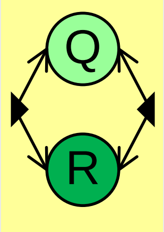




Functional Determinant

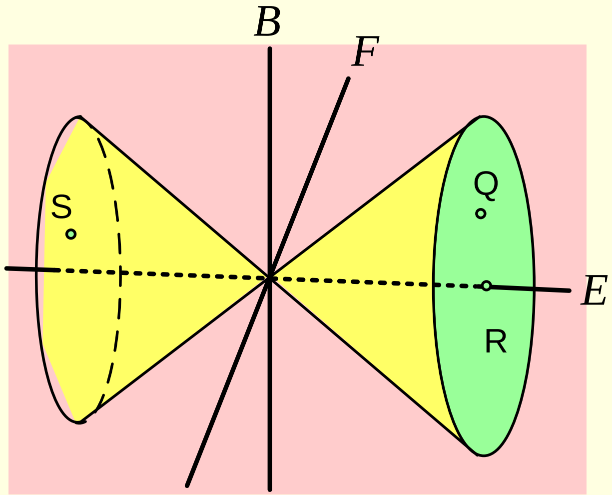


$$\mathbf{Q} = \begin{bmatrix} A_Q & B_Q \\ B_Q & C_Q \end{bmatrix}, \mathbf{R} = \begin{bmatrix} A_R & B_R \\ B_R & C_R \end{bmatrix}$$

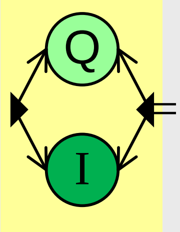


$$= -A_Q C_R + 2B_Q B_R - C_Q A_R$$

$$= \begin{bmatrix} A_Q & B_Q & C_Q \end{bmatrix} \begin{bmatrix} -C_R \\ 2B_R \\ -A_R \end{bmatrix}$$

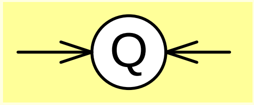


$$\mathbf{Q} = \begin{bmatrix} A_Q & B_Q \\ B_Q & C_Q \end{bmatrix}, \mathbf{R} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

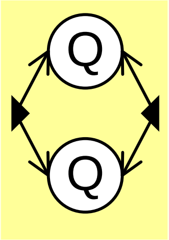


$$= -\left(A_Q + C_Q\right) = -2E_Q = -2trace\left(\mathbf{Q}\right)$$

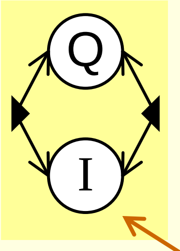
Categorizing Equivalence Classes



$$= \begin{bmatrix} A & B \\ B & C \end{bmatrix}$$



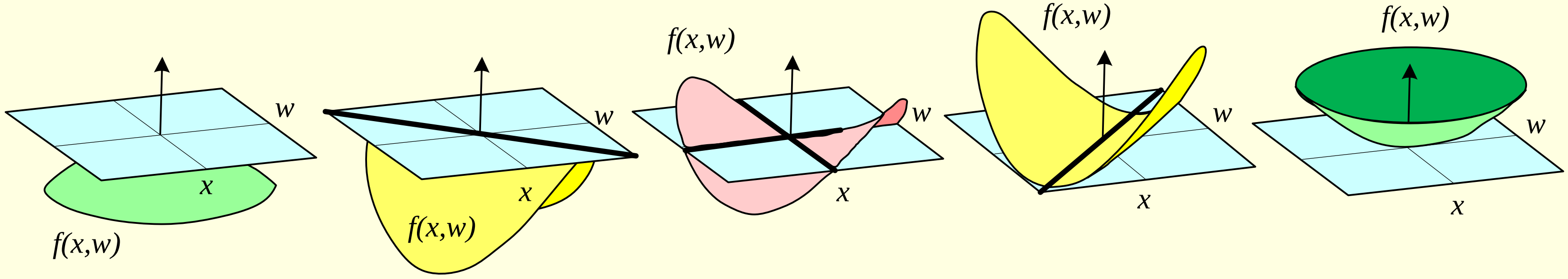
$$= -2\left(AC - B^2\right) = -2\,\mathrm{det}\left(\mathbf{Q}\right)$$

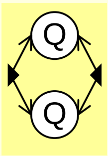
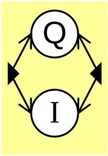


$$= -\left(A + C\right) = -2E = -2\,\mathrm{trace}\left(\mathbf{Q}\right)$$

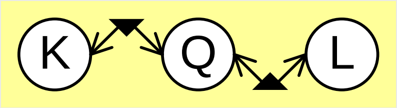
But could use any quadratic in positive cone

Categorizing Equivalence Classes



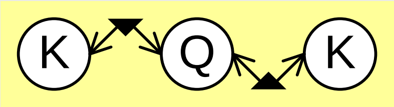
	$\frac{1}{1}-$	2-	11	2+	$\frac{1}{1}+$
	–	0	+	0	–
	+	+		–	–

What does this mean?



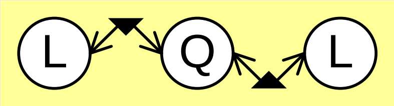
$= 0$

We already know the meaning of:



$= 0$

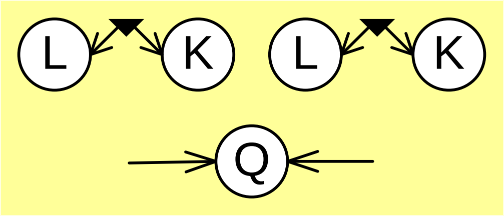
K is a factor of Q (Q is type 11)



$= 0$

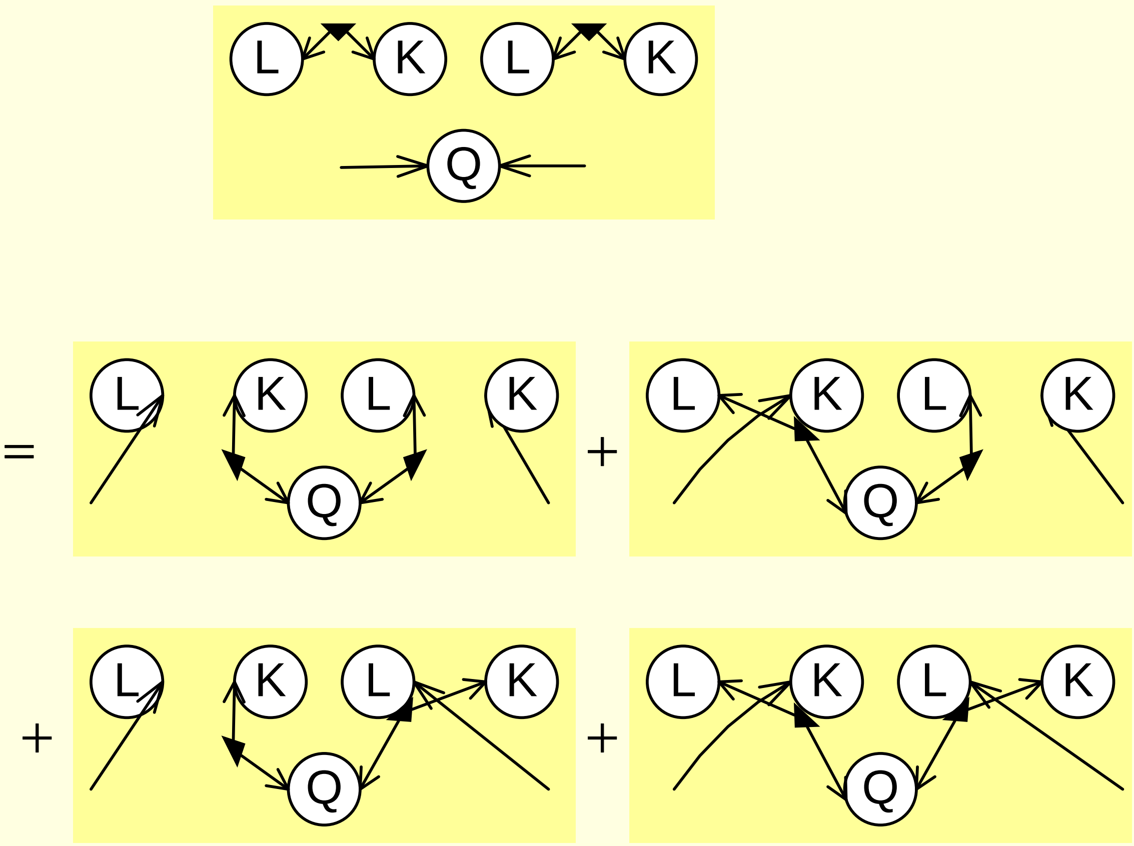
L is a factor of Q (Q is type 11)

An Identity

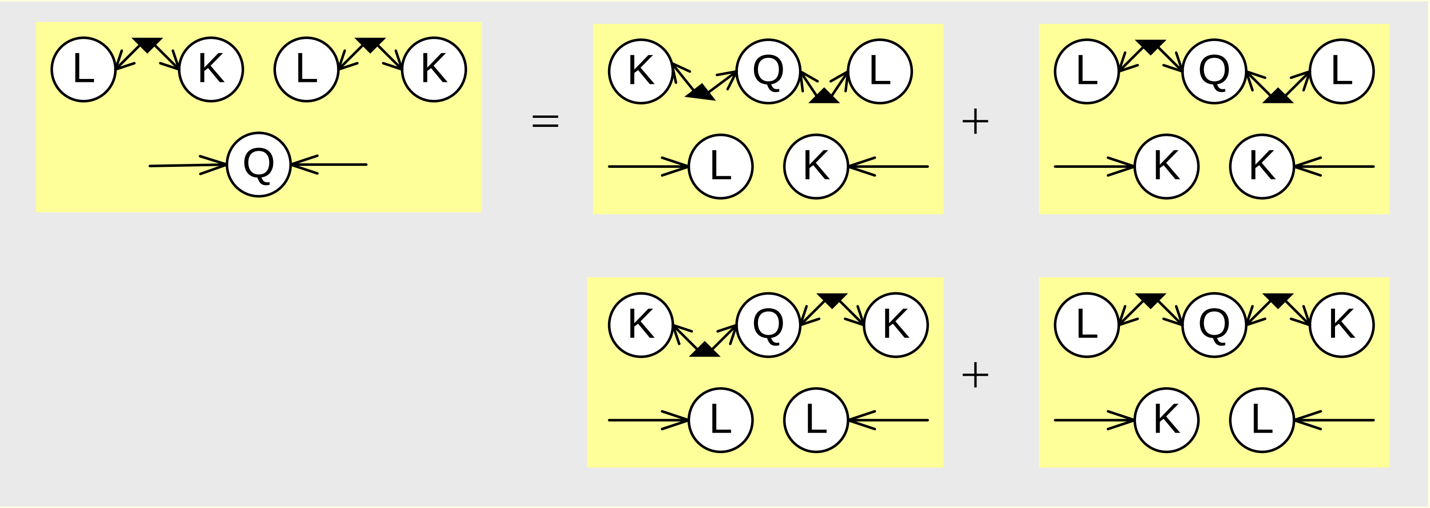


Visio Demo

An Identity



An Identity

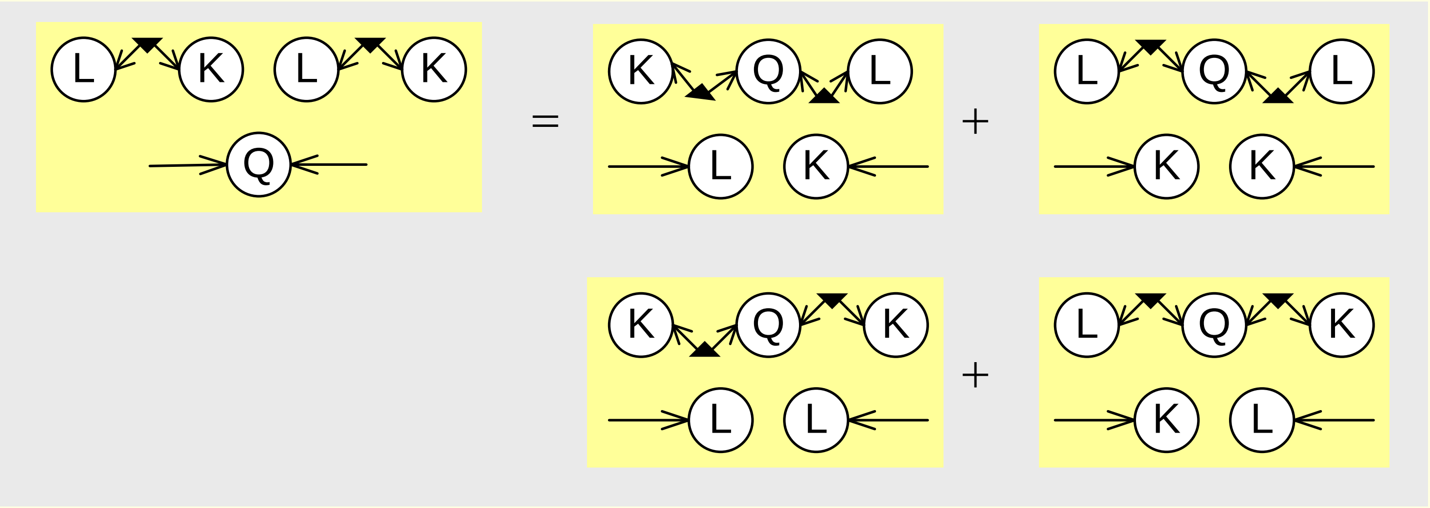


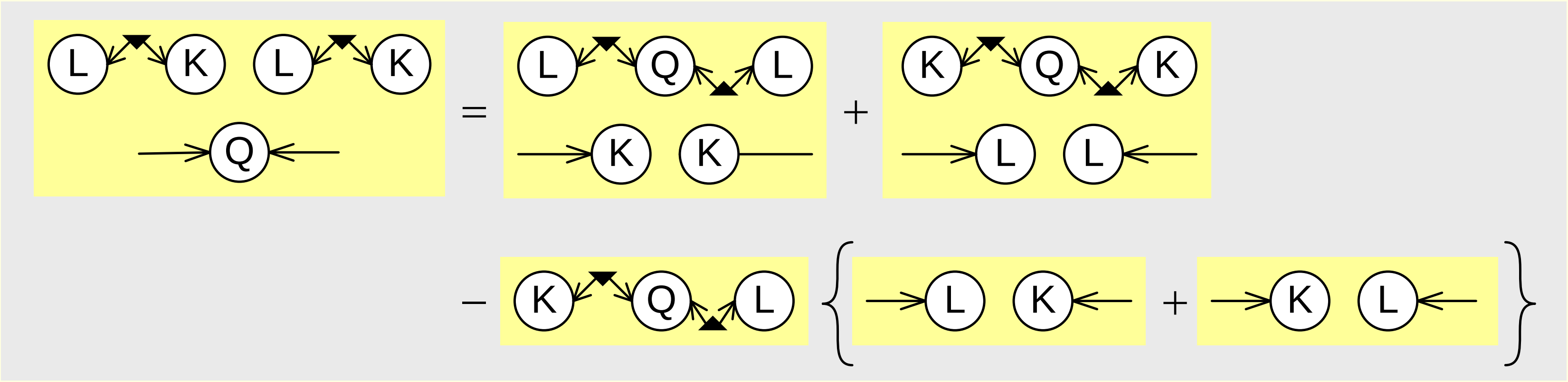
True for all Q, K, L

Each term has same number of Q, K, L
Just connected differently

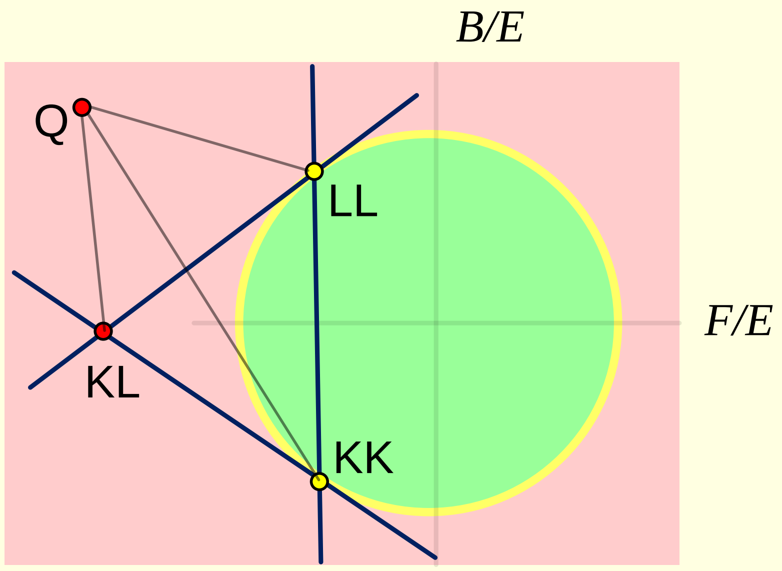
Called a “Syzygy”

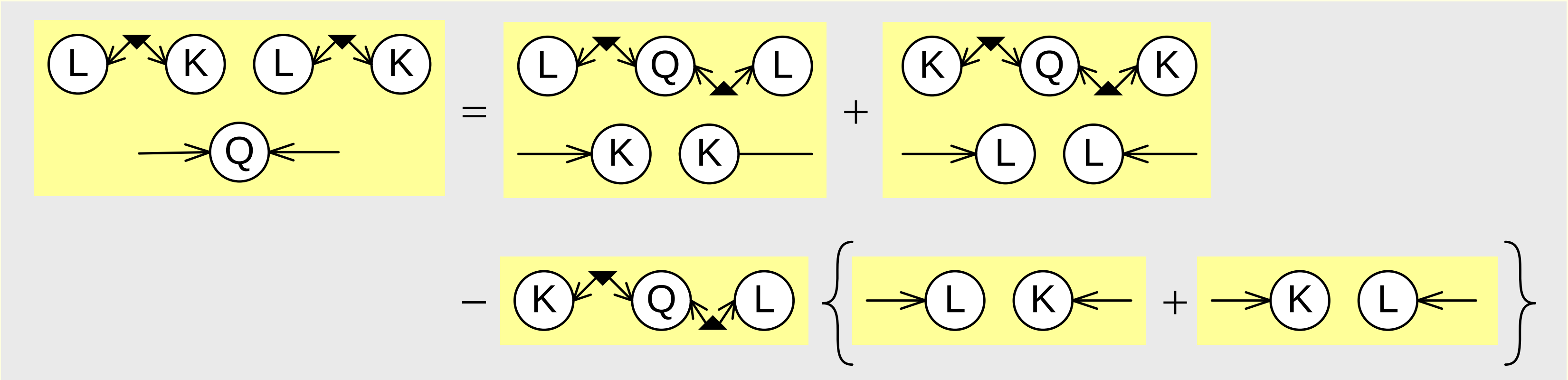
Rearrange Syzygy



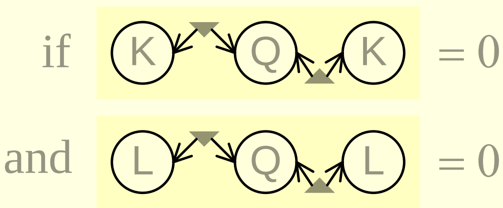


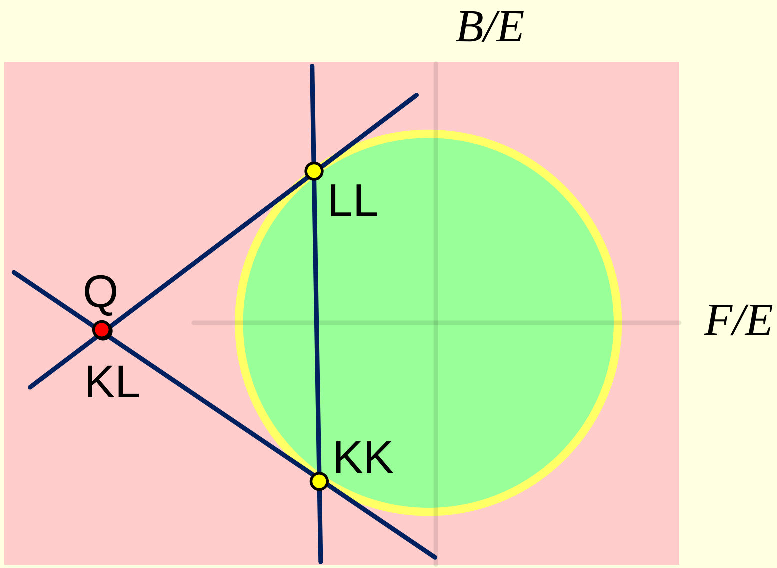
Interpret Syzygy

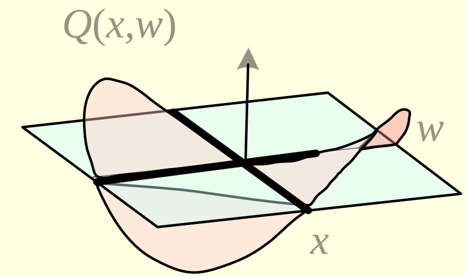


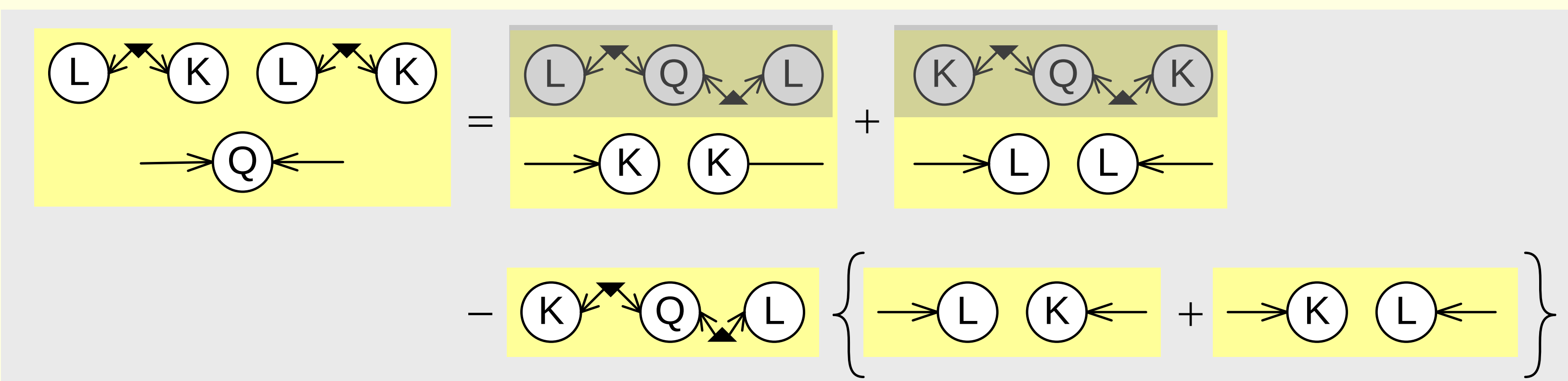


Interpret Syzygy

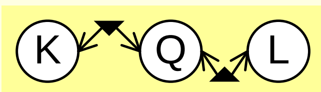


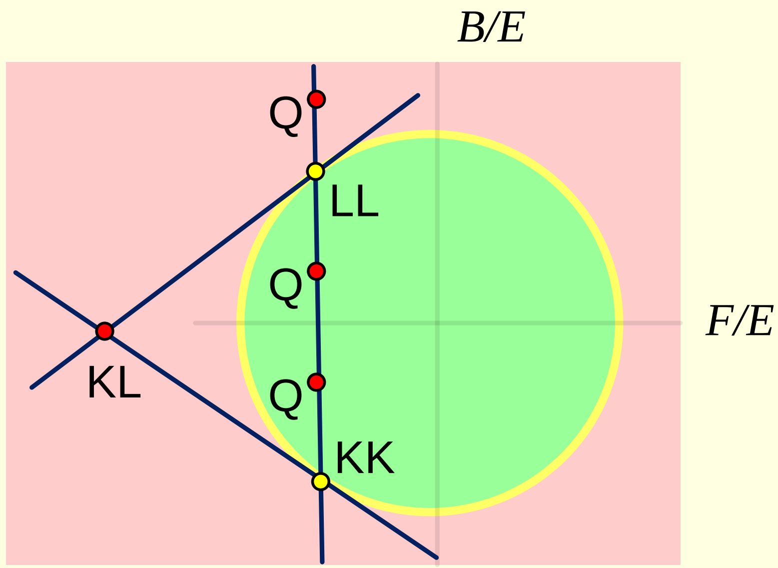


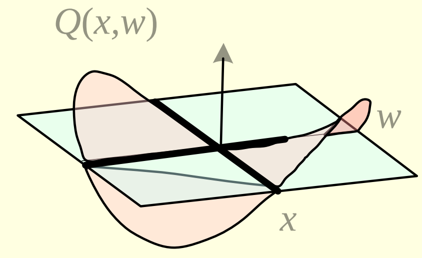


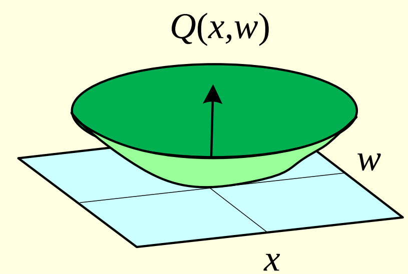


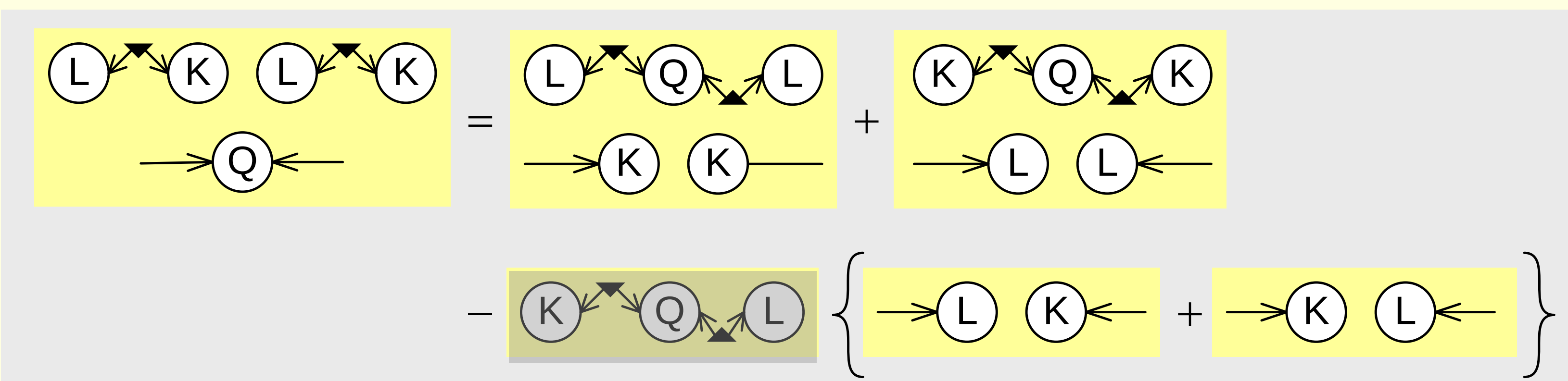
Interpret Syzygy

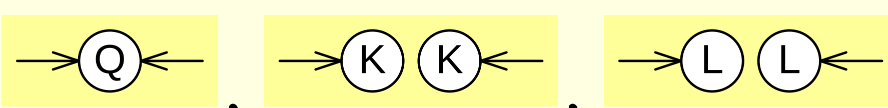
if  = 0





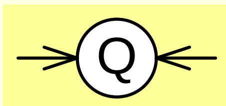




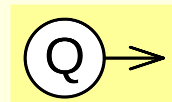


Are linearly dependent

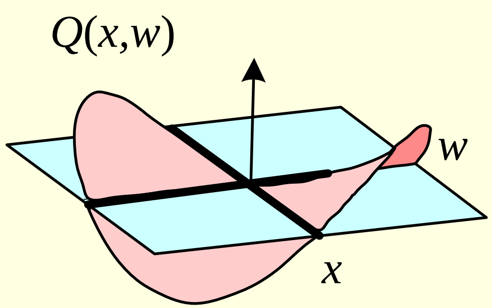
Two ways to look at Q

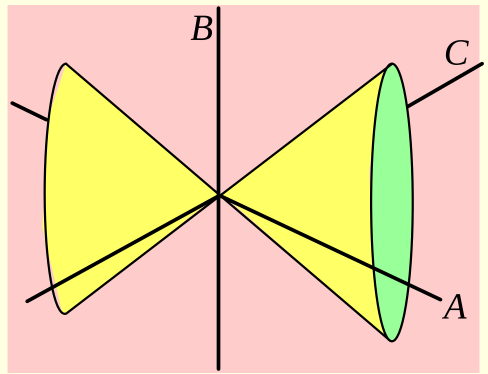


$$= \begin{bmatrix} A & B \\ B & C \end{bmatrix}$$

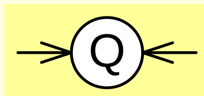


$$= \begin{bmatrix} A & B & C \end{bmatrix}$$

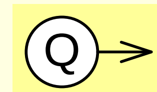




Two ways to look at Q

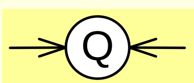


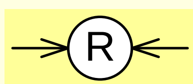
$$= \begin{bmatrix} A & B \\ B & C \end{bmatrix}$$

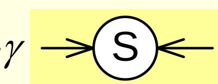


$$= \begin{bmatrix} A & B & C \end{bmatrix}$$

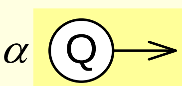
Linearly dependent

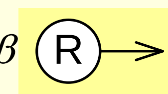
α 

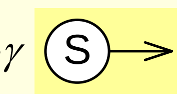
$+\beta$ 

$+\gamma$ 

$= 0$

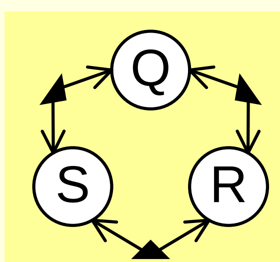
α 

$+\beta$ 

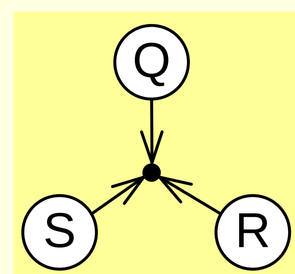
$+\gamma$ 

$= 0$

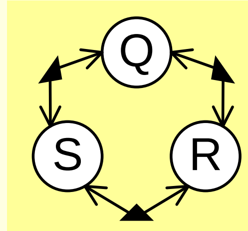
If

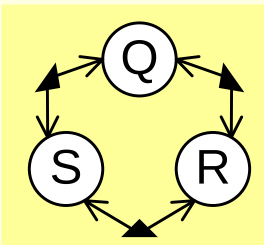


$= 0$



$= 0$

What is 

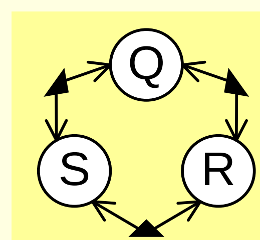


$$= trace\left(\varepsilon \mathbf{Q} \varepsilon \mathbf{R} \varepsilon \mathbf{S}\right) \qquad \mathbf{Q} = \begin{bmatrix} A_Q & B_Q \\ B_Q & C_Q \end{bmatrix}, \mathbf{R} = \begin{bmatrix} A_R & B_R \\ B_R & C_R \end{bmatrix}, \mathbf{S} = \begin{bmatrix} A_S & B_S \\ B_S & C_S \end{bmatrix}$$

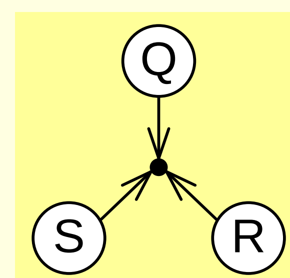
$$= trace\left\{ \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} A_Q & B_Q \\ B_Q & C_Q \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} A_R & B_R \\ B_R & C_R \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} A_S & B_S \\ B_S & C_S \end{bmatrix} \right\}$$

$$= -A_Q B_R C_R - B_Q C_R A_S - C_Q A_R B_S + A_Q C_R B_S + B_Q A_R C_R + C_Q B_R A_S$$

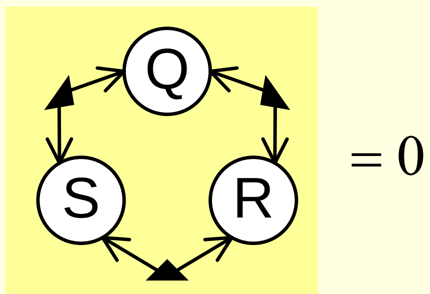
$$= -\det \begin{bmatrix} A_Q & B_Q & C_Q \\ A_R & B_R & C_R \\ A_S & B_S & C_R \end{bmatrix}$$

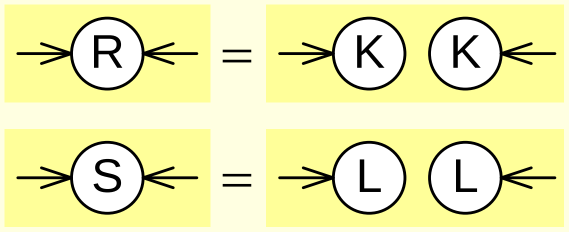


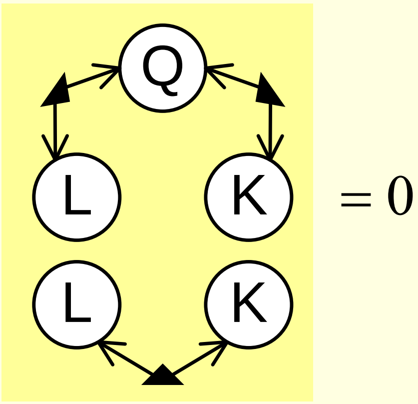
$=$



Linear Dependence of Q,KK,LL

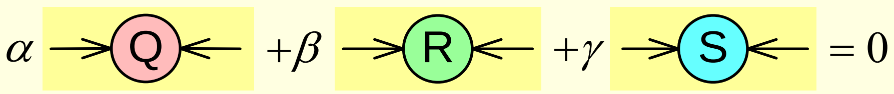


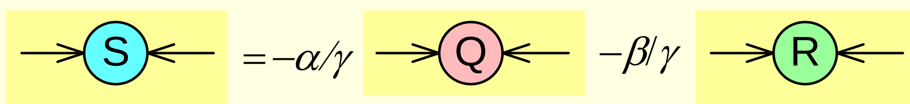




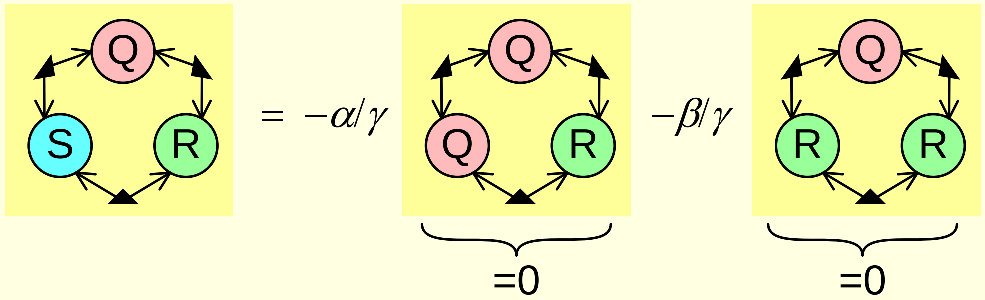
The reverse direction

If

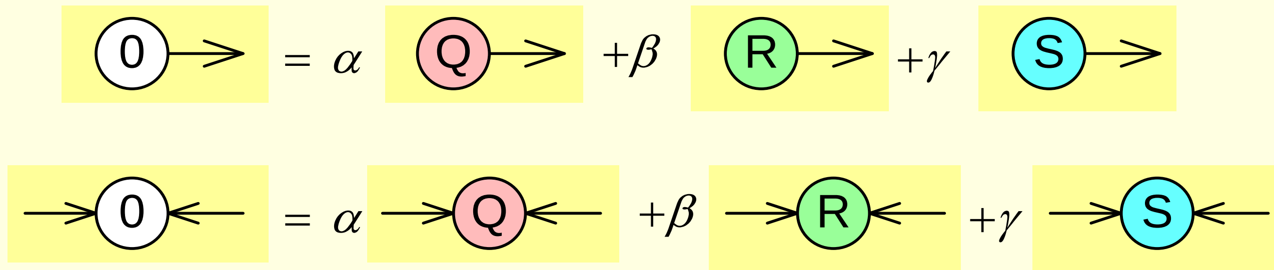




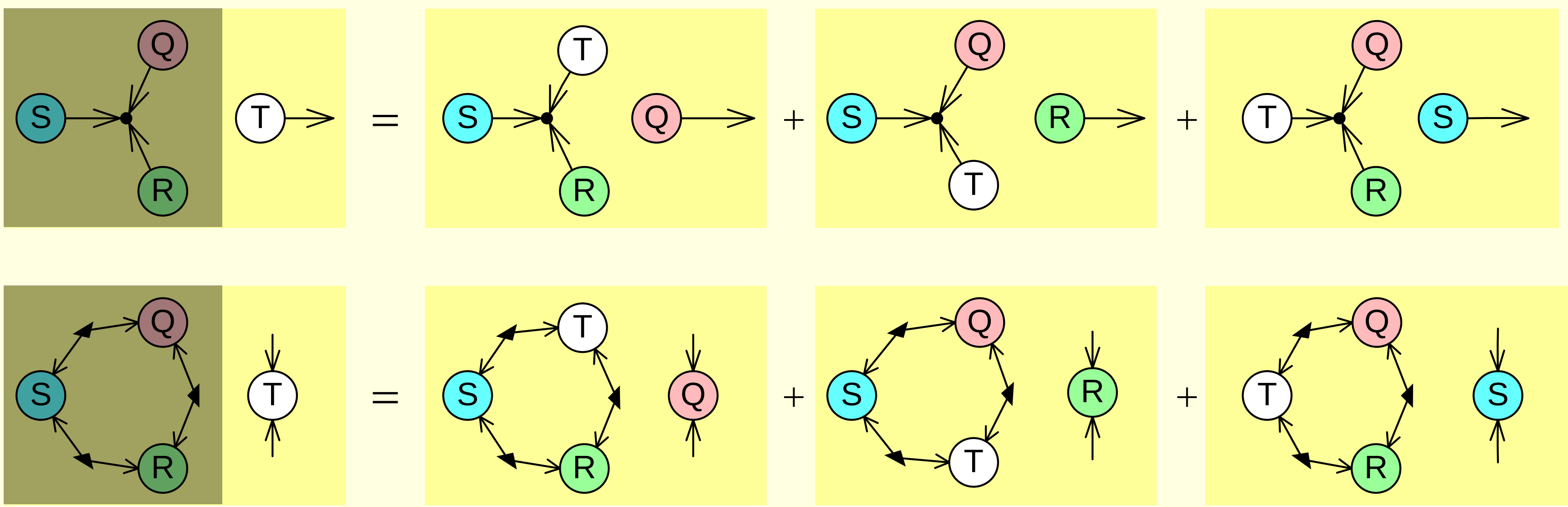
Then



What are the scale factors?

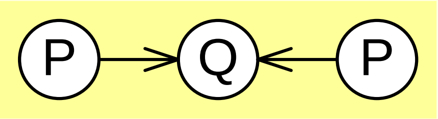


Use the Cramer's Rule identity



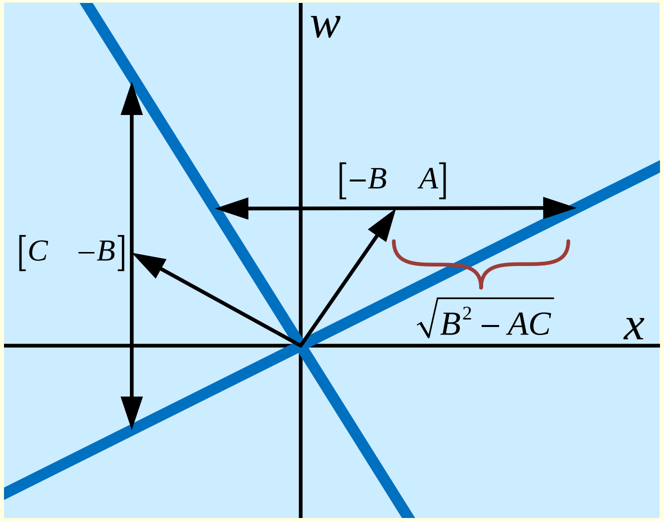
Roots of Q

$$Q(x,w) = Ax^2 + 2Bxw + Cw^2 =$$



$$\frac{x}{w} = \frac{-2B \pm \sqrt{(2B)^2 - 4AC}}{2A} \qquad \begin{bmatrix} x & w \end{bmatrix} = \begin{bmatrix} -B \pm \sqrt{B^2 - AC} & A \end{bmatrix}$$

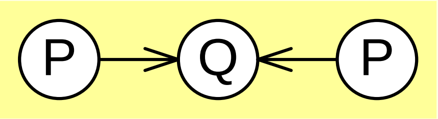
$$\frac{w}{x} = \frac{-2B \pm \sqrt{(2B)^2 - 4CA}}{2C} \qquad \begin{bmatrix} x & w \end{bmatrix} = \begin{bmatrix} C & -B \pm \sqrt{B^2 - AC} \end{bmatrix}$$

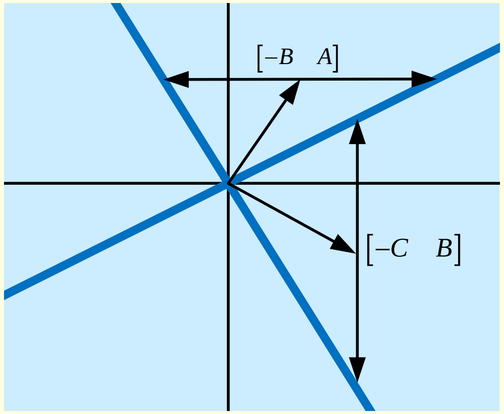


$$\begin{bmatrix} -B + \sqrt{B^2 - AC} & A \end{bmatrix} \quad \begin{bmatrix} C & -B - \sqrt{B^2 - AC} \end{bmatrix} = 0$$

Note: different signs

Roots of Q

$$Q(x,w) = Ax^2 + 2Bxw + Cw^2 =$$ 

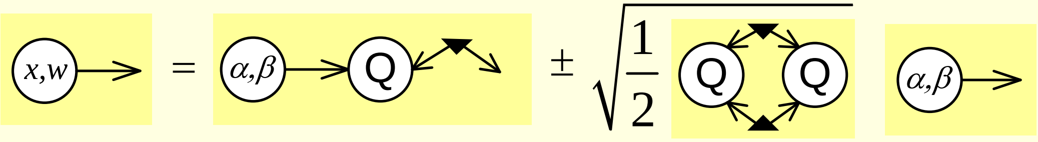


$$\begin{bmatrix} x & w \end{bmatrix} = \begin{bmatrix} -B \pm \sqrt{B^2 - AC} & A \end{bmatrix}$$

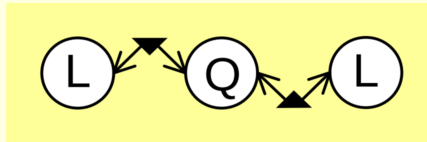
$$\begin{bmatrix} x & w \end{bmatrix} = \begin{bmatrix} -C & B \pm \sqrt{B^2 - AC} \end{bmatrix}$$

$$\begin{bmatrix} x & w \end{bmatrix} = \alpha \begin{bmatrix} -B \pm \sqrt{B^2 - AC} & A \end{bmatrix} + \beta \begin{bmatrix} -C & B \pm \sqrt{B^2 - AC} \end{bmatrix}$$

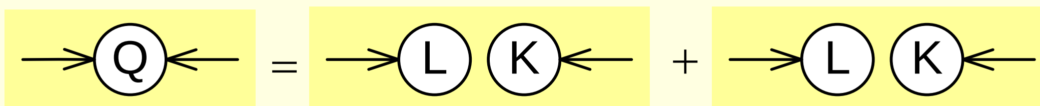
$$\begin{bmatrix} x & w \end{bmatrix} = \begin{bmatrix} \alpha & \beta \end{bmatrix} \left\{ \begin{bmatrix} -B & A \\ -C & B \end{bmatrix} \pm \sqrt{B^2 - AC} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$



Division

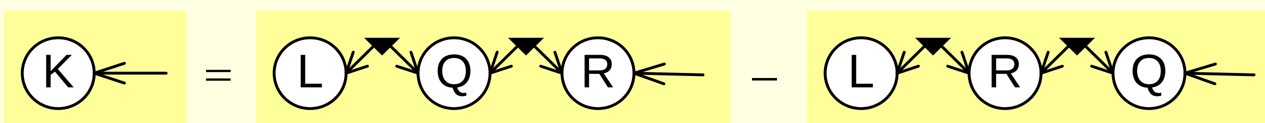
If  = 0

Then L is a factor of Q so



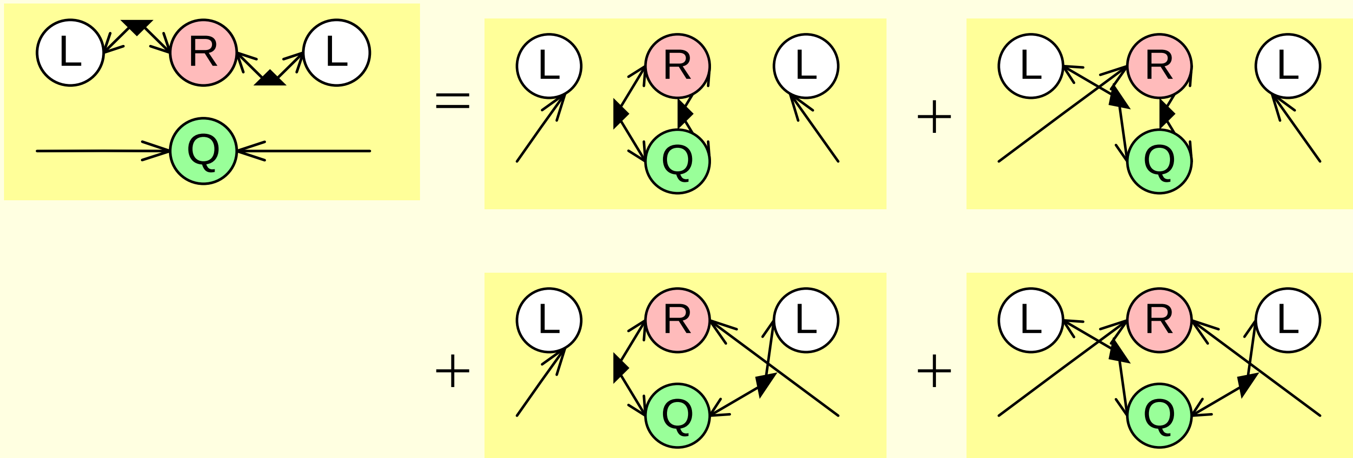
What is K?

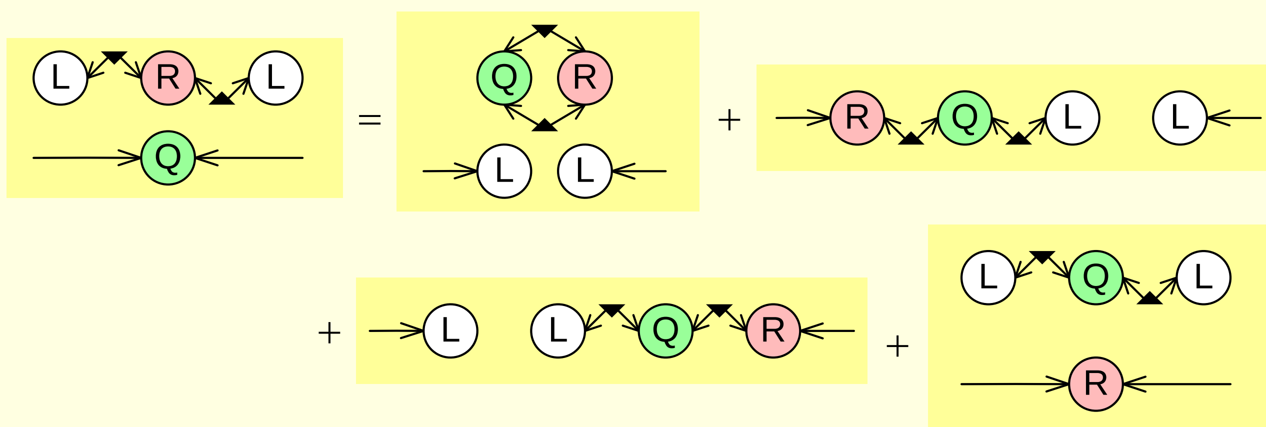
Answer:



Why does this work?
Where does R come from?

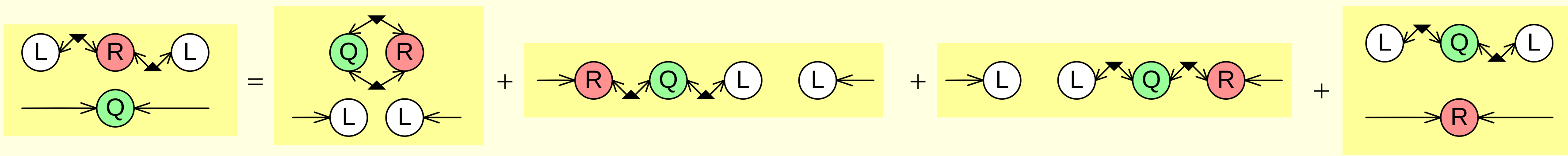
Another Syzygy



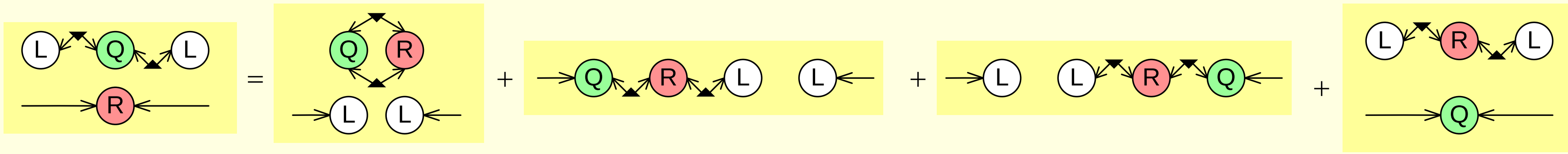


Syzygy continued

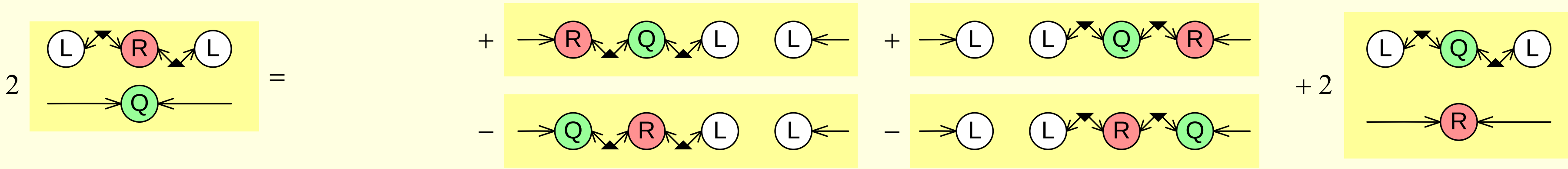
True for all Q,R,L



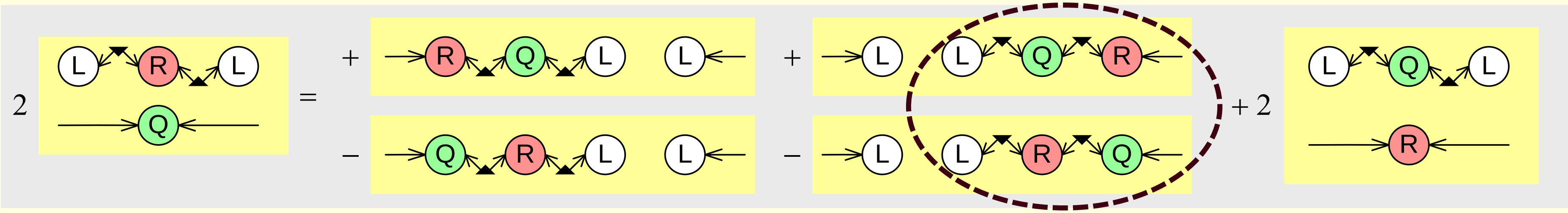
Swap Q,R

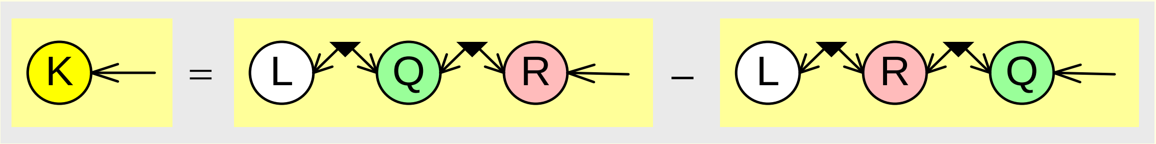


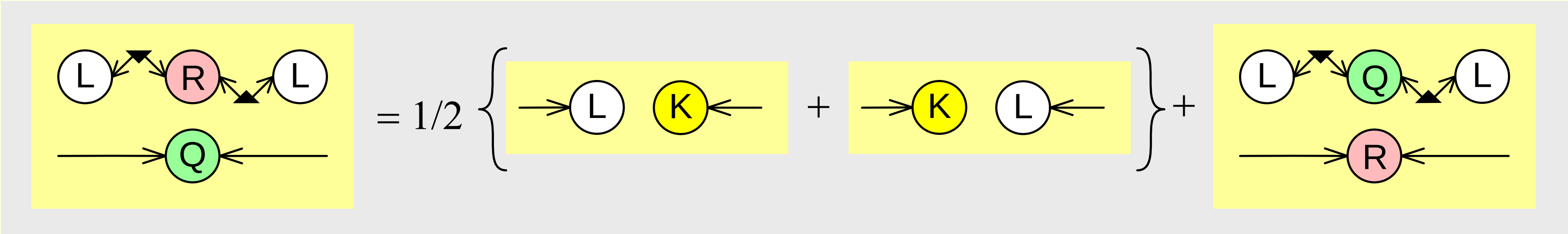
Subtract and rearrange



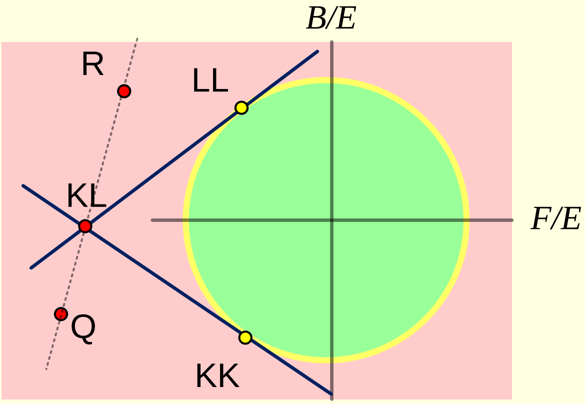
Syzygy continued





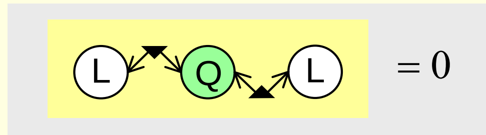


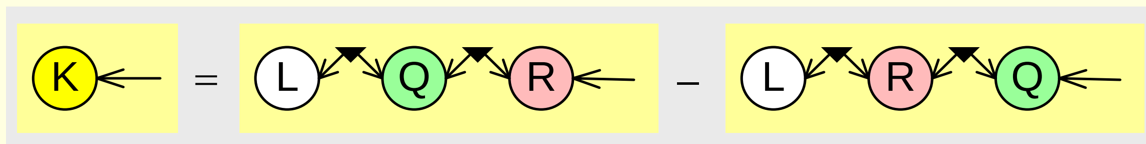
True for all Q,R,L

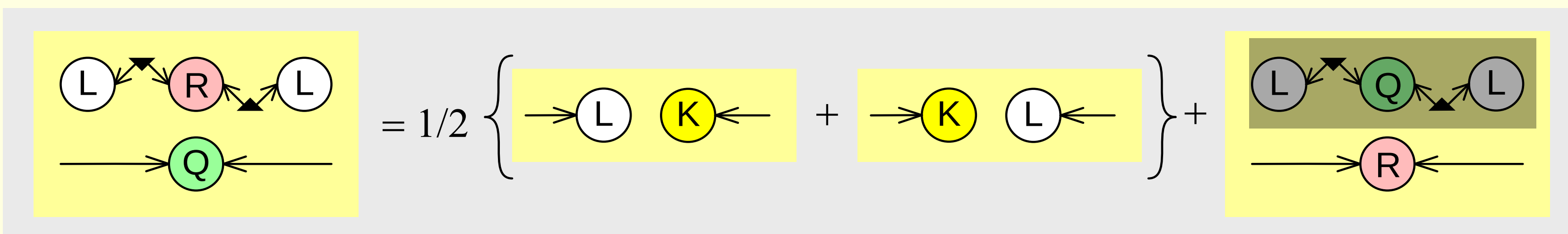


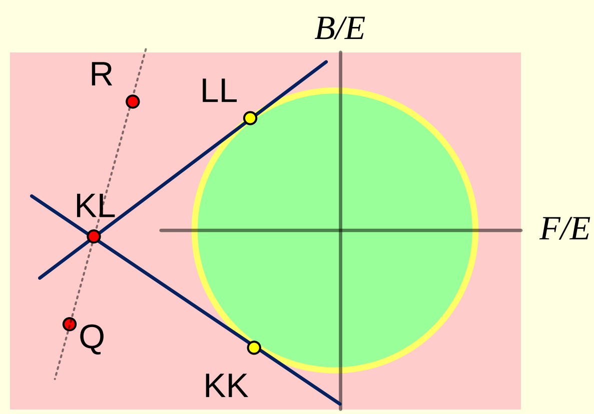
Division

If

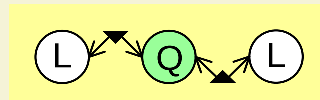




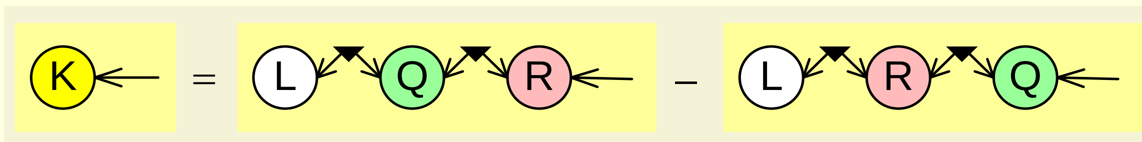


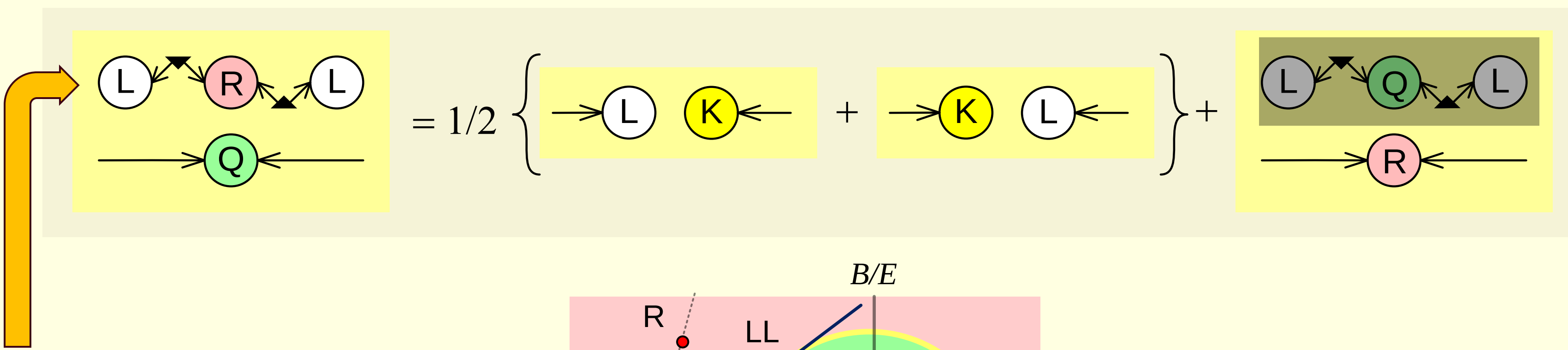


Division

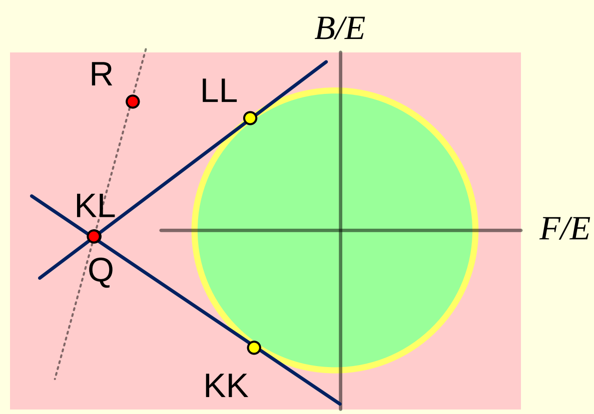
If 

= 0



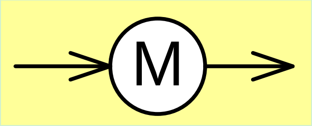


As long as R doesn't have L as a factor, so this is nonzero



Transformations

$$\mathbf{M} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

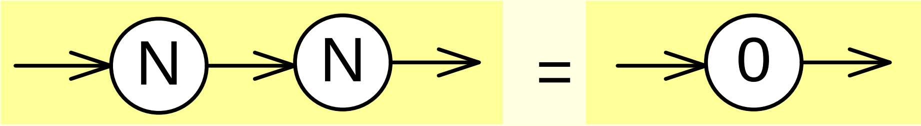


$$\begin{bmatrix} x & w \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} \tilde{x} & \tilde{w} \end{bmatrix}$$

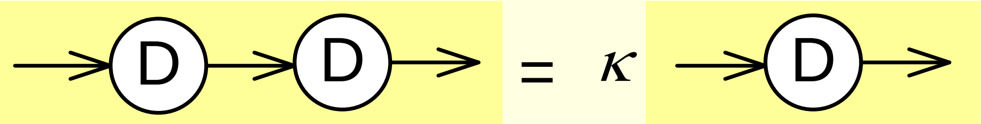
$$\frac{\tilde{x}}{\tilde{w}} = \frac{Ax + Cw}{Bx + Dw}$$

Special Matrices

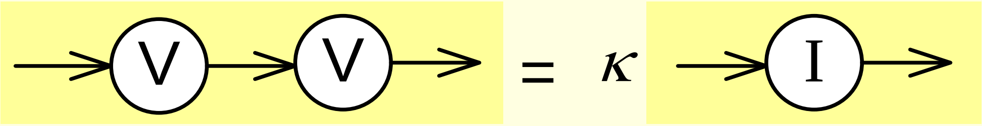
Nilpotent



Idempotent

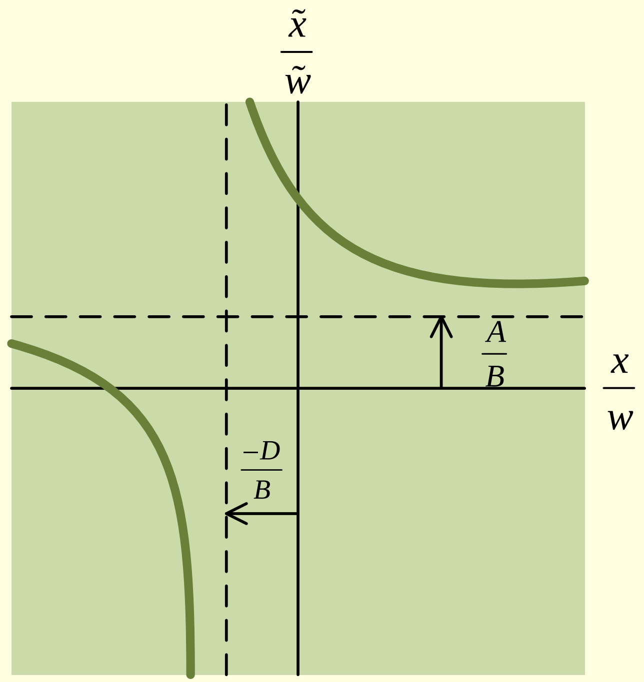


Involution



The Function

$$\frac{\tilde{x}}{\tilde{w}} = \frac{Ax + Cw}{Bx + Dw}$$



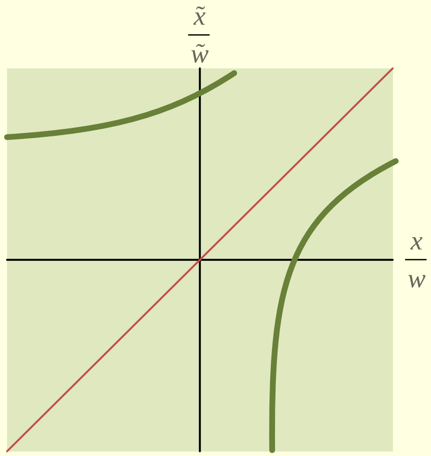
$$w = 0 \quad \Rightarrow \quad \frac{\tilde{x}}{\tilde{w}} = \frac{A}{B}$$

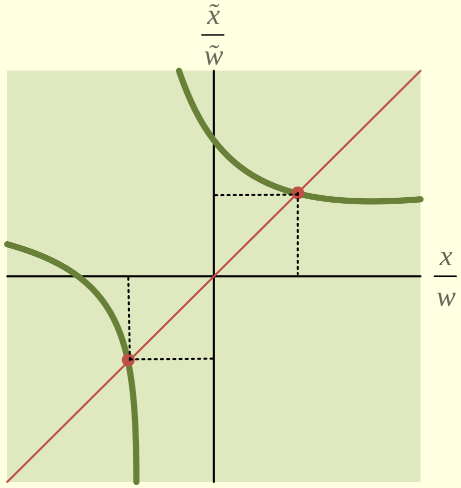
$$\tilde{w} = 0 \quad \Rightarrow \quad Bx + Dw = 0 \quad \Rightarrow \quad \frac{x}{w} = \frac{-D}{B}$$

Examples of function

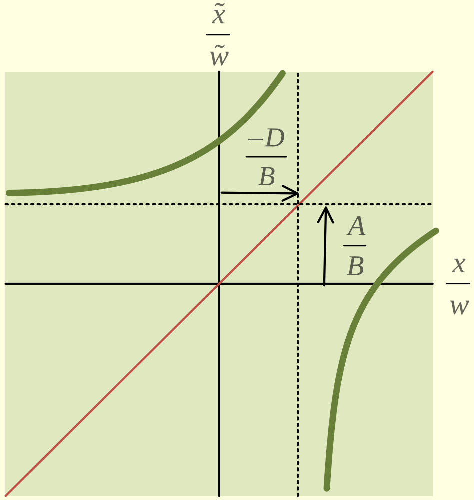
$$T\left(\frac{x}{w}\right)=\frac{A\frac{x}{w}+C}{B\frac{x}{w}+D}$$

$$T'\left(\frac{x}{w}\right)=\left(\frac{A\frac{x}{w}+C}{B\frac{x}{w}+D}\right)'=\frac{AD-BC}{\left(B\frac{x}{w}+D\right)^2}$$





$$\begin{bmatrix}x_e & w_e\end{bmatrix}\begin{bmatrix}A & B\\ C & D\end{bmatrix}=\lambda\begin{bmatrix}x_e & w_e\end{bmatrix}$$



$A=-D$ trace=0

Three Invariants

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

Determinant $$\Delta = AD - BC$$

trace $$t = A + D$$

Characteristic equation

$$\det\begin{bmatrix} A-\lambda & B \\ C & D-\lambda \end{bmatrix} = \lambda^2 + (-A-D)\lambda + (AD-BC) = 0$$

$$\begin{aligned}\delta &= (-A-D)^2 - 4(AD-BC) \\ &= (A-D)^2 + 4BC\end{aligned}$$

Three Invariants

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

Determinant $\Delta = AD - BC$

trace $t = A + D$

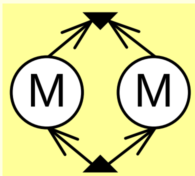
Characteristic equation discriminant

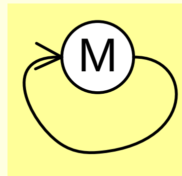
$$\delta = A^2 - 2AD + D^2 + 4BC$$

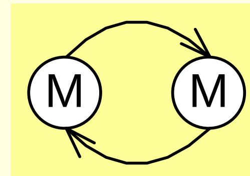
Relation between them:

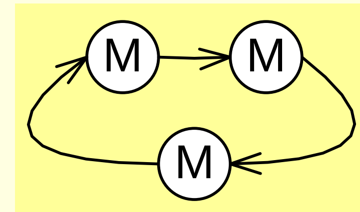
$$4\Delta + \delta = t^2$$

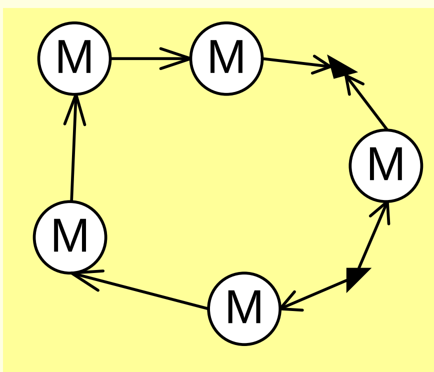
Diagrams

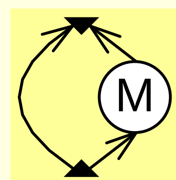


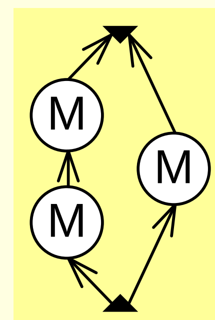




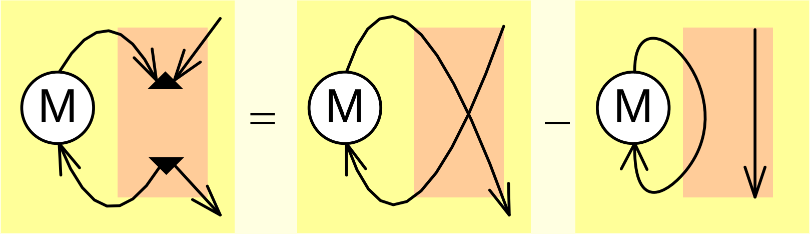


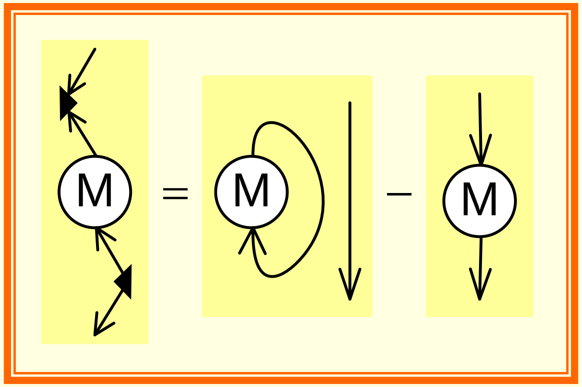






Identity 1





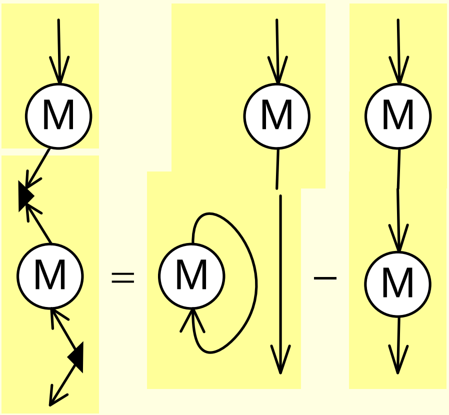
$$adj\mathbf{M} = (trace\mathbf{M})\mathbf{I} - \mathbf{M}$$

$$\begin{bmatrix} D & -B \\ -C & A \end{bmatrix}$$

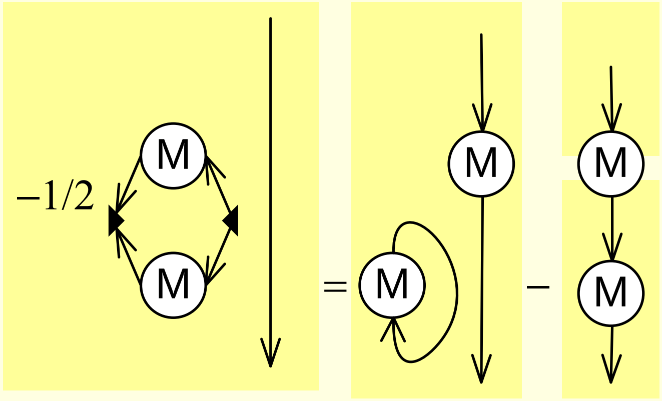
$$\begin{bmatrix} A+D & 0 \\ 0 & A+D \end{bmatrix} - \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

$$= \begin{bmatrix} D & -B \\ -C & A \end{bmatrix}$$

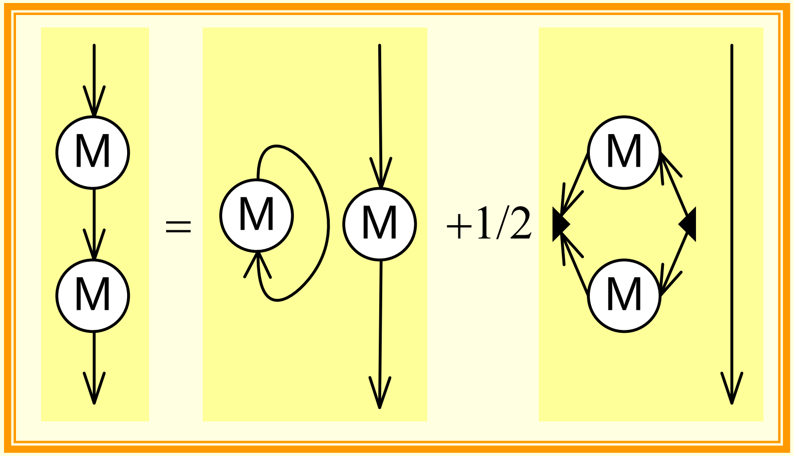
Identity 2



Identity 2



Identity 2



$$\mathbf{MM} = (trace\,\mathbf{M})\mathbf{M} - (\det \mathbf{M})\mathbf{I}$$

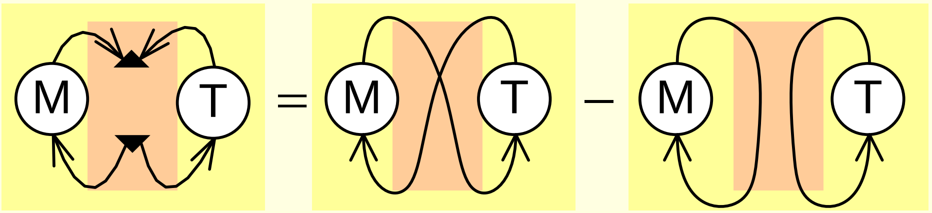
$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} AA+BC & AB+BD \\ AC+DC & BC+DD \end{bmatrix}$$

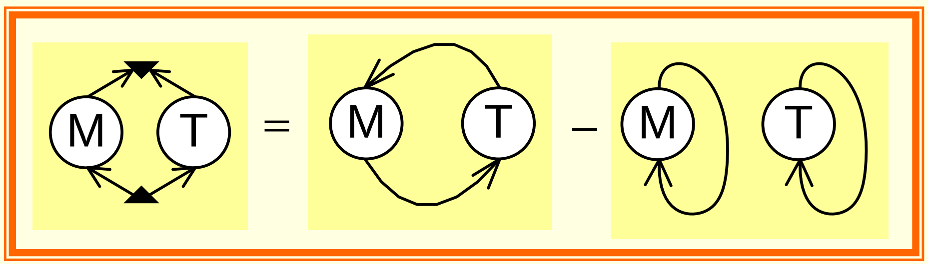
$$= (A+D)\begin{bmatrix} A & B \\ C & D \end{bmatrix} - (AD-CB)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

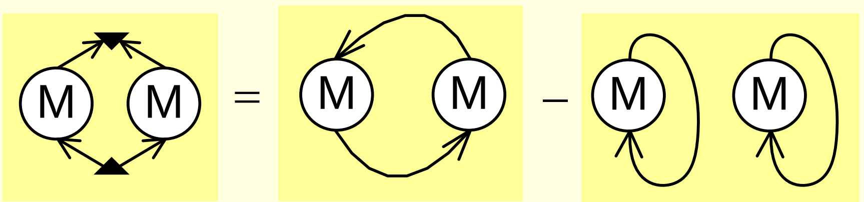
$$= \begin{bmatrix} AA+AD & BA+BD \\ CA+CD & AD+DD \end{bmatrix} + \begin{bmatrix} BC-AD & 0 \\ 0 & BC-AD \end{bmatrix}$$

$$= \begin{bmatrix} AA+BC & AB+BD \\ AC+DC & BC+DD \end{bmatrix}$$

Identity 3

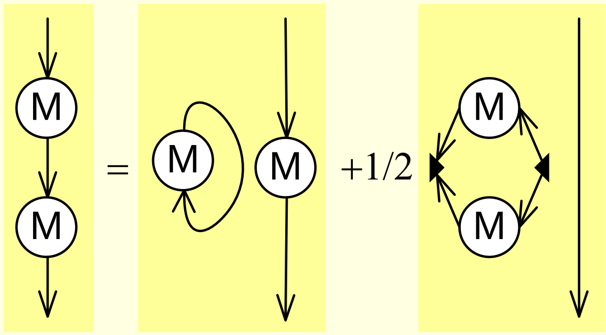


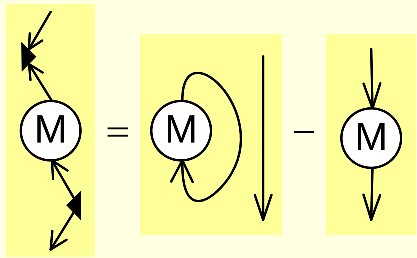


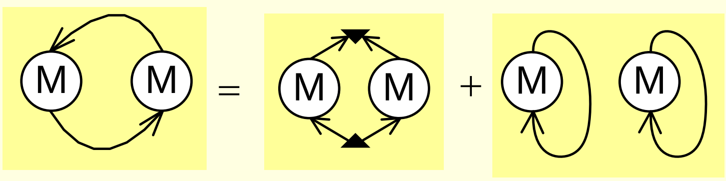


Reducing complex diagrams

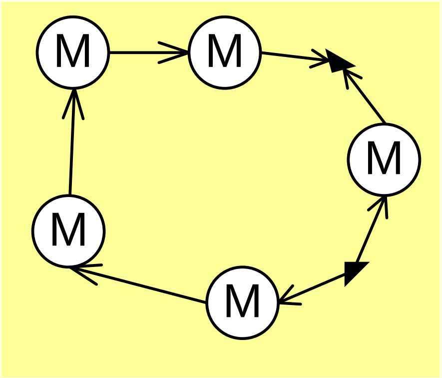
Tools:



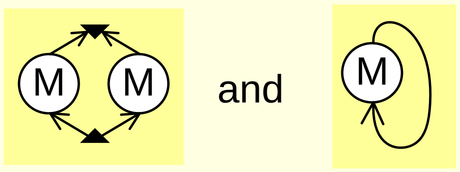




To reduce this:

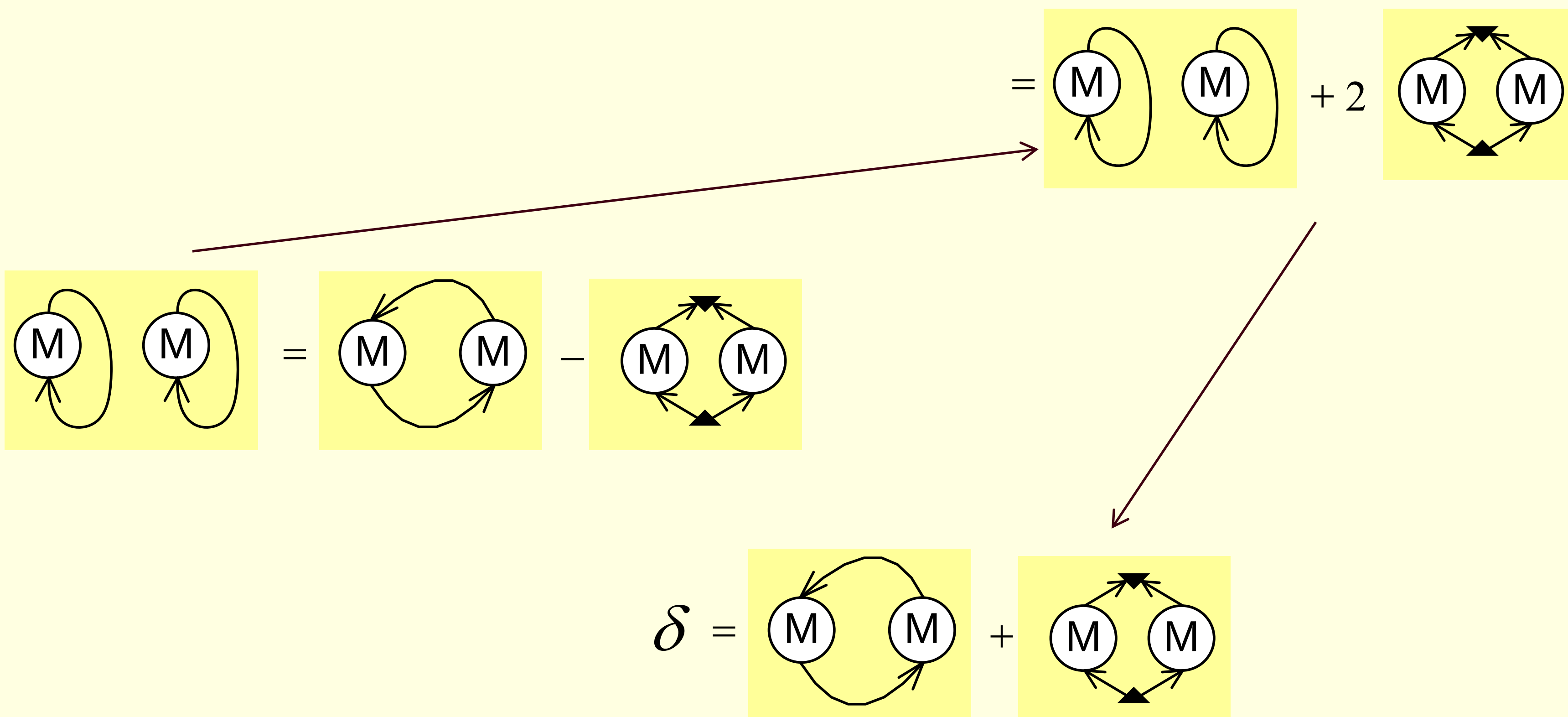


To simple combinations of:

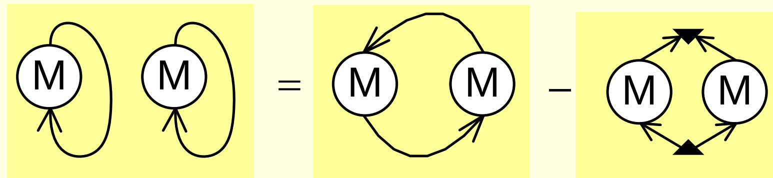


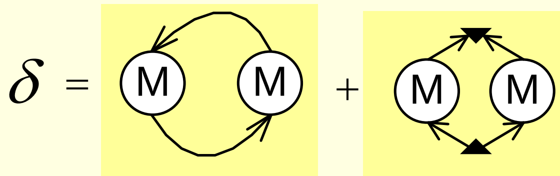
Discriminant of Characteristic Equation

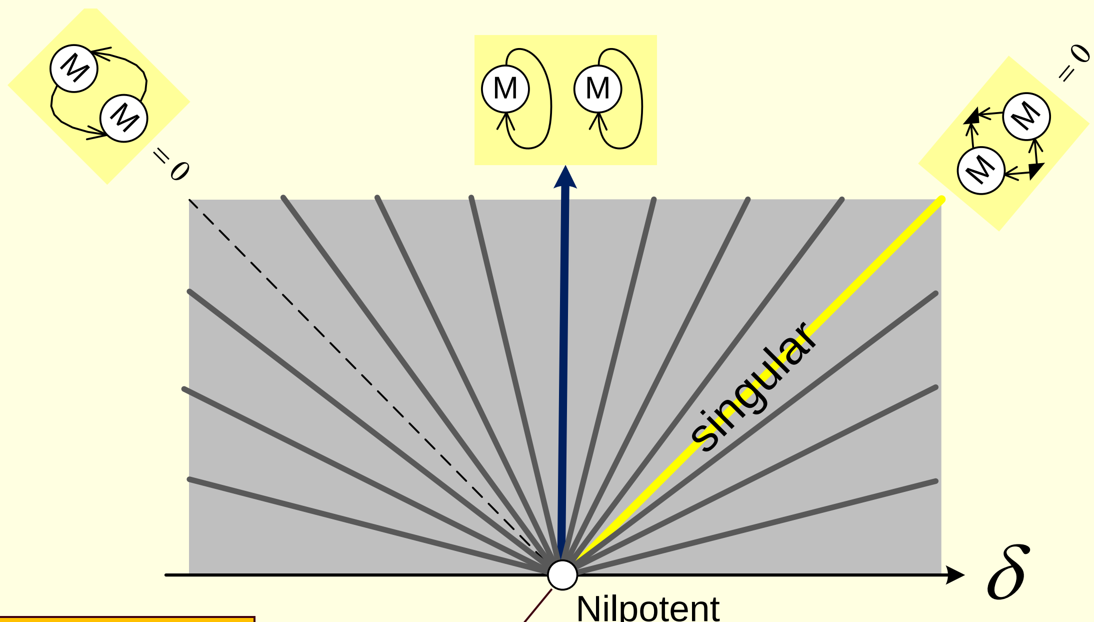
$$\delta = t^2 - 4\Delta$$



The “Phase Space” of M







A valid matrix

Numeric “signature”

$$\frac{\delta}{t^2} = \chi$$

$$\tan^{-1}\left(\delta, t^2\right) = \phi$$

Plotting Invariants in ABCD Space

$$\Delta = AD - BC$$

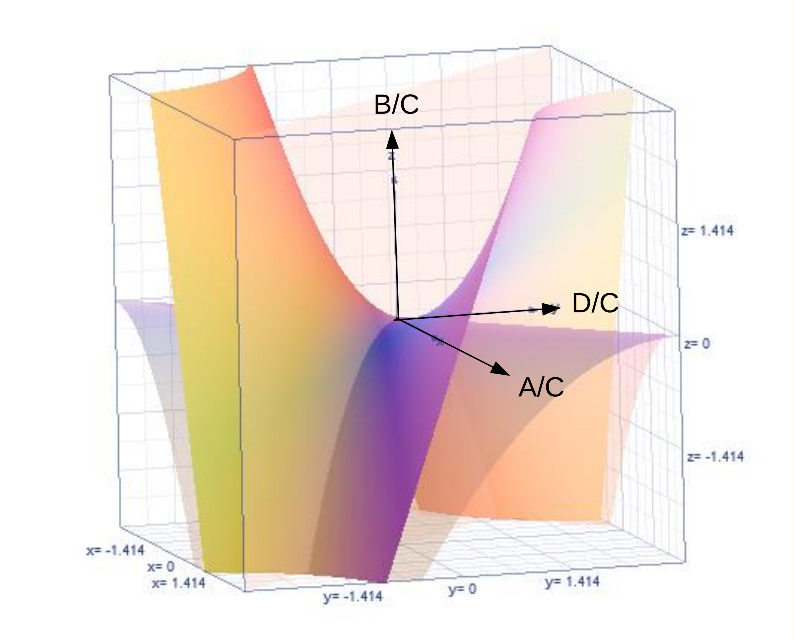
$$t = A + D$$

$$\delta = A^2 - 2AD + D^2 + 4BC$$

$$\Delta = 0 \quad \Rightarrow \quad \frac{B}{C} = \frac{A}{C}\frac{D}{C}$$

$$t = 0 \quad \Rightarrow \quad \frac{A}{C} + \frac{D}{C} = 0$$

$$\delta = 0 \quad \Rightarrow \quad \frac{B}{C} = -\frac{1}{4}\left(\frac{A}{C} - \frac{D}{C}\right)^2$$



New Coordinate System

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} E+F & G+H \\ G-H & E-F \end{bmatrix}$$

$$\begin{aligned} \Delta &= AD - BC \\ &= (E+F)(E-F) - (G+H)(G-H) \\ &= (E^2 + H^2) - (F^2 + G^2) \end{aligned}$$

$$t = A + D = 2E$$

$$\begin{aligned} \delta &= (-A-D)^2 - 4(AD - BC) \\ &= 4(F^2 + G^2 - H^2) \end{aligned}$$

$$\Delta = E^2 + H^2 - F^2 - G^2$$

$$\tfrac{1}{4} t^2 = E^2$$

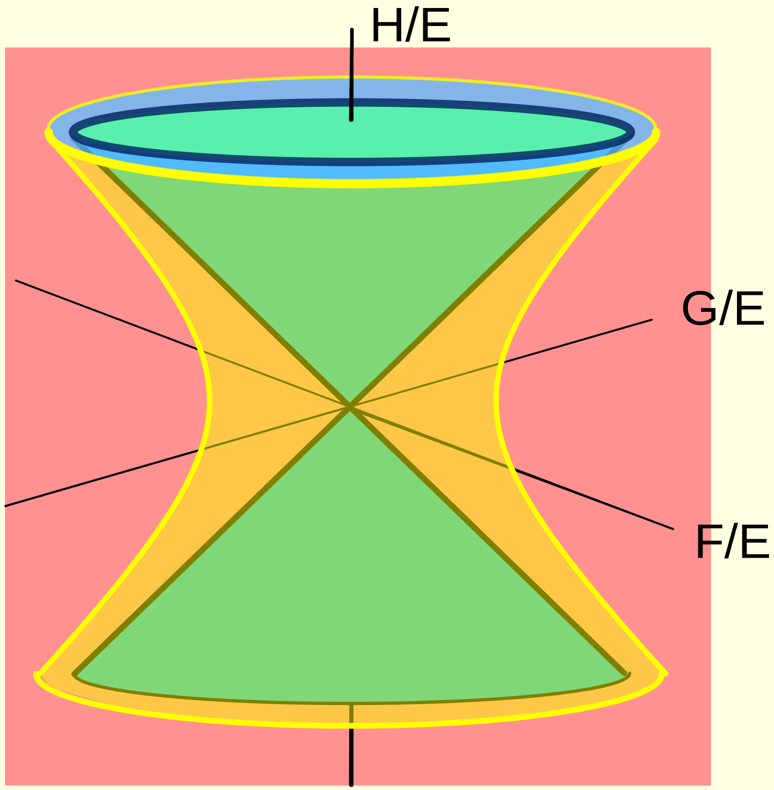
$$\tfrac{1}{4}\delta = \quad -H^2 + F^2 + G^2$$

Plot in EFGH space

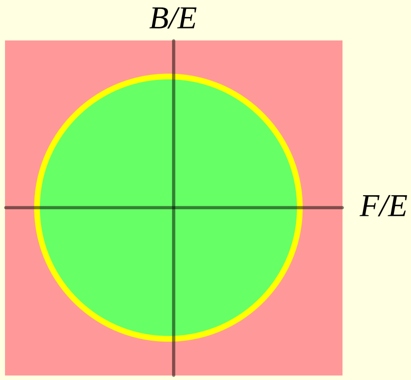
$$\Delta = 0 \quad \left(\frac{H}{E}\right)^2 = \left(\frac{F}{E}\right)^2 + \left(\frac{G}{E}\right)^2 - 1$$

$$\delta = 0 \quad \left(\frac{H}{E}\right)^2 = \left(\frac{F}{E}\right)^2 + \left(\frac{G}{E}\right)^2$$

$t = 0$ plane at infinity



Compare with Q version



Plot in EFGH space

$\Delta = 0 \quad \left(\frac{H}{E}\right)^2 = \left(\frac{F}{E}\right)^2 + \left(\frac{G}{E}\right)^2 - 1$

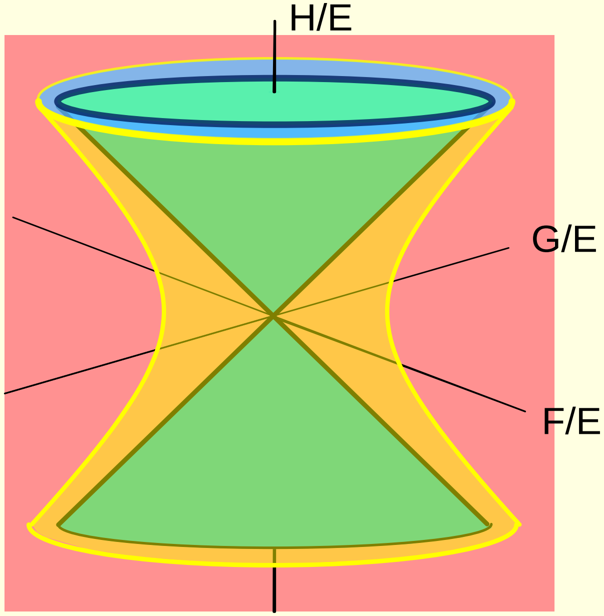
$t = 0$ plane at infinity

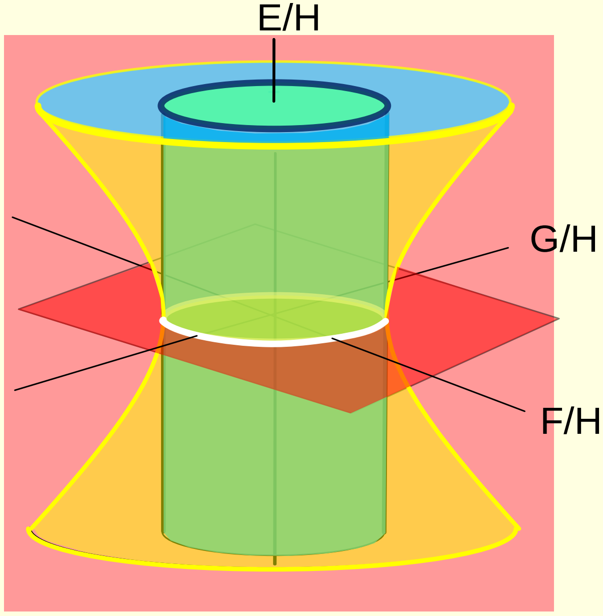
$\delta = 0 \quad \left(\frac{F}{E}\right)^2 + \left(\frac{G}{E}\right)^2 = \left(\frac{H}{E}\right)^2$

$\Delta = 0 \quad \left(\frac{E}{H}\right)^2 = \left(\frac{F}{H}\right)^2 + \left(\frac{G}{H}\right)^2 - 1$

$t = 0 \quad \left(\frac{E}{H}\right) = 0$

$\delta = 0 \quad \left(\frac{F}{H}\right)^2 + \left(\frac{G}{H}\right)^2 = 1$

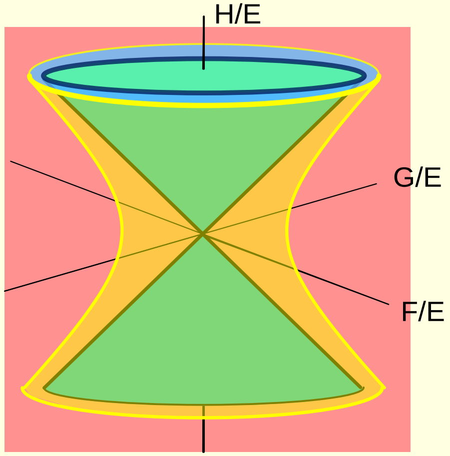




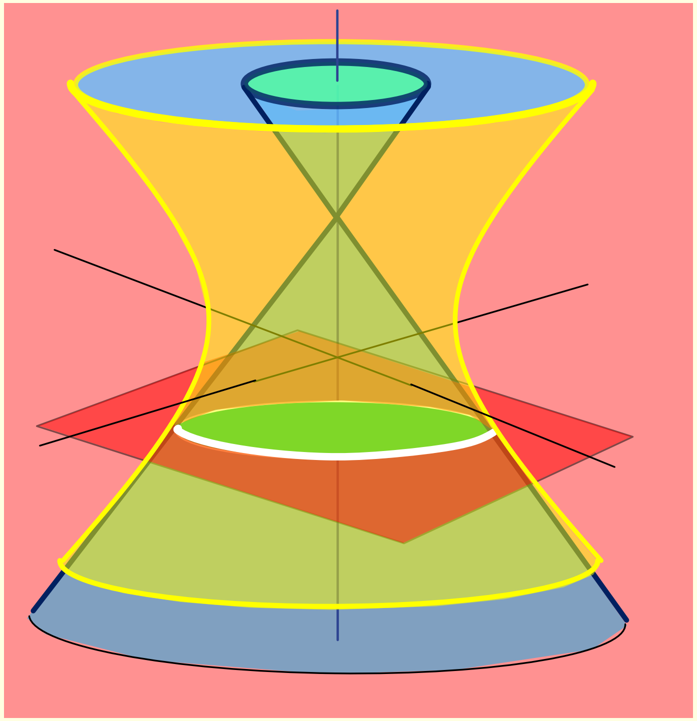
More generally, rotate along E,H axis

$$\begin{bmatrix} E \\ H \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \hat{E} \\ \hat{H} \end{bmatrix}$$

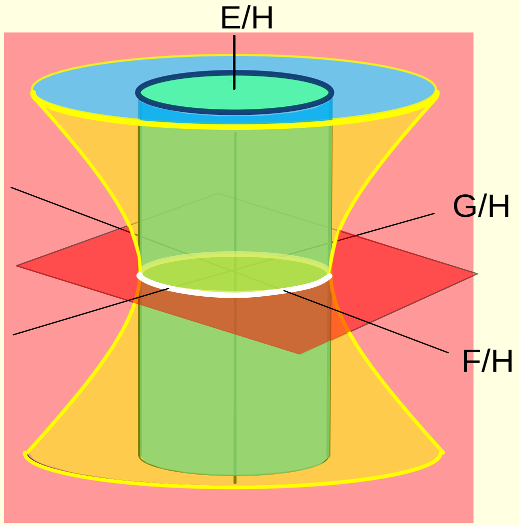
$\theta = 0^\circ$



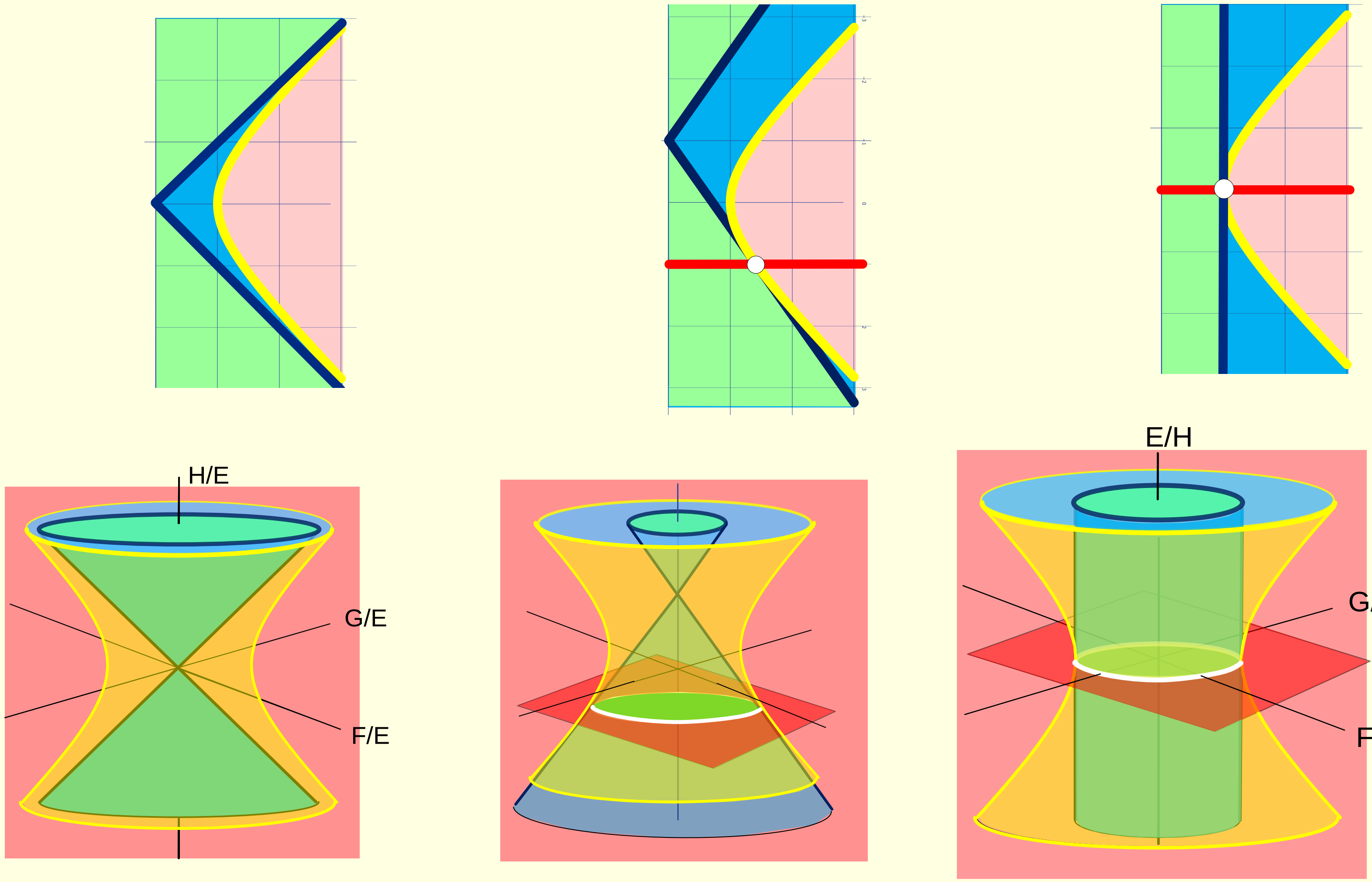
$\theta = 45^\circ$



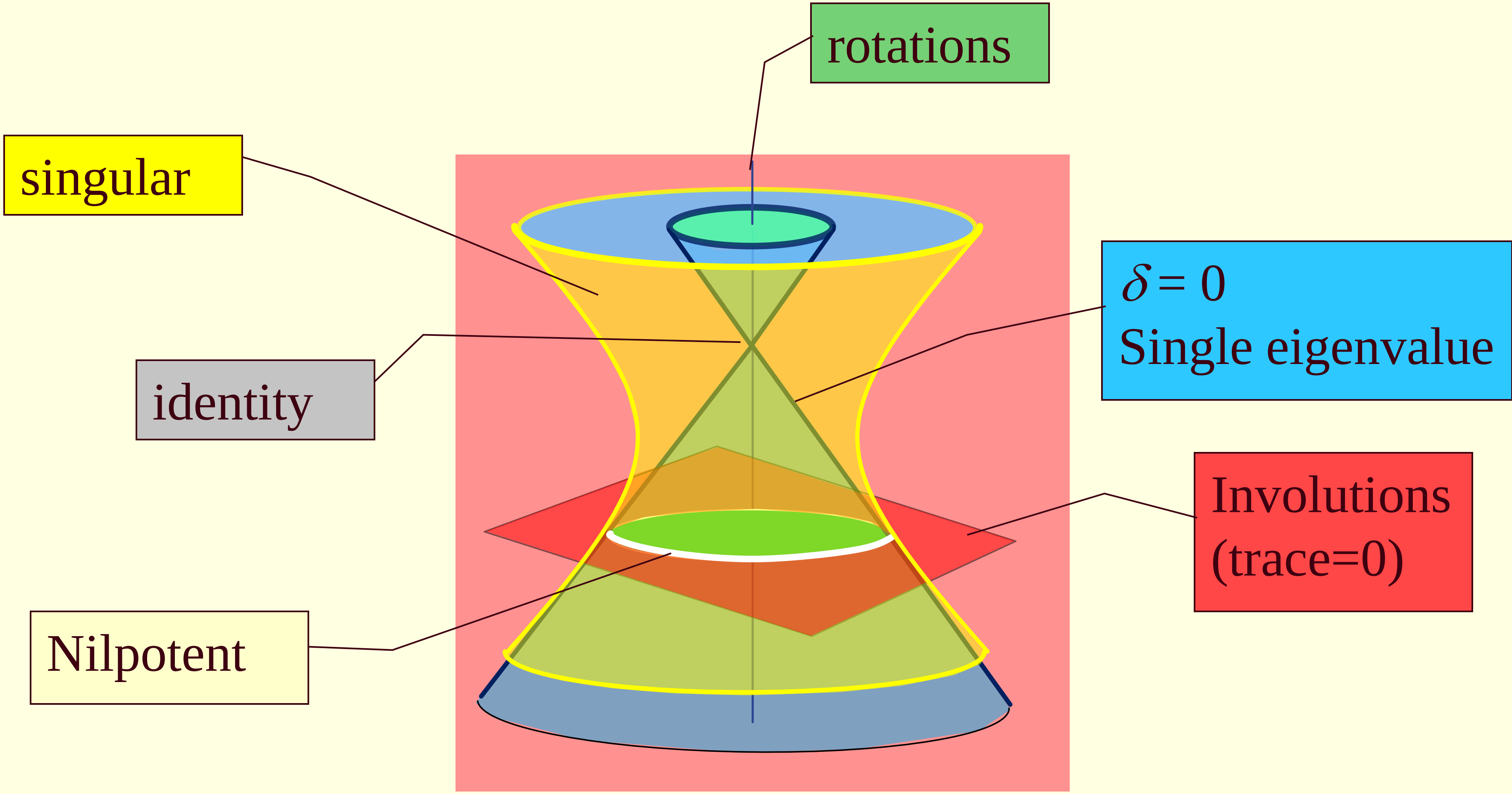
$\theta = 90^\circ$



Cross section



Roadmap of M



rotations
singular
$\delta = 0$
Single eigenvalue
identity
Involutions
(trace=0)
Nilpotent