

CSE590B Lecture 3

More about P^1

Resultants, Division, Syzygies
and Transformations

James F. Blinn

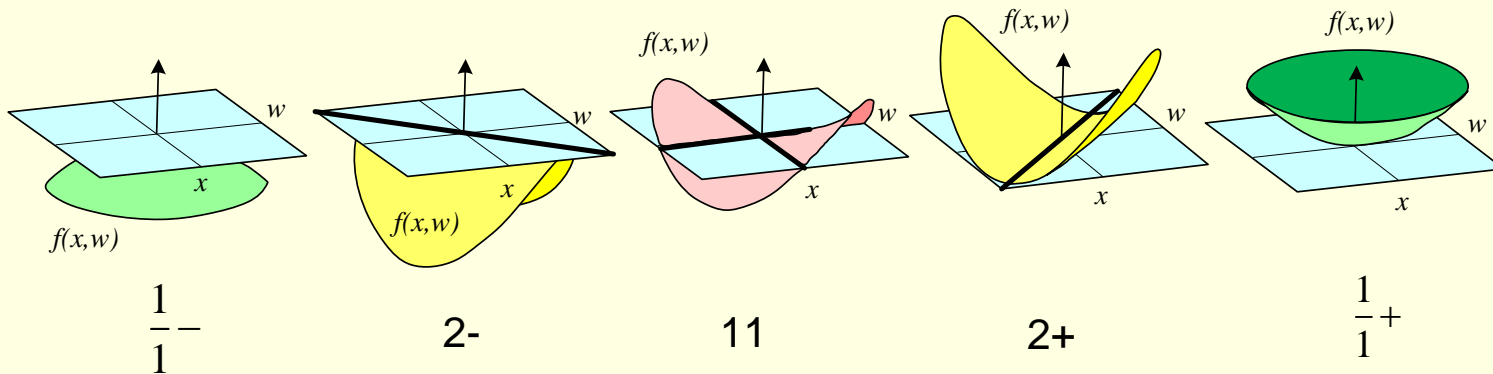
JimBlinn.Com

<http://courses.cs.washington.edu/courses/cse590b/13au/>

Previously On CSE590b

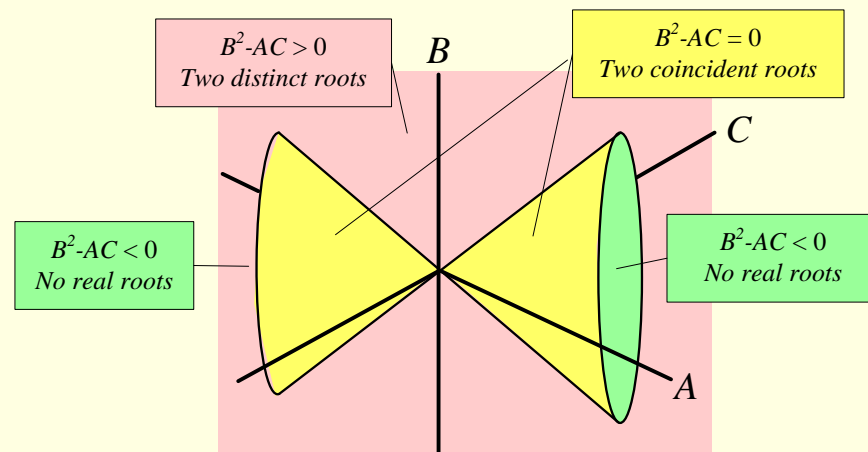
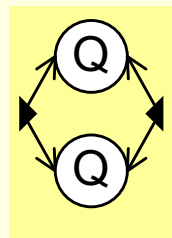
Quadratic Polynomial

$$Q(x, w) = Ax^2 + 2Bxw + Cw^2 = \textcircled{P} \rightarrow \textcircled{Q} \leftarrow \textcircled{P}$$



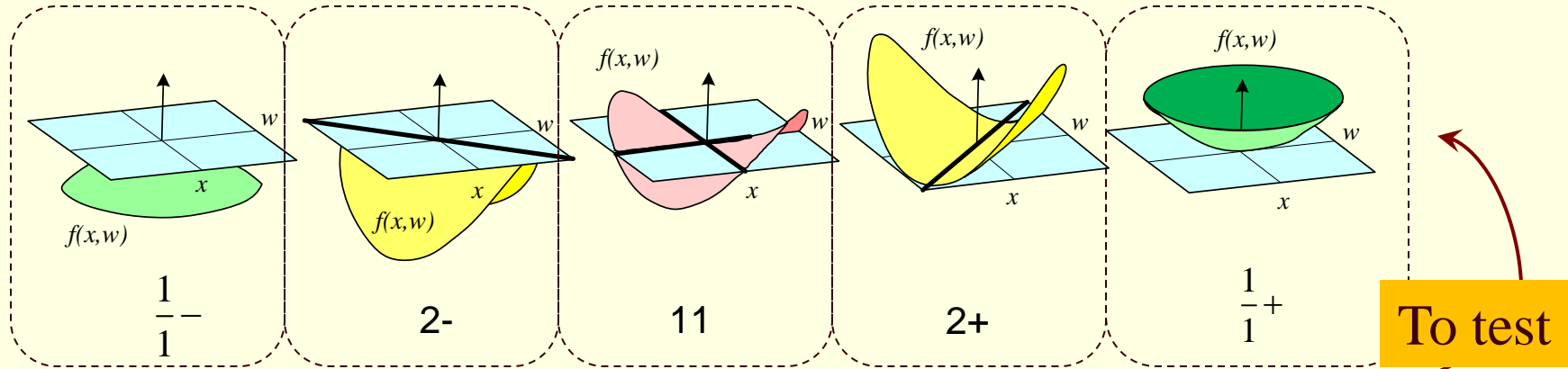
$$\text{discr}(Q) = B^2 - AC$$

$$= -\det \begin{bmatrix} A & B \\ B & C \end{bmatrix} = 1/2$$

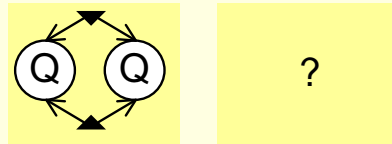


Topics

Equivalence Classes

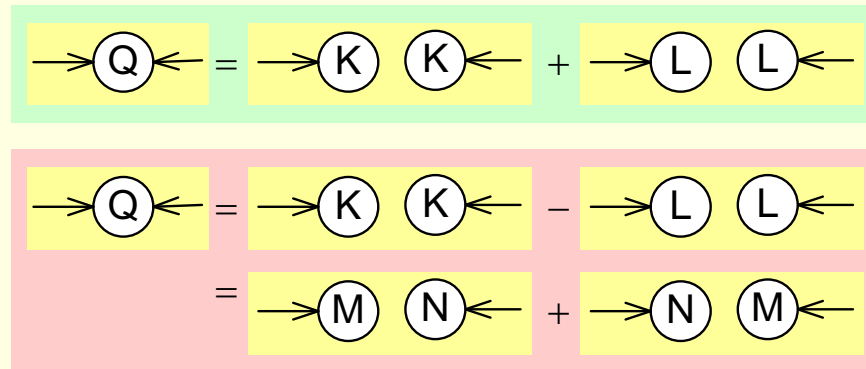


Invariant Diagrams



Plug in

Internal Structure



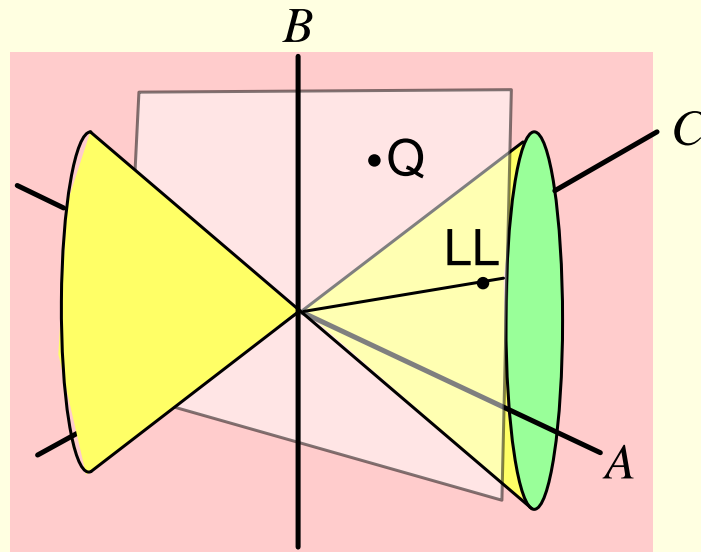
And Now

Locus of Q's containing L as factor

$$Q = KL$$

$$= (K_1x + K_2w)(L_1x + L_2w)$$

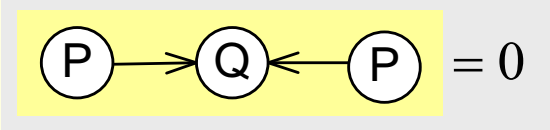
$$= K_1L_1x^2 + (K_1L_2 + K_2L_1)xw + K_2L_2w^2$$



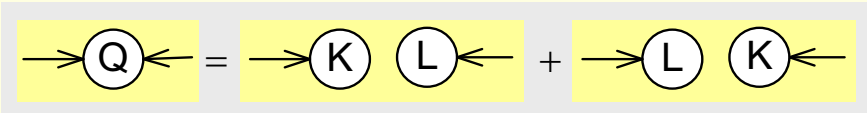
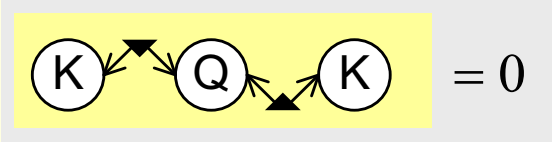
Plane tangent to cone
along line LL

Root vs Factor

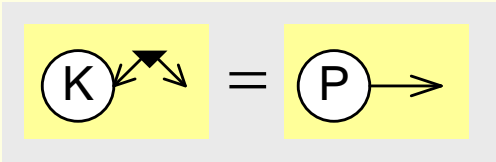
Root



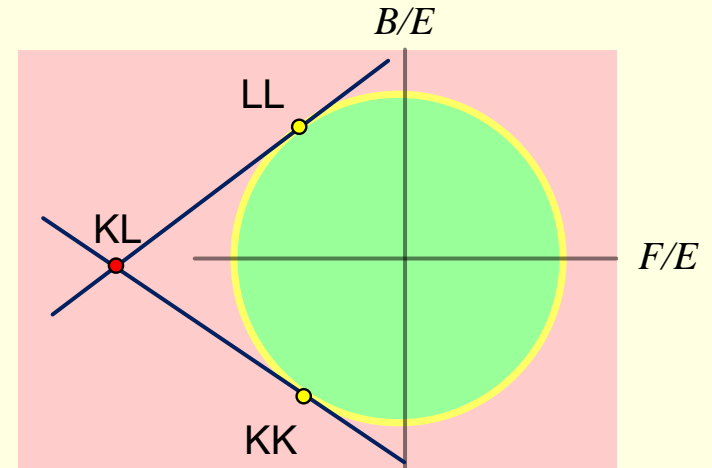
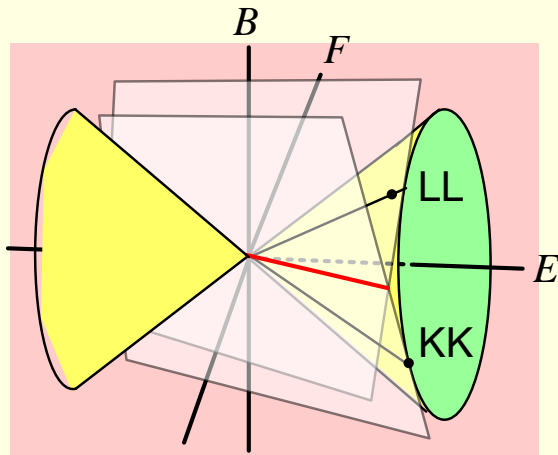
Factor



Relation
between



Intersection of 2 loci

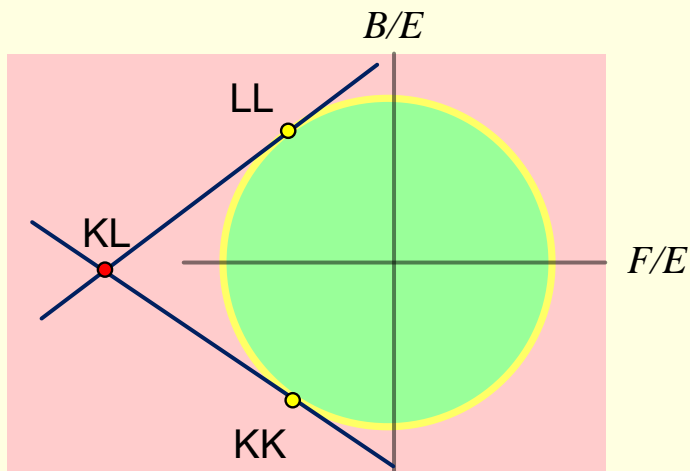


$$\rightarrow \textcircled{Q} \leftarrow = \rightarrow \textcircled{K} \textcircled{L} \leftarrow + \rightarrow \textcircled{L} \textcircled{K} \leftarrow$$

Given Q, find K,L

→ Draw tangents from Q to cone

Another Invariant Test



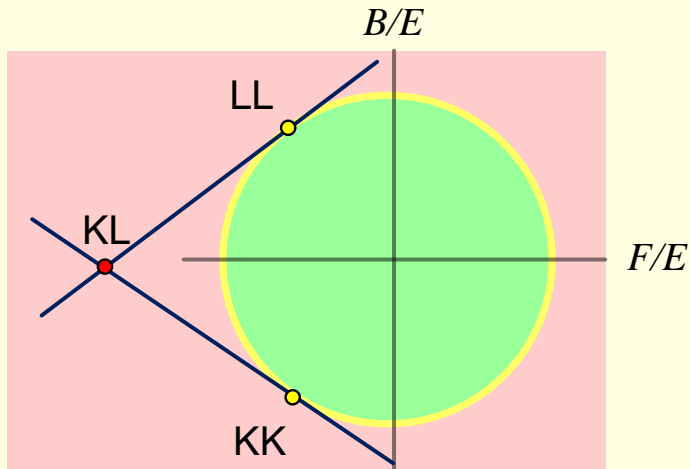
$$\begin{array}{c} \rightarrow \\ \text{Q} \\ \leftarrow \end{array} = \begin{array}{c} \rightarrow \\ \text{K} \quad \text{L} \\ \leftarrow \end{array} + \begin{array}{c} \rightarrow \\ \text{L} \quad \text{K} \\ \leftarrow \end{array}$$

$$\begin{array}{c} \text{Q} \\ \updownarrow \\ \text{Q} \end{array} = \begin{array}{c} \text{K} \quad \text{L} \\ \updownarrow \\ \text{K} \quad \text{L} \end{array} + \begin{array}{c} \text{K} \quad \text{L} \\ \updownarrow \\ \text{L} \quad \text{K} \end{array}$$

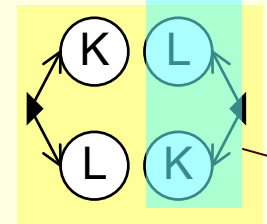
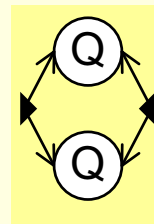
$$+ \begin{array}{c} \text{L} \quad \text{K} \\ \updownarrow \\ \text{K} \quad \text{L} \end{array} + \begin{array}{c} \text{L} \quad \text{K} \\ \updownarrow \\ \text{L} \quad \text{K} \end{array}$$

Grayed out indicates “identically zero”

Another Invariant Test

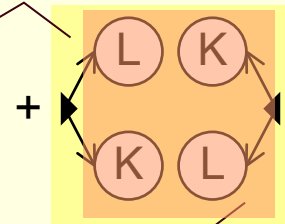


$$\rightarrow \text{Q} \leftarrow = \rightarrow \text{K} \text{ L} \leftarrow + \rightarrow \text{L} \text{ K} \leftarrow$$



rotate

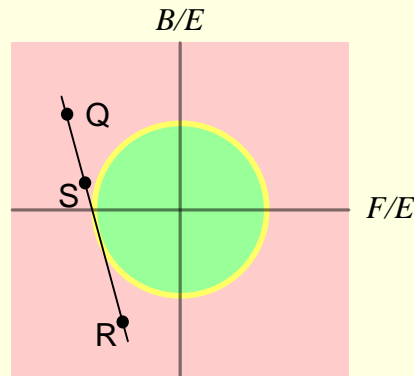
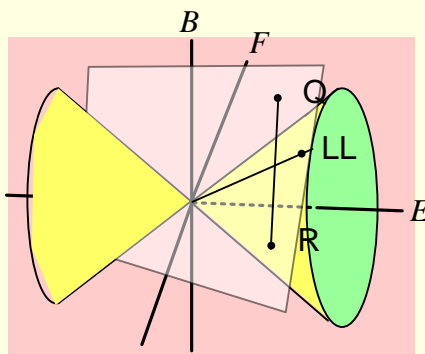
Flip vertically



Flip horizontally

$$\begin{matrix} \text{Q} \\ \text{Q} \end{matrix} = 2 \begin{matrix} \text{K} & \text{K} \\ \text{L} & \text{L} \end{matrix} > 0$$

Do Two Quadratics share a root?

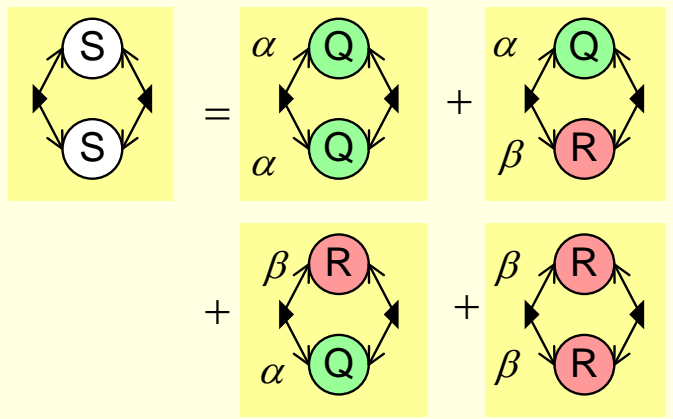


Is line from Q to R

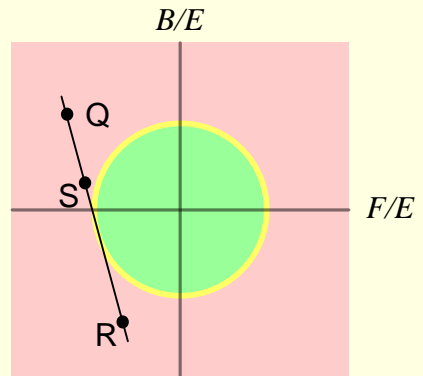
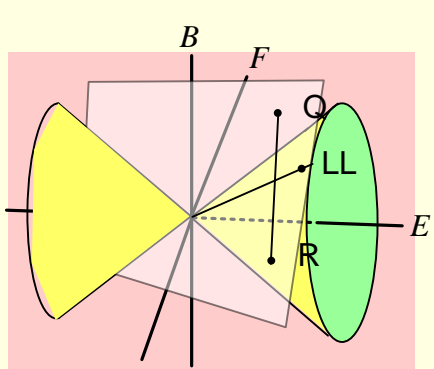
$$\rightarrow \textcircled{S} \leftarrow = \alpha \rightarrow \textcircled{Q} \leftarrow + \beta \rightarrow \textcircled{R} \leftarrow$$

tangent to cone?

$\det(S(\alpha, \beta))$ has a double root



Do Two Quadratics share a root?

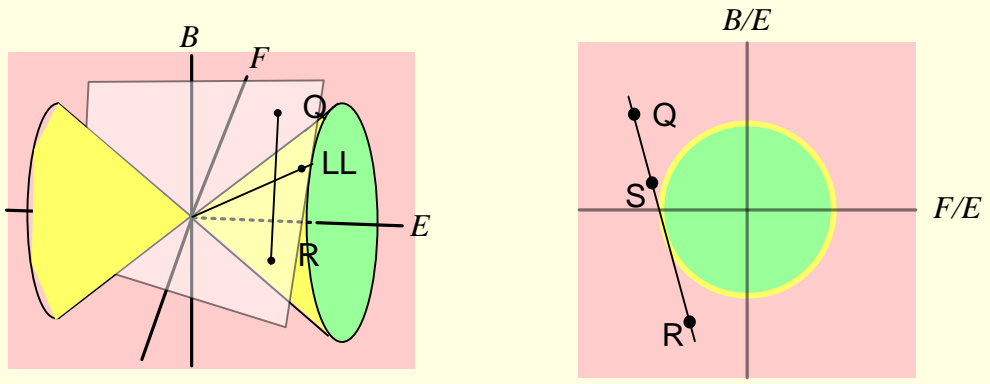


$$\begin{array}{c}
 \begin{array}{|c|} \hline \text{S} \\ \hline \text{S} \\ \hline \end{array} \\
 = \\
 \begin{array}{|c|} \hline \alpha \text{ Q} \\ \hline \alpha \text{ Q} \\ \hline \end{array} + \begin{array}{|c|} \hline \alpha \text{ Q} \\ \hline \beta \text{ R} \\ \hline \end{array} \\
 + \\
 \begin{array}{|c|} \hline \beta \text{ R} \\ \hline \alpha \text{ Q} \\ \hline \end{array} + \begin{array}{|c|} \hline \beta \text{ R} \\ \hline \beta \text{ R} \\ \hline \end{array} \\
 = \alpha^2 \begin{array}{|c|} \hline \text{Q} \\ \hline \text{Q} \\ \hline \end{array} + 2\alpha\beta \begin{array}{|c|} \hline \text{Q} \\ \hline \text{R} \\ \hline \end{array} + \beta^2 \begin{array}{|c|} \hline \text{R} \\ \hline \text{R} \\ \hline \end{array}
 \end{array}$$

Discriminant:

$$\begin{array}{|c|} \hline \text{Q} \\ \hline \text{R} \\ \hline \end{array} \begin{array}{|c|} \hline \text{Q} \\ \hline \text{R} \\ \hline \end{array} - \begin{array}{|c|} \hline \text{Q} \\ \hline \text{Q} \\ \hline \end{array} \begin{array}{|c|} \hline \text{R} \\ \hline \text{R} \\ \hline \end{array}$$

Do Two Quadratics share a root?



Is line from Q to R
tangent to cone?

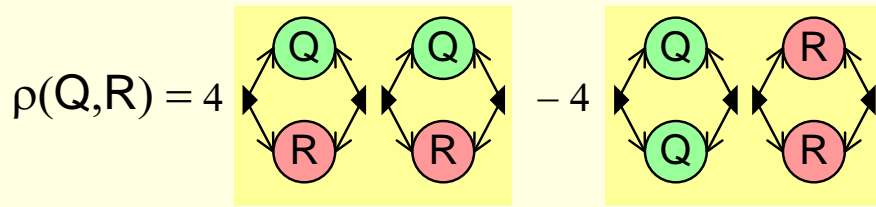
$$\rightarrow \textcircled{S} \leftarrow = \alpha \rightarrow \textcircled{Q} \leftarrow + \beta \rightarrow \textcircled{R} \leftarrow$$

$\det S(\alpha, \beta)$ has a double root

“Resultant”

$$\rho(Q, R) = \begin{matrix} \textcircled{Q} & \textcircled{Q} \\ \textcircled{R} & \textcircled{R} \end{matrix} - \begin{matrix} \textcircled{Q} & \textcircled{R} \\ \textcircled{Q} & \textcircled{R} \end{matrix} = 0$$

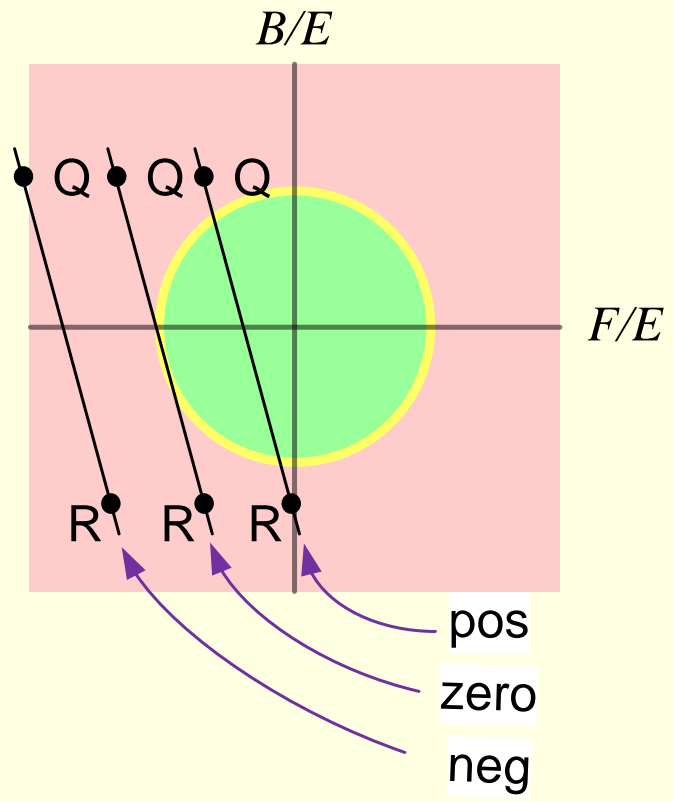
Resultant



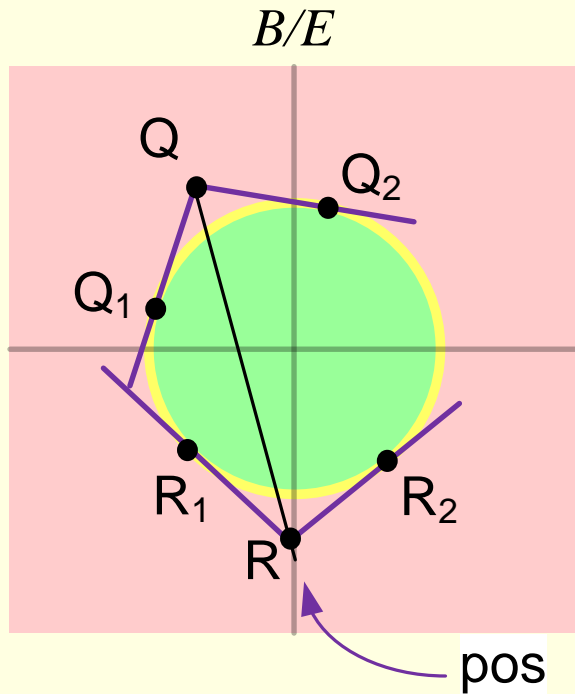
$$\mathbf{Q} = \begin{bmatrix} A_Q & B_Q \\ B_Q & C_Q \end{bmatrix}, \mathbf{R} = \begin{bmatrix} A_R & B_R \\ B_R & C_R \end{bmatrix}$$

$$\rho(\mathbf{Q}, \mathbf{R}) = \det \begin{bmatrix} A_Q & \frac{1}{2}B_Q & C_Q & 0 \\ 0 & A_Q & \frac{1}{2}B_Q & C_Q \\ A_R & \frac{1}{2}B_R & C_R & 0 \\ 0 & A_R & \frac{1}{2}B_R & C_R \end{bmatrix}$$

Sylvester Matrix

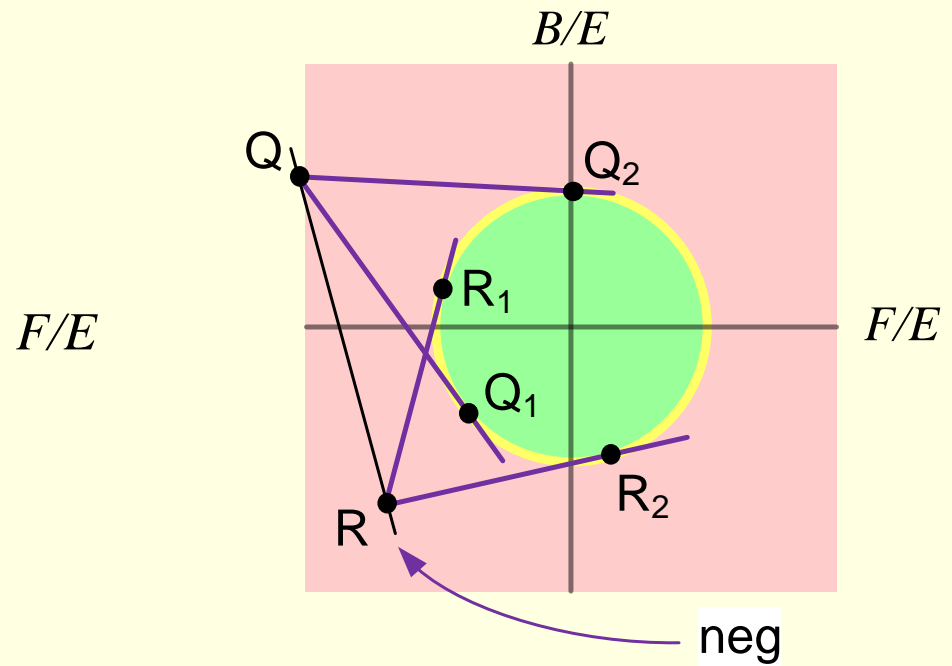


Resultant and root interleaving



$$\rho(Q, R) > 0$$

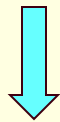
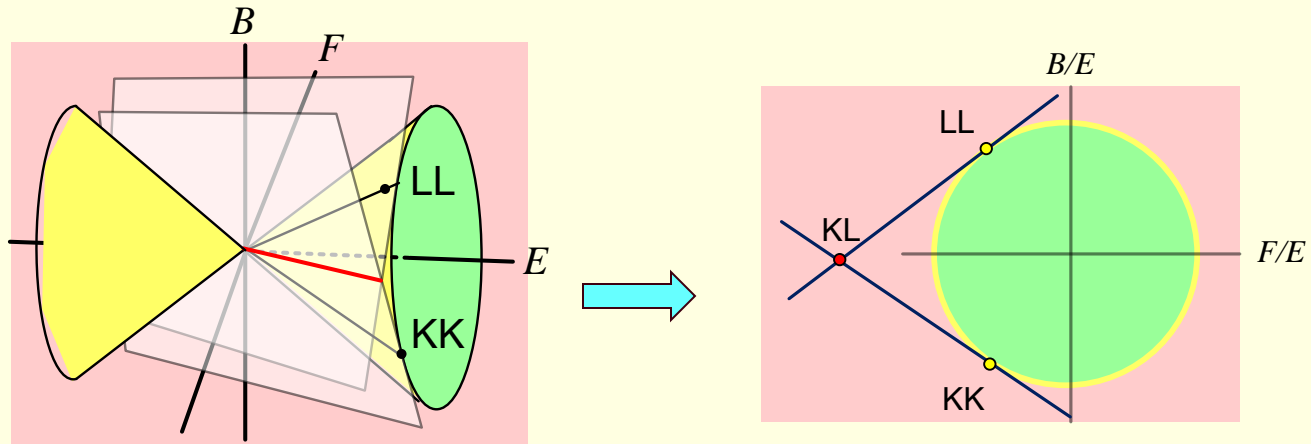
Roots disjoint



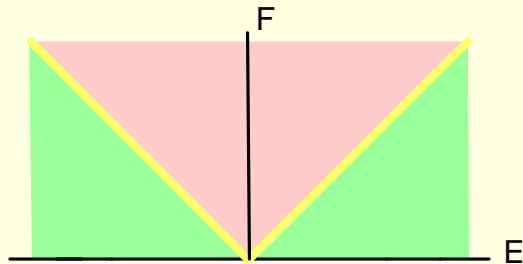
$$\rho(Q, R) < 0$$

Roots interleaved

Two possible mappings 3D->2D



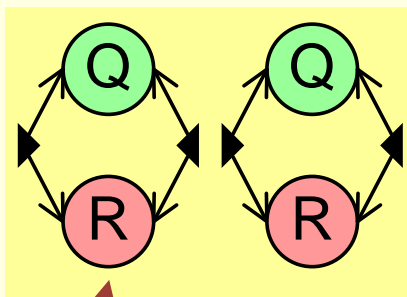
- ☺ Preserves lines
- ☹ Maps + and - cones together



- ☺ + and - cones distinct
- ☹ Lines not preserved

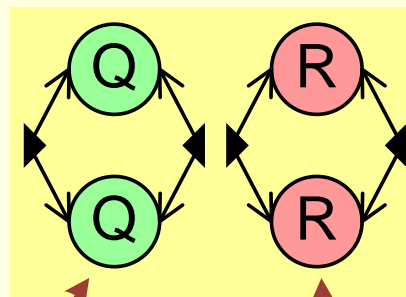
Resultant

$$\rho(Q,R) = 4$$



functional determinant

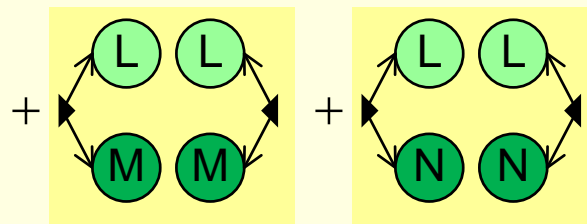
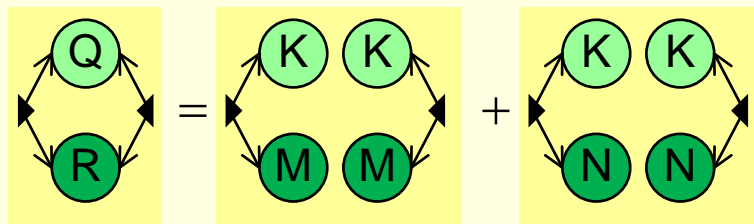
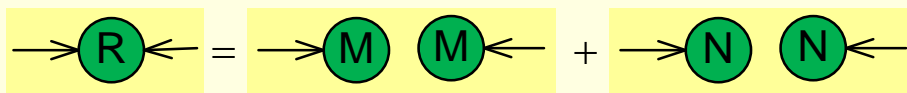
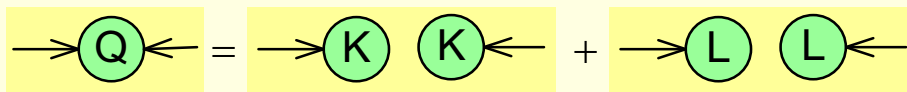
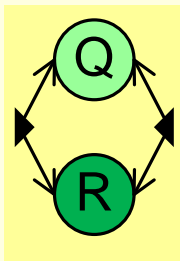
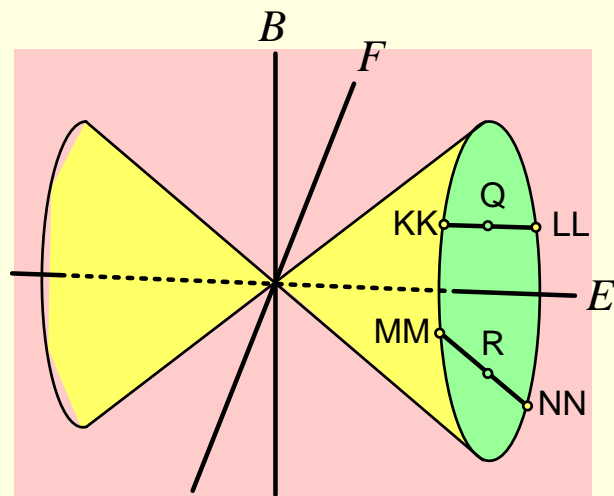
$$-4$$



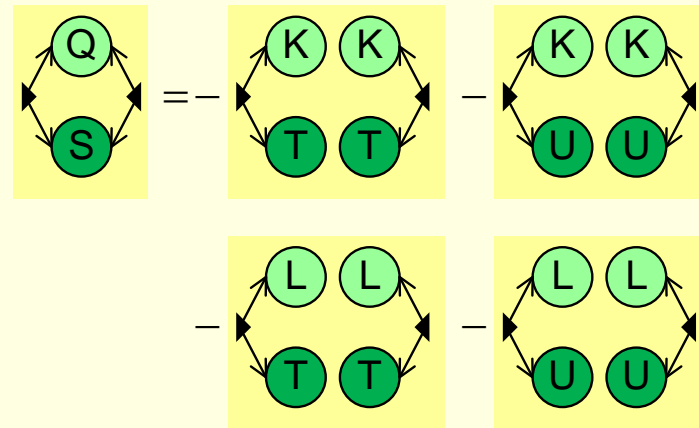
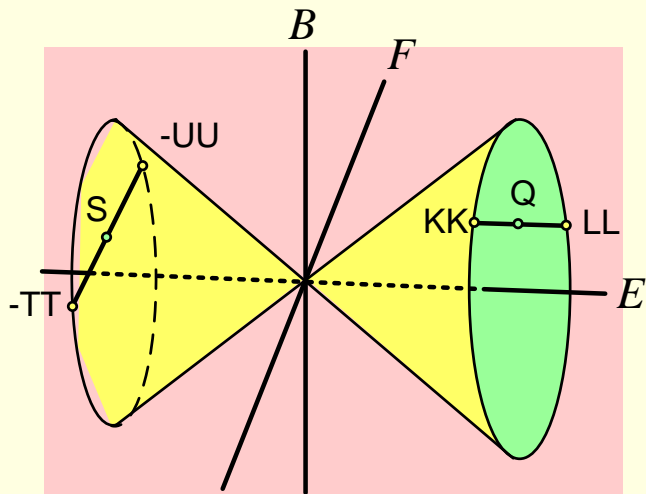
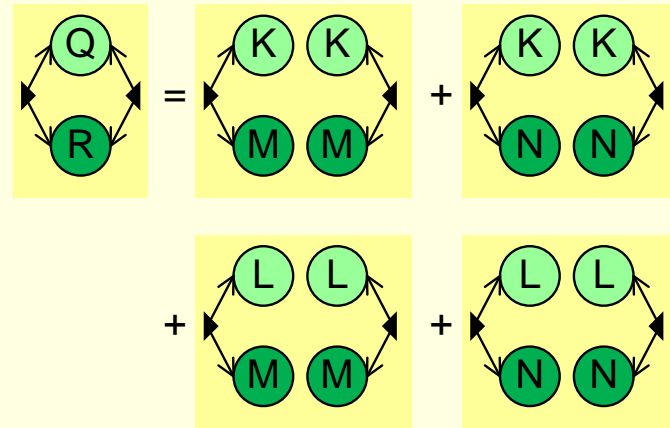
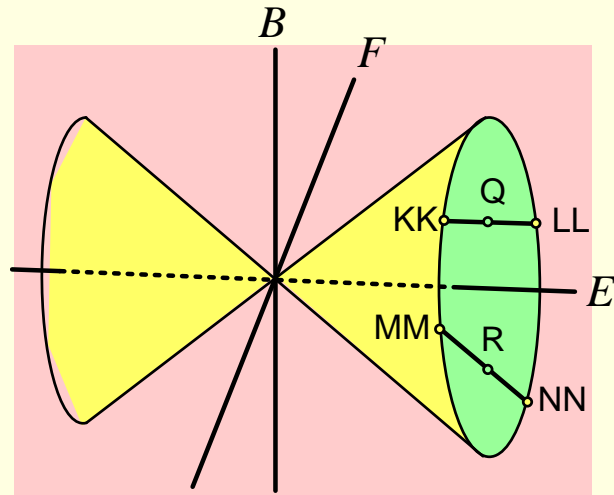
$$-2 \det Q$$

$$-2 \det R$$

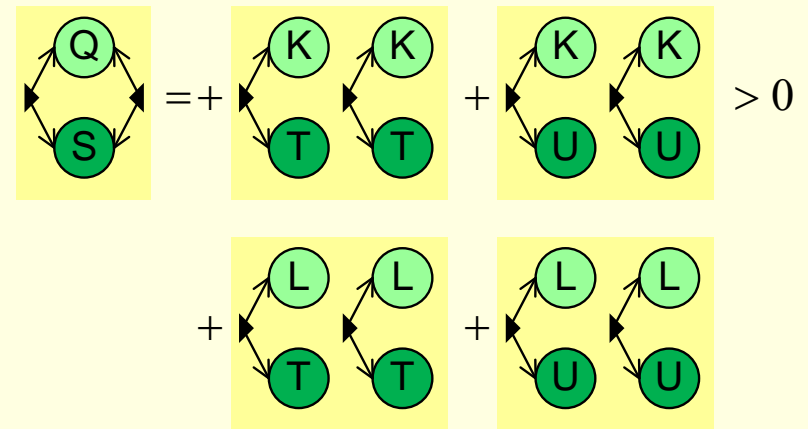
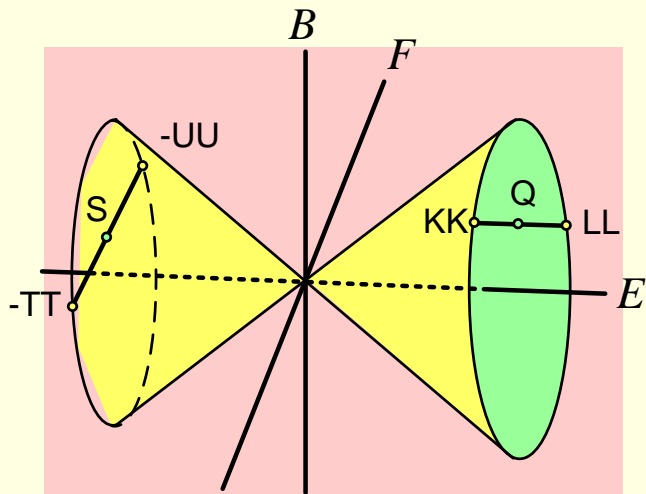
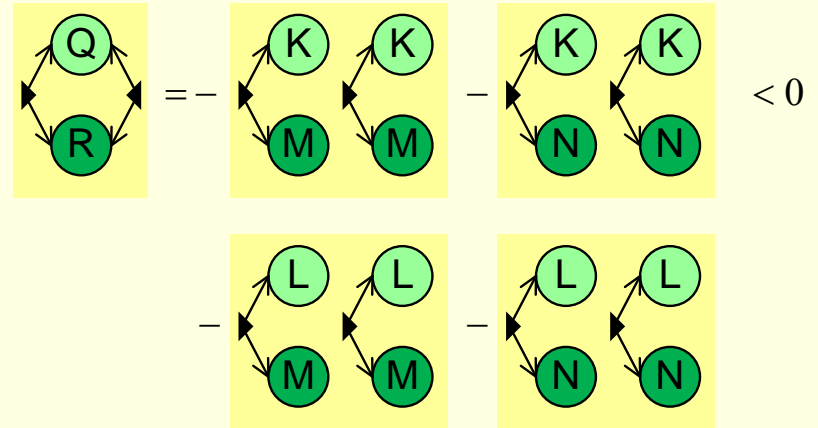
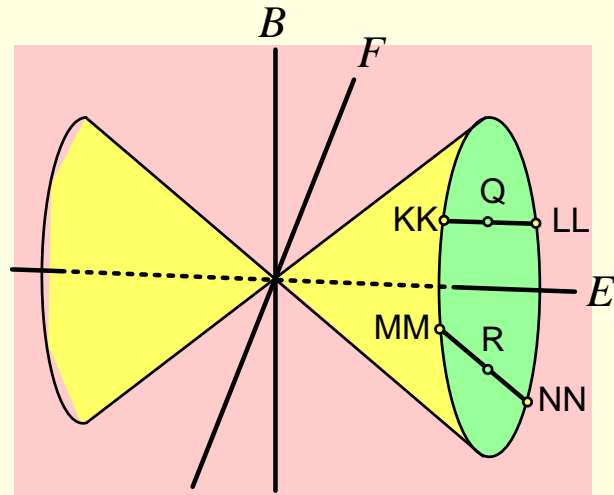
Functional Determinant



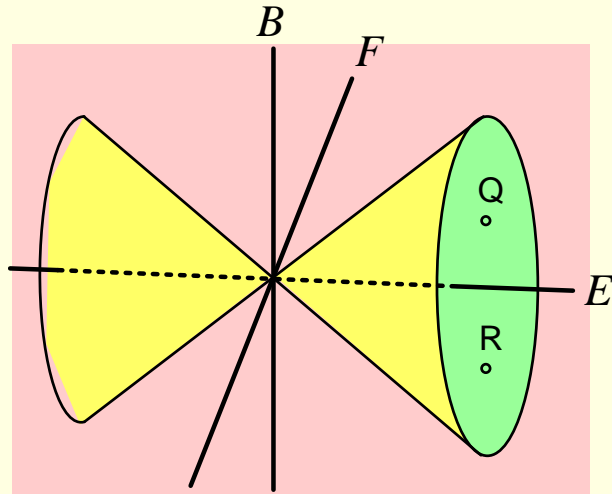
Functional Determinant



Functional Determinant



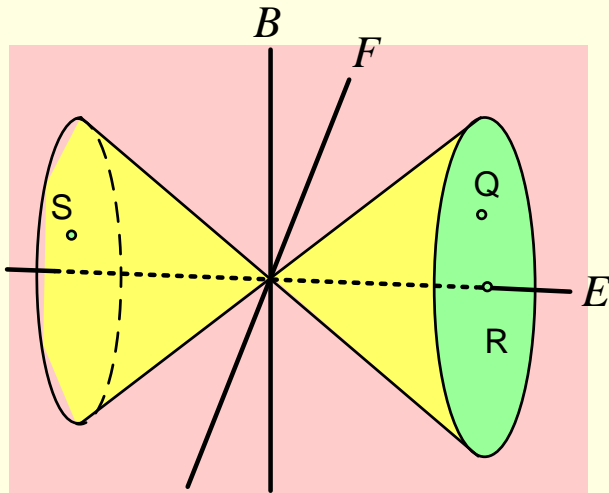
Functional Determinant



$$\mathbf{Q} = \begin{bmatrix} A_Q & B_Q \\ B_Q & C_Q \end{bmatrix}, \mathbf{R} = \begin{bmatrix} A_R & B_R \\ B_R & C_R \end{bmatrix}$$

$$= -A_Q C_R + 2B_Q B_R - C_Q A_R$$

$$= \begin{bmatrix} A_Q & B_Q & C_Q \end{bmatrix} \begin{bmatrix} -C_R \\ 2B_R \\ -A_R \end{bmatrix}$$



$$\mathbf{Q} = \begin{bmatrix} A_Q & B_Q \\ B_Q & C_Q \end{bmatrix}, \mathbf{R} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= -(A_Q + C_Q) = -2E_Q = -2\text{trace}(\mathbf{Q})$$

Categorizing Equivalence Classes

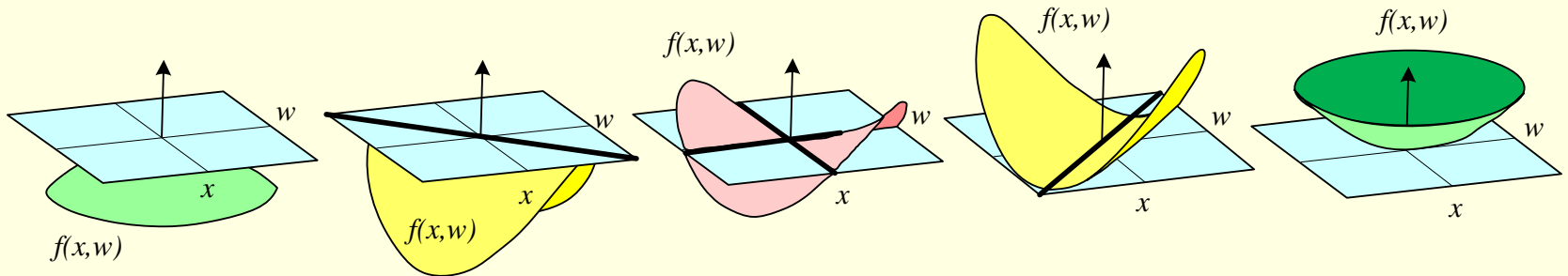
$$\begin{array}{c} \rightarrow \\ \text{Q} \\ \leftarrow \end{array} = \begin{bmatrix} A & B \\ B & C \end{bmatrix}$$

$$\begin{array}{c} \text{Q} \\ \swarrow \quad \searrow \\ \text{Q} \\ \swarrow \quad \searrow \end{array} = -2(AC - B^2) = -2 \det(\mathbf{Q})$$

$$\begin{array}{c} \text{Q} \\ \swarrow \quad \searrow \\ \text{I} \\ \swarrow \quad \searrow \end{array} = -(A + C) = -2E = -2 \text{trace}(\mathbf{Q})$$

But could use any quadratic
in positive cone

Categorizing Equivalence Classes



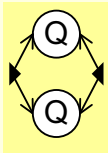
$\frac{1}{1}^-$

2-

11

2+

$\frac{1}{1}^+$



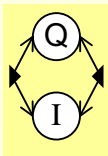
-

0

+

0

-



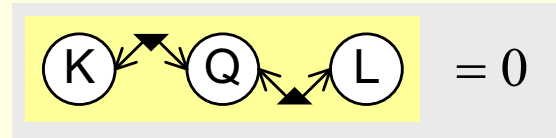
+

+

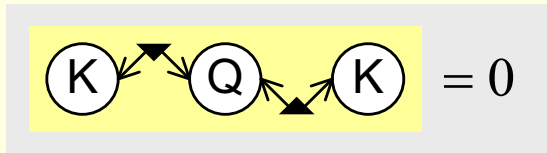
-

-

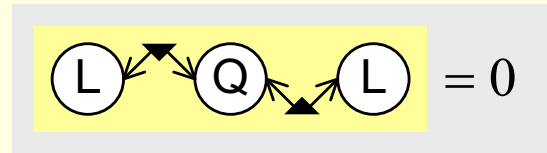
What does this mean?



We already know the meaning of:

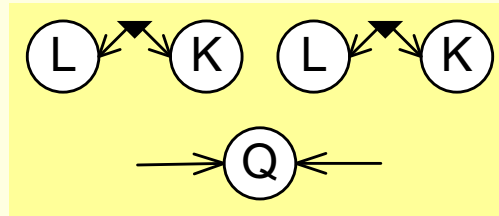


K is a factor of Q (Q is type 11)



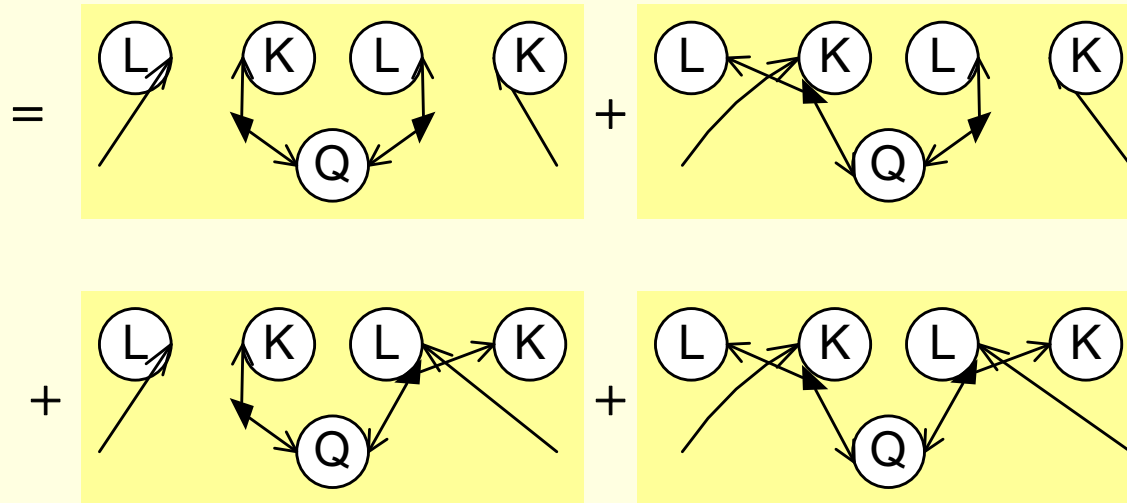
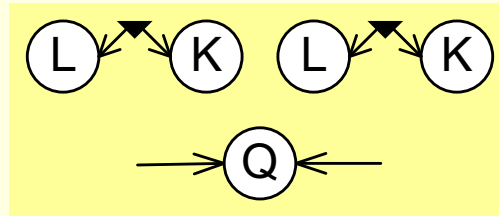
L is a factor of Q (Q is type 11)

An Identity

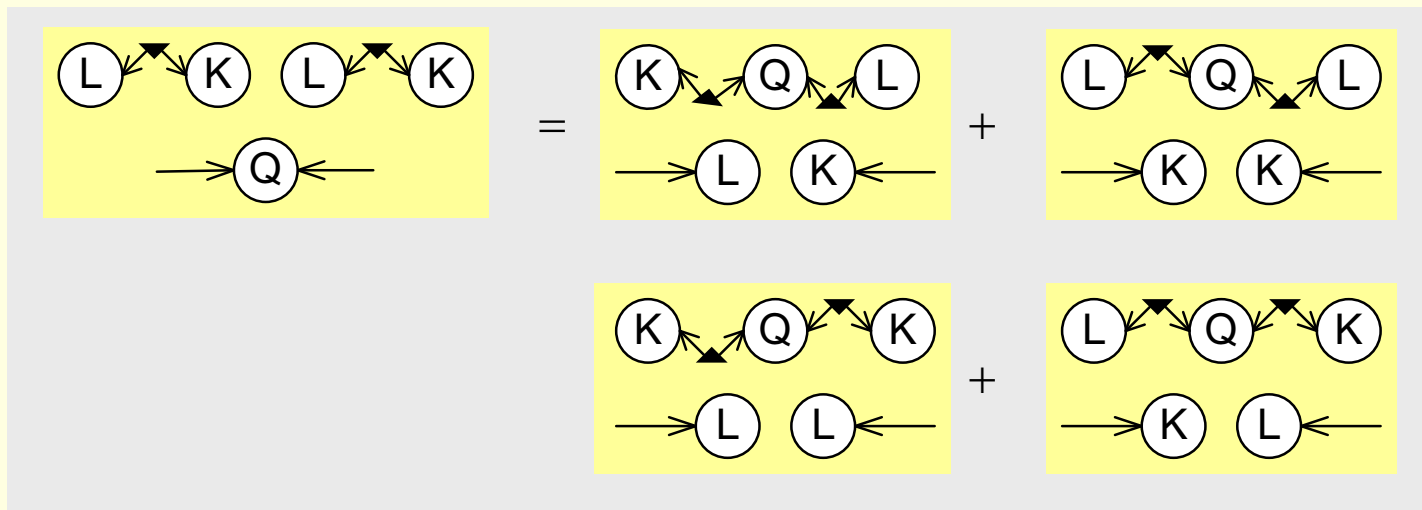


Visio Demo

An Identity



An Identity

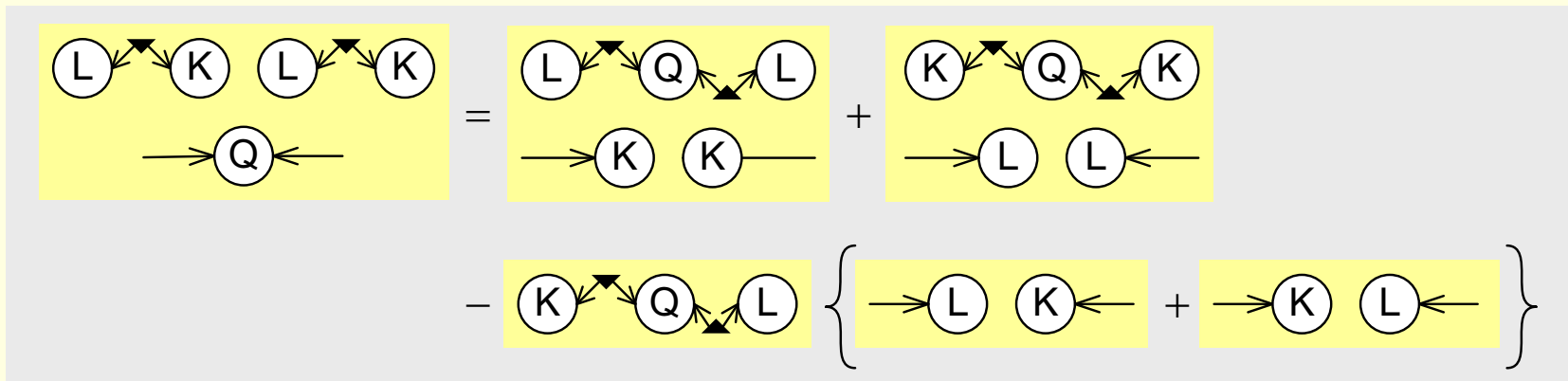
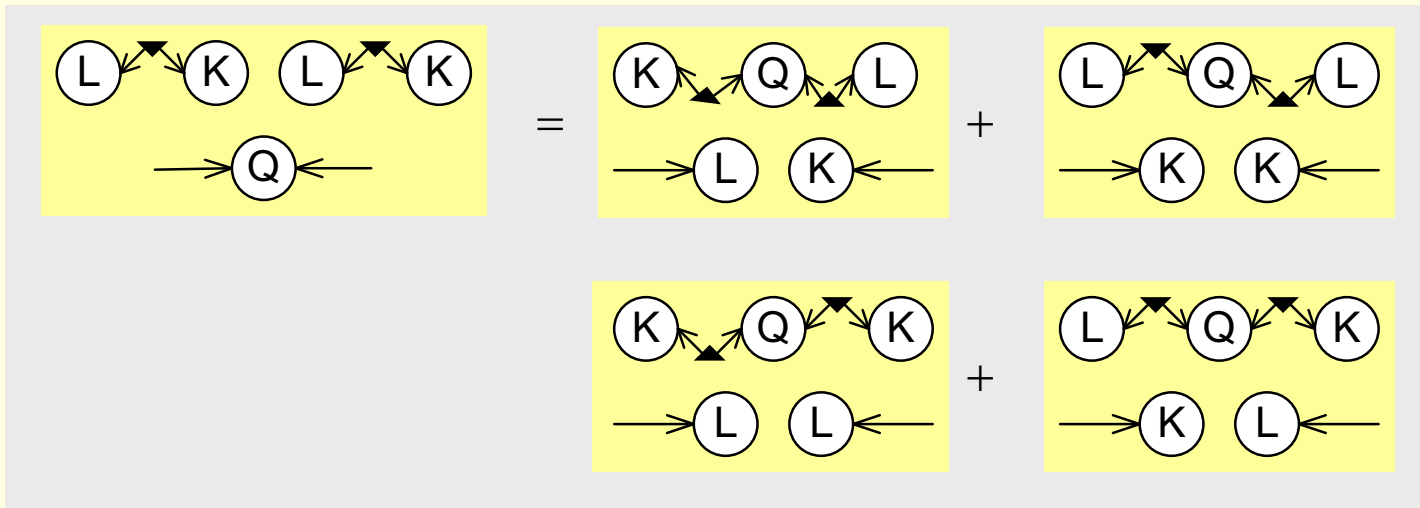


True for all Q, K, L

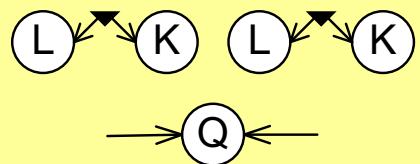
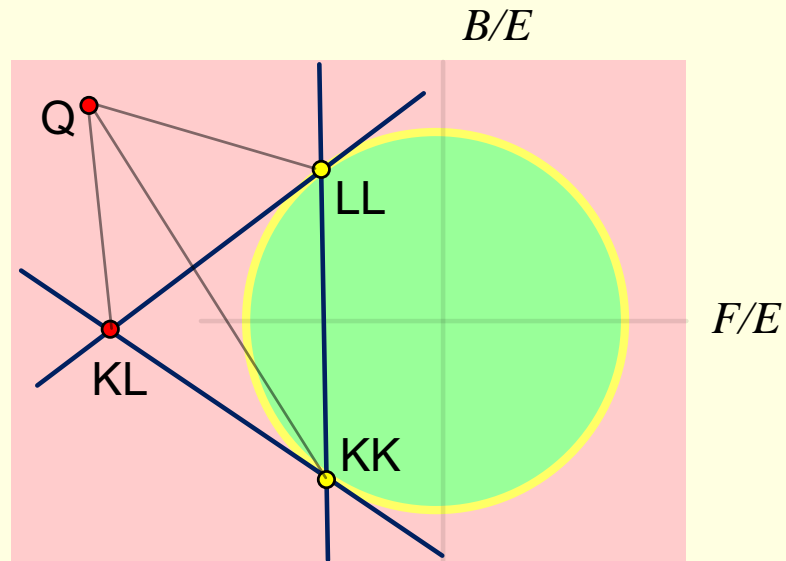
Each term has same number of Q, K, L
Just connected differently

Called a “Syzygy”

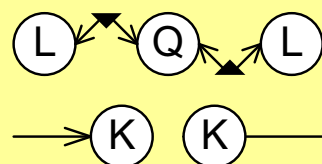
Rearrange Syzygy



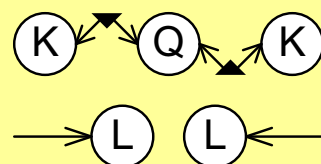
Interpret Syzygy



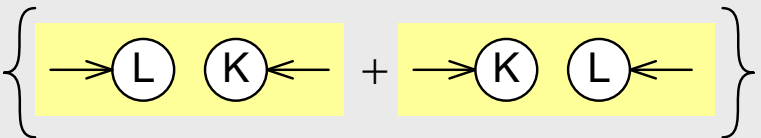
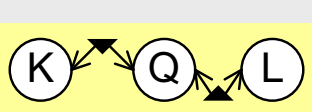
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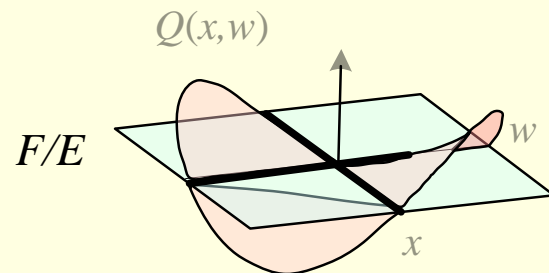
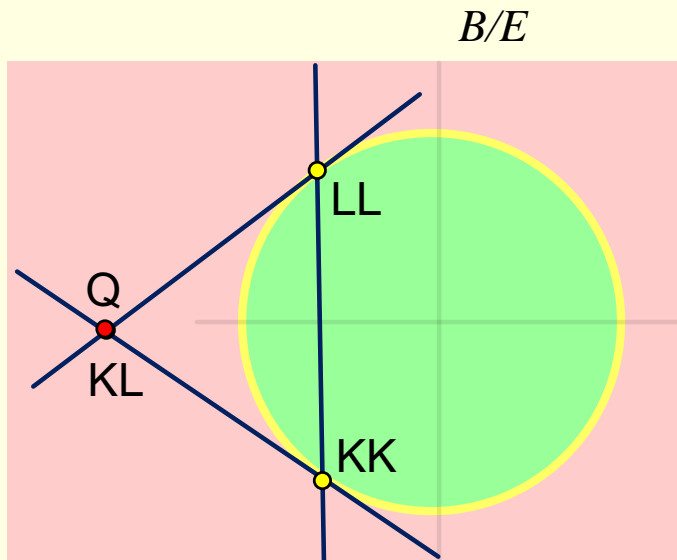


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Interpret Syzygy

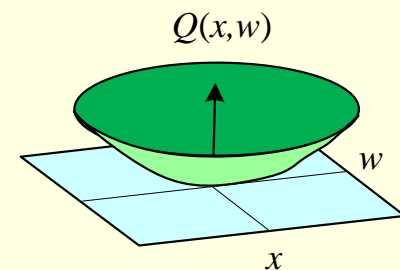
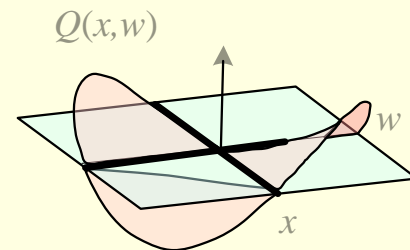
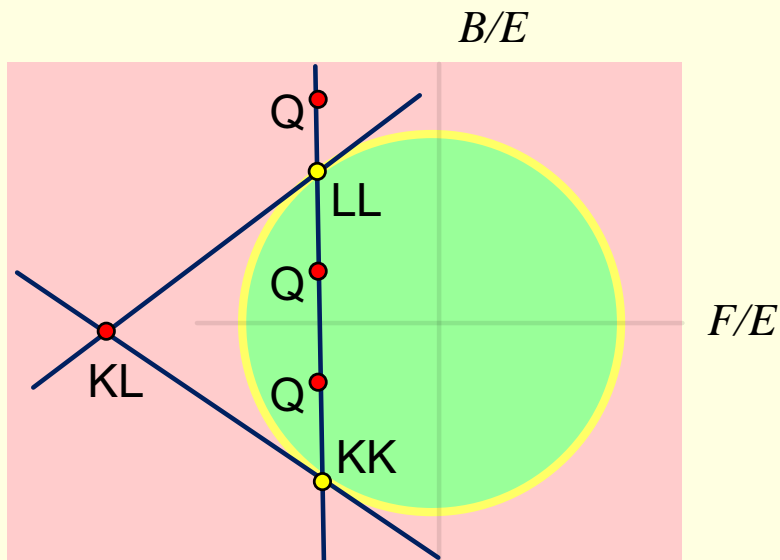
if $\begin{array}{c} \textcircled{K} \rightleftarrows \textcircled{Q} \rightleftarrows \textcircled{K} \\ \textcircled{L} \rightleftarrows \textcircled{Q} \rightleftarrows \textcircled{L} \end{array} = 0$
 and $\begin{array}{c} \textcircled{L} \rightleftarrows \textcircled{Q} \rightleftarrows \textcircled{L} \\ \textcircled{K} \rightleftarrows \textcircled{Q} \rightleftarrows \textcircled{K} \end{array} = 0$



$$\begin{array}{c} \textcircled{L} \rightleftarrows \textcircled{K} \quad \textcircled{L} \rightleftarrows \textcircled{K} \\ \textcircled{Q} \end{array} = \begin{array}{c} \textcircled{L} \rightleftarrows \textcircled{Q} \rightleftarrows \textcircled{L} \\ \textcircled{K} \quad \textcircled{K} \end{array} + \begin{array}{c} \textcircled{K} \rightleftarrows \textcircled{Q} \rightleftarrows \textcircled{K} \\ \textcircled{L} \quad \textcircled{L} \end{array} - \left\{ \begin{array}{c} \textcircled{K} \rightleftarrows \textcircled{Q} \rightleftarrows \textcircled{L} \\ \textcircled{L} \quad \textcircled{K} \end{array} + \begin{array}{c} \textcircled{K} \rightleftarrows \textcircled{Q} \rightleftarrows \textcircled{L} \\ \textcircled{K} \quad \textcircled{L} \end{array} \right\}$$

Interpret Syzygy

if $\begin{matrix} \text{K} & \text{Q} & \text{L} \\ \swarrow & \nearrow & \swarrow \\ & \text{Q} & \\ \searrow & \nwarrow & \searrow \\ \text{K} & \text{L} & \end{matrix} = 0$



$$\begin{matrix} \text{L} & \text{K} & \text{L} & \text{K} \\ \swarrow & \nearrow & \swarrow & \nearrow \\ & \text{Q} & \\ \searrow & \nwarrow & \searrow & \nearrow \\ \text{L} & \text{K} & \text{L} & \text{K} \end{matrix} = \begin{matrix} \text{L} & \text{Q} & \text{L} \\ \swarrow & \nearrow & \swarrow \\ & \text{K} & \text{K} \\ \searrow & \nwarrow & \searrow \\ \text{K} & \text{K} & \end{matrix} + \begin{matrix} \text{K} & \text{Q} & \text{K} \\ \swarrow & \nearrow & \swarrow \\ & \text{L} & \text{L} \\ \searrow & \nwarrow & \searrow \\ \text{L} & \text{L} & \end{matrix}$$

$$- \begin{matrix} \text{K} & \text{Q} & \text{L} \\ \swarrow & \nearrow & \swarrow \\ & \text{Q} & \\ \searrow & \nwarrow & \searrow \\ \text{K} & \text{L} & \end{matrix} \left\{ \begin{matrix} \text{L} & \text{K} \\ \swarrow & \nearrow \\ & \text{Q} \\ \searrow & \nwarrow \\ \text{K} & \text{L} \end{matrix} + \begin{matrix} \text{K} & \text{L} \\ \swarrow & \nearrow \\ & \text{Q} \\ \searrow & \nwarrow \\ \text{L} & \text{K} \end{matrix} \right\}$$

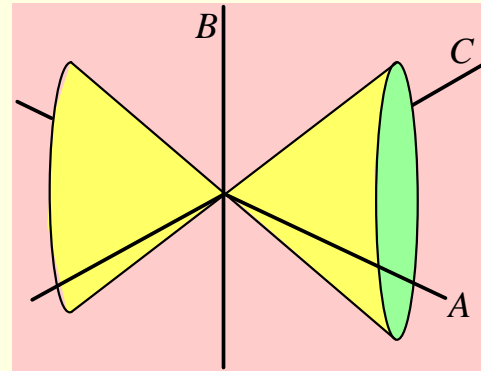
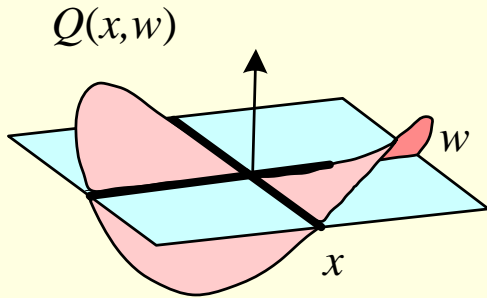
$$\begin{matrix} \text{Q} \\ \swarrow & \nearrow \\ & \text{Q} \\ \searrow & \nwarrow \\ \text{K} & \text{L} \end{matrix}, \quad \begin{matrix} \text{K} & \text{K} \\ \swarrow & \nearrow \\ & \text{Q} \\ \searrow & \nwarrow \\ \text{K} & \text{K} \end{matrix}, \quad \begin{matrix} \text{L} & \text{L} \\ \swarrow & \nearrow \\ & \text{Q} \\ \searrow & \nwarrow \\ \text{L} & \text{L} \end{matrix}$$

Are linearly dependent

Two ways to look at Q

$$\left. \begin{array}{c} \rightarrow \\ \text{Q} \\ \leftarrow \end{array} \right\} = \begin{bmatrix} A & B \\ B & C \end{bmatrix}$$

$$\text{Q} \rightarrow = [A \quad B \quad C]$$



Two ways to look at Q

$$\rightarrow \text{Q} \leftarrow = \begin{bmatrix} A & B \\ B & C \end{bmatrix}$$

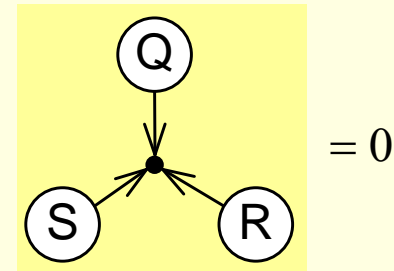
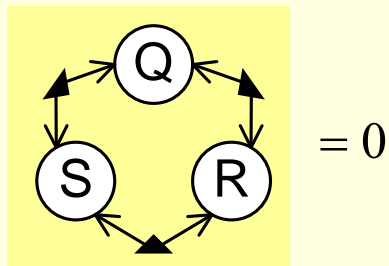
$$\text{Q} \rightarrow = [A \quad B \quad C]$$

Linearly dependent

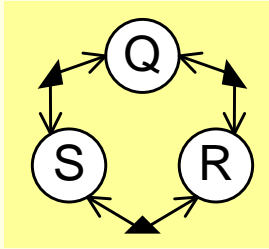
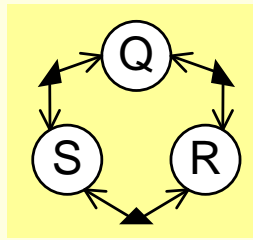
$$\alpha \rightarrow \text{Q} \leftarrow + \beta \rightarrow \text{R} \leftarrow + \gamma \rightarrow \text{S} \leftarrow = 0$$

$$\alpha \text{Q} \rightarrow + \beta \text{R} \rightarrow + \gamma \text{S} \rightarrow = 0$$

If



What is



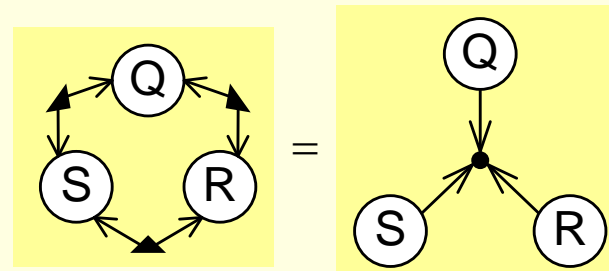
$$= \text{trace}(\varepsilon \mathbf{Q} \varepsilon \mathbf{R} \varepsilon \mathbf{S})$$

$$\mathbf{Q} = \begin{bmatrix} A_Q & B_Q \\ B_Q & C_Q \end{bmatrix}, \mathbf{R} = \begin{bmatrix} A_R & B_R \\ B_R & C_R \end{bmatrix}, \mathbf{S} = \begin{bmatrix} A_S & B_S \\ B_S & C_S \end{bmatrix}$$

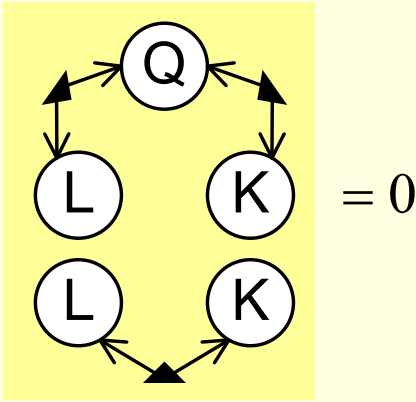
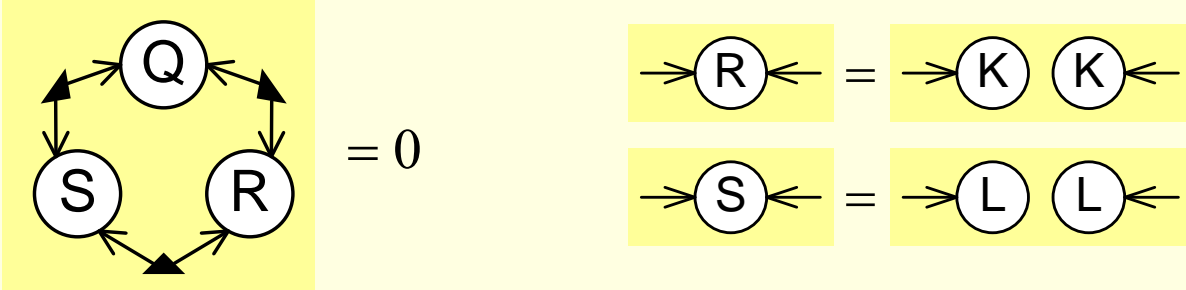
$$= \text{trace} \left\{ \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} A_Q & B_Q \\ B_Q & C_Q \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} A_R & B_R \\ B_R & C_R \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} A_S & B_S \\ B_S & C_S \end{bmatrix} \right\}$$

$$= -A_Q B_R C_R - B_Q C_R A_S - C_Q A_R B_S + A_Q C_R B_S + B_Q A_R C_R + C_Q B_R A_S$$

$$= -\det \begin{bmatrix} A_Q & B_Q & C_Q \\ A_R & B_R & C_R \\ A_S & B_S & C_R \end{bmatrix}$$



Linear Dependence of Q, KK, LL



The reverse direction

If $\alpha \begin{array}{c} \rightarrow \\ \text{Q} \\ \leftarrow \end{array} + \beta \begin{array}{c} \rightarrow \\ \text{R} \\ \leftarrow \end{array} + \gamma \begin{array}{c} \rightarrow \\ \text{S} \\ \leftarrow \end{array} = 0$

$$\begin{array}{c} \rightarrow \\ \text{S} \\ \leftarrow \end{array} = -\alpha/\gamma \begin{array}{c} \rightarrow \\ \text{Q} \\ \leftarrow \end{array} - \beta/\gamma \begin{array}{c} \rightarrow \\ \text{R} \\ \leftarrow \end{array}$$

Then

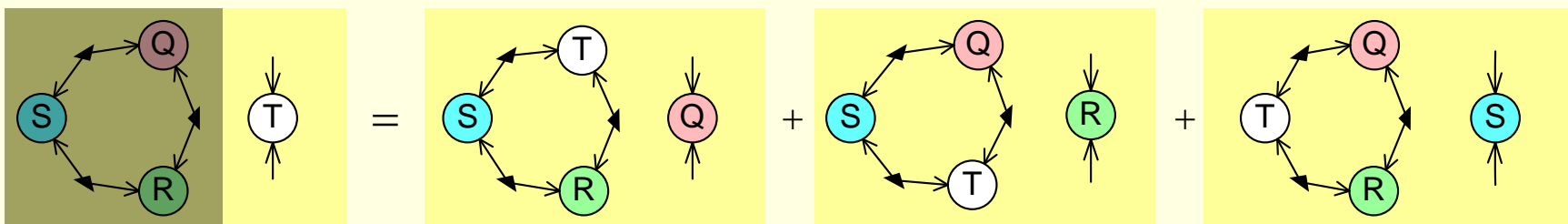
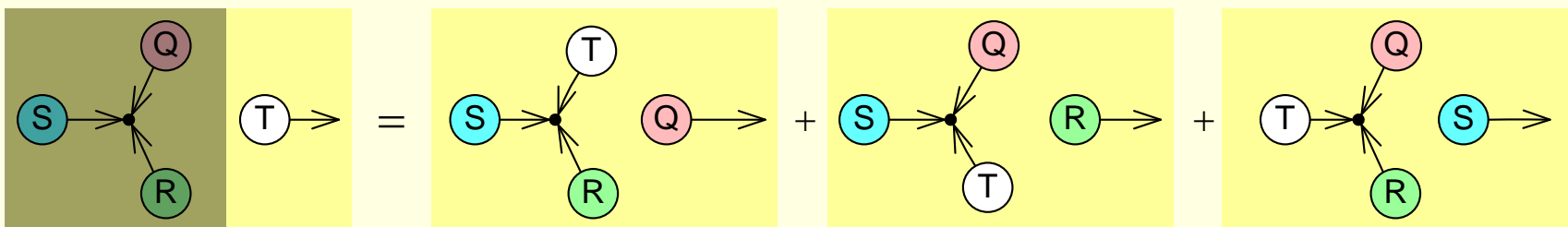
$$\begin{array}{c} \text{Q} \\ \updownarrow \\ \text{S} \quad \text{R} \end{array} = -\alpha/\gamma \underbrace{\begin{array}{c} \text{Q} \\ \updownarrow \\ \text{Q} \quad \text{R} \end{array}}_{=0} - \beta/\gamma \underbrace{\begin{array}{c} \text{Q} \\ \updownarrow \\ \text{R} \quad \text{R} \end{array}}_{=0}$$

What are the scale factors?

$$\boxed{0 \rightarrow} = \alpha \boxed{Q \rightarrow} + \beta \boxed{R \rightarrow} + \gamma \boxed{S \rightarrow}$$

$$\boxed{\rightarrow 0 \leftarrow} = \alpha \boxed{\rightarrow Q \leftarrow} + \beta \boxed{\rightarrow R \leftarrow} + \gamma \boxed{\rightarrow S \leftarrow}$$

Use the Cramer's Rule identity



Roots of Q

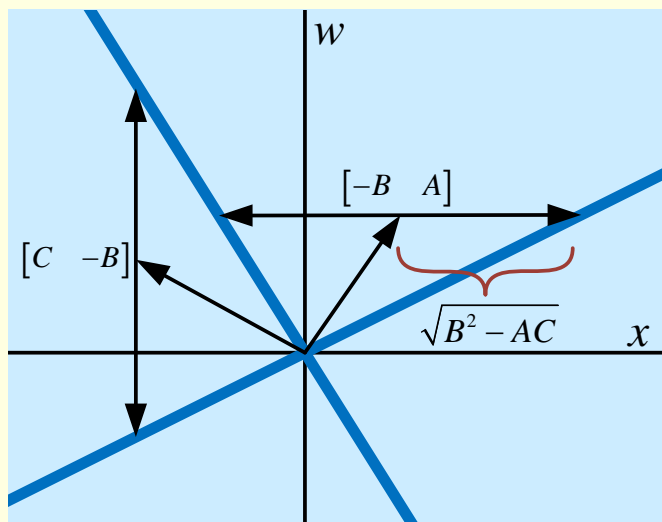
$$Q(x, w) = Ax^2 + 2Bxw + Cw^2 = \textcircled{P} \rightarrow \textcircled{Q} \leftarrow \textcircled{P}$$

$$\frac{x}{w} = \frac{-2B \pm \sqrt{(2B)^2 - 4AC}}{2A}$$

$$\begin{bmatrix} x & w \end{bmatrix} = \begin{bmatrix} -B \pm \sqrt{B^2 - AC} & A \end{bmatrix}$$

$$\frac{w}{x} = \frac{-2B \pm \sqrt{(2B)^2 - 4CA}}{2C}$$

$$\begin{bmatrix} x & w \end{bmatrix} = \begin{bmatrix} C & -B \pm \sqrt{B^2 - AC} \end{bmatrix}$$

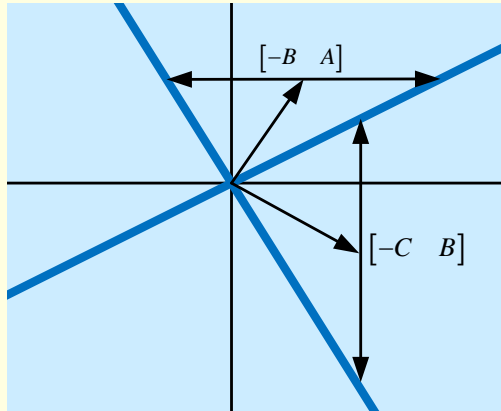


$$\begin{bmatrix} -B + \sqrt{B^2 - AC} & A \end{bmatrix} \leftrightarrow \begin{bmatrix} C & -B - \sqrt{B^2 - AC} \end{bmatrix} = 0$$

Note: different signs

Roots of Q

$$Q(x, w) = Ax^2 + 2Bxw + Cw^2 = \textcircled{P} \rightarrow \textcircled{Q} \leftarrow \textcircled{P}$$



$$\begin{bmatrix} x & w \end{bmatrix} = \begin{bmatrix} -B \pm \sqrt{B^2 - AC} & A \end{bmatrix}$$

$$\begin{bmatrix} x & w \end{bmatrix} = \begin{bmatrix} -C & B \pm \sqrt{B^2 - AC} \end{bmatrix}$$

$$\begin{bmatrix} x & w \end{bmatrix} = \alpha \begin{bmatrix} -B \pm \sqrt{B^2 - AC} & A \end{bmatrix} + \beta \begin{bmatrix} -C & B \pm \sqrt{B^2 - AC} \end{bmatrix}$$

$$\begin{bmatrix} x & w \end{bmatrix} = \begin{bmatrix} \alpha & \beta \end{bmatrix} \left\{ \begin{bmatrix} -B & A \\ -C & B \end{bmatrix} \pm \sqrt{B^2 - AC} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

$$\textcircled{x, w} \rightarrow = \textcircled{\alpha, \beta} \rightarrow \textcircled{Q} \leftarrow \rightarrow \pm \sqrt{\frac{1}{2}} \begin{matrix} \textcircled{Q} \leftarrow \rightarrow \\ \leftarrow \rightarrow \textcircled{Q} \end{matrix} \textcircled{\alpha, \beta} \rightarrow$$

Division

If $\begin{array}{c} \text{L} \quad \text{Q} \quad \text{L} \\ \swarrow \quad \searrow \quad \swarrow \quad \searrow \\ \text{Q} \end{array} = 0$

Then L is a factor of Q so

$$\begin{array}{c} \text{Q} \\ \swarrow \quad \searrow \end{array} = \begin{array}{c} \text{L} \quad \text{K} \\ \swarrow \quad \searrow \end{array} + \begin{array}{c} \text{L} \quad \text{K} \\ \swarrow \quad \searrow \end{array}$$

What is K?

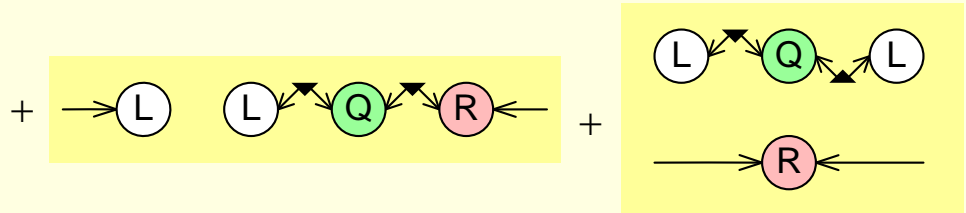
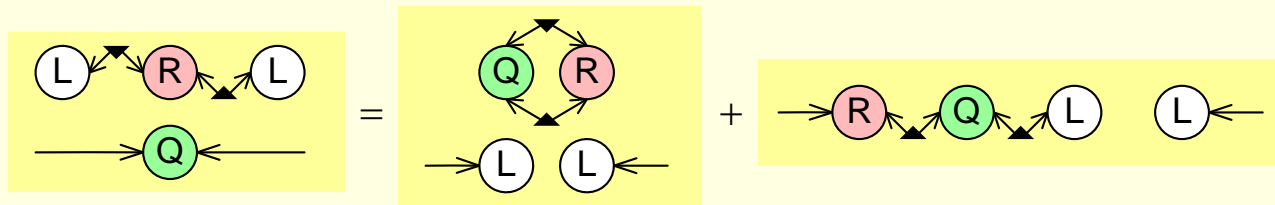
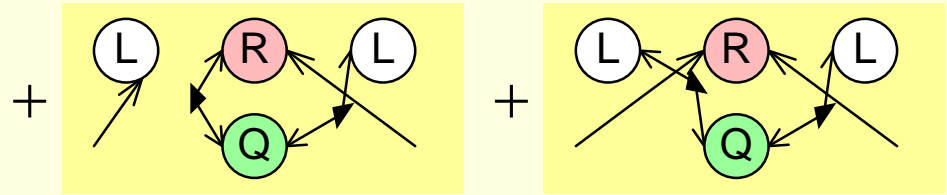
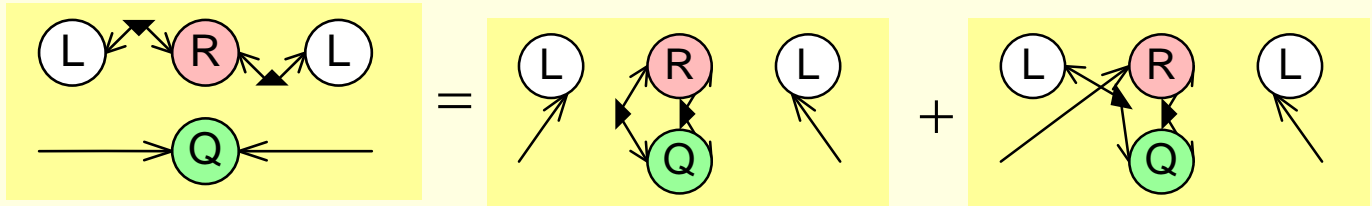
Answer:

$$\begin{array}{c} \text{K} \\ \swarrow \quad \searrow \end{array} = \begin{array}{c} \text{L} \quad \text{Q} \quad \text{R} \\ \swarrow \quad \searrow \quad \swarrow \quad \searrow \\ \text{Q} \end{array} - \begin{array}{c} \text{L} \quad \text{R} \quad \text{Q} \\ \swarrow \quad \searrow \quad \swarrow \quad \searrow \\ \text{Q} \end{array}$$

Why does this work?

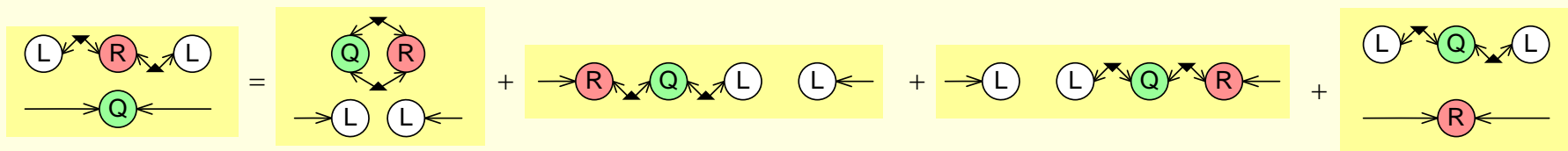
Where does R come from?

Another Syzygy

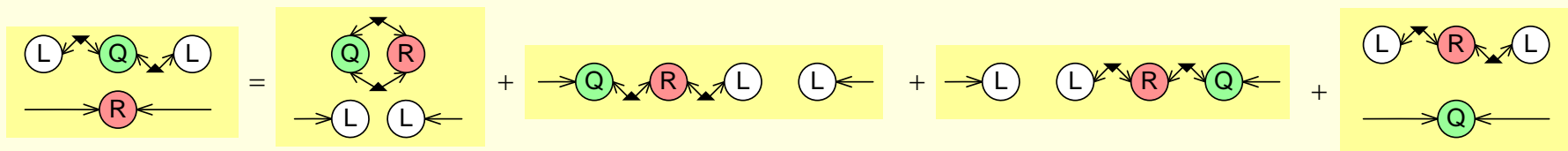


Syzygy continued

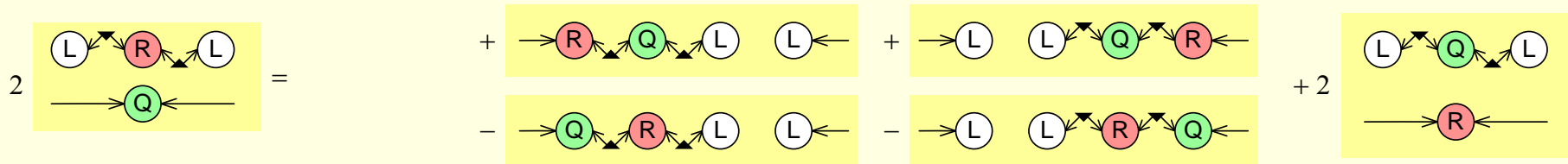
True for all Q,R,L



Swap Q,R



Subtract and rearrange



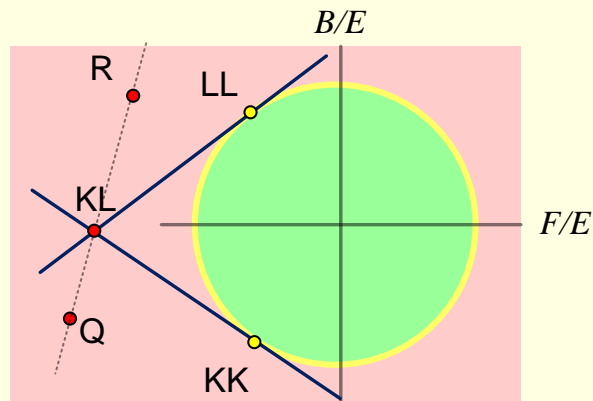
Syzygy continued

$$2 \begin{array}{c} \text{L} \nearrow \text{R} \searrow \text{L} \\ \text{---} \text{Q} \text{---} \end{array} = + \begin{array}{c} \text{R} \text{---} \text{Q} \text{---} \text{L} \text{---} \text{L} \leftarrow \\ \text{---} \text{Q} \text{---} \text{R} \text{---} \text{L} \text{---} \text{L} \leftarrow \end{array} + \begin{array}{c} \text{---} \text{L} \text{---} \text{L} \nearrow \text{Q} \nearrow \text{R} \text{---} \\ \text{---} \text{L} \text{---} \text{L} \searrow \text{R} \searrow \text{Q} \text{---} \end{array} + 2 \begin{array}{c} \text{L} \nearrow \text{Q} \searrow \text{L} \\ \text{---} \text{R} \text{---} \end{array}$$

$$\text{K} \leftarrow = \begin{array}{c} \text{L} \nearrow \text{Q} \nearrow \text{R} \leftarrow \\ \text{---} \text{R} \text{---} \text{Q} \text{---} \end{array} - \begin{array}{c} \text{L} \nearrow \text{R} \nearrow \text{Q} \leftarrow \\ \text{---} \text{Q} \text{---} \text{R} \text{---} \end{array}$$

$$\begin{array}{c} \text{L} \nearrow \text{R} \searrow \text{L} \\ \text{---} \text{Q} \text{---} \end{array} = 1/2 \left\{ \begin{array}{c} \text{---} \text{L} \text{---} \text{K} \leftarrow \\ \text{---} \text{K} \leftarrow \text{---} \text{L} \leftarrow \end{array} \right\} + \begin{array}{c} \text{L} \nearrow \text{Q} \searrow \text{L} \\ \text{---} \text{R} \text{---} \end{array}$$

True for all Q,R,L



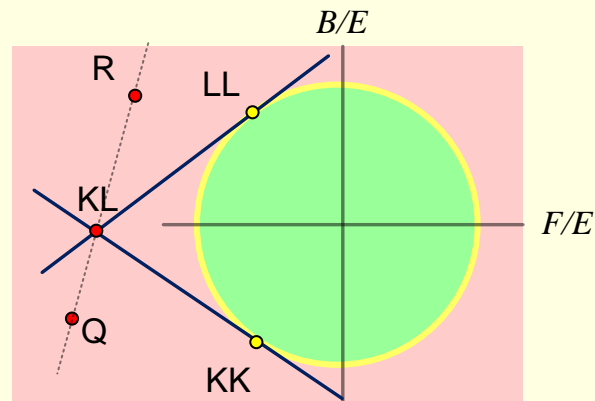
Division

If

$$\begin{array}{c} \text{L} \rightleftarrows \text{Q} \rightleftarrows \text{L} \\ \hline \end{array} = 0$$

$$\begin{array}{c} \text{K} \leftarrow \\ \hline \end{array} = \begin{array}{c} \text{L} \rightleftarrows \text{Q} \rightleftarrows \text{R} \leftarrow \\ \hline \end{array} - \begin{array}{c} \text{L} \rightleftarrows \text{R} \rightleftarrows \text{Q} \leftarrow \\ \hline \end{array}$$

$$\begin{array}{c} \text{L} \rightleftarrows \text{R} \rightleftarrows \text{L} \\ \hline \text{Q} \leftarrow \text{R} \leftarrow \end{array} = \frac{1}{2} \left\{ \begin{array}{c} \text{L} \text{ K} \leftarrow \\ \hline \end{array} + \begin{array}{c} \text{K} \text{ L} \leftarrow \\ \hline \end{array} \right\} + \begin{array}{c} \text{L} \rightleftarrows \text{Q} \rightleftarrows \text{L} \\ \hline \text{R} \leftarrow \end{array}$$



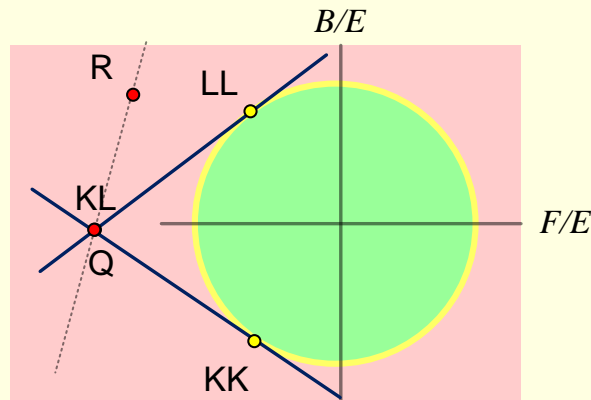
Division

If $\begin{array}{c} \text{L} \rightleftarrows \text{Q} \rightleftarrows \text{L} \\ \text{---} \end{array} = 0$

$$\begin{array}{c} \text{K} \leftarrow \\ \text{---} \end{array} = \begin{array}{c} \text{L} \rightleftarrows \text{Q} \rightleftarrows \text{R} \leftarrow \\ \text{---} \end{array} - \begin{array}{c} \text{L} \rightleftarrows \text{R} \rightleftarrows \text{Q} \leftarrow \\ \text{---} \end{array}$$

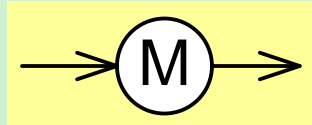
$$\begin{array}{c} \text{L} \rightleftarrows \text{R} \rightleftarrows \text{L} \\ \text{---} \end{array} \begin{array}{c} \text{Q} \leftarrow \\ \text{---} \end{array} = \frac{1}{2} \left\{ \begin{array}{c} \text{L} \text{ K} \leftarrow \\ \text{---} \end{array} + \begin{array}{c} \text{K} \text{ L} \leftarrow \\ \text{---} \end{array} \right\} + \begin{array}{c} \text{L} \rightleftarrows \text{Q} \rightleftarrows \text{L} \\ \text{---} \end{array} \begin{array}{c} \text{R} \leftarrow \\ \text{---} \end{array}$$

As long as R doesn't have L as a factor, so this is nonzero



Transformations

$$\mathbf{M} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

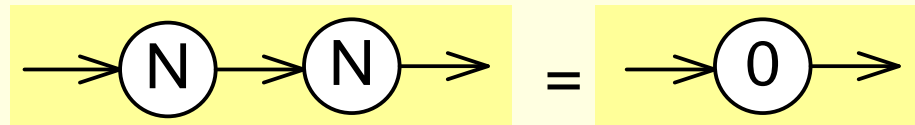


$$\begin{bmatrix} x & w \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} \tilde{x} & \tilde{w} \end{bmatrix}$$

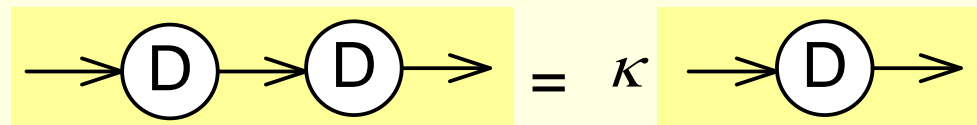
$$\frac{\tilde{x}}{\tilde{w}} = \frac{Ax + Cw}{Bx + Dw}$$

Special Matrices

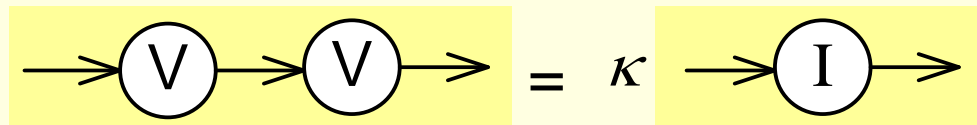
Nilpotent



Idempotent

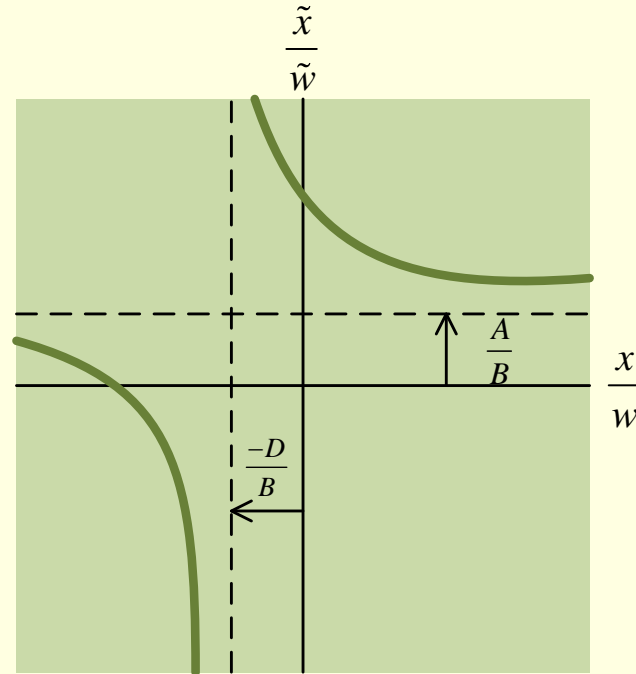


Involution



The Function

$$\frac{\tilde{x}}{\tilde{w}} = \frac{Ax + Cw}{Bx + Dw}$$



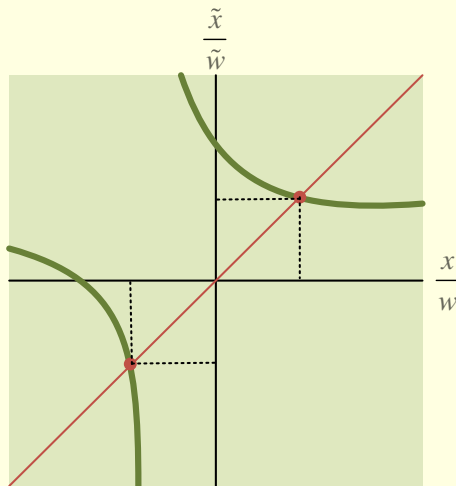
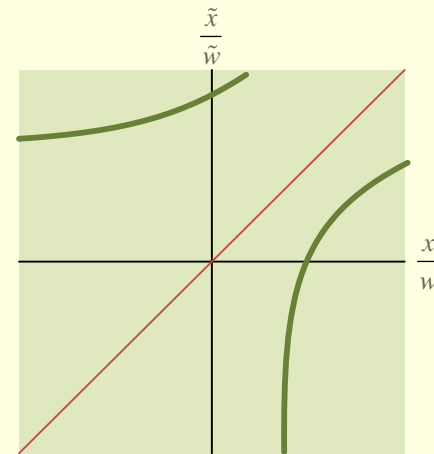
$$w = 0 \Rightarrow \frac{\tilde{x}}{\tilde{w}} = \frac{A}{B}$$

$$\tilde{w} = 0 \Rightarrow Bx + Dw = 0 \Rightarrow \frac{x}{w} = \frac{-D}{B}$$

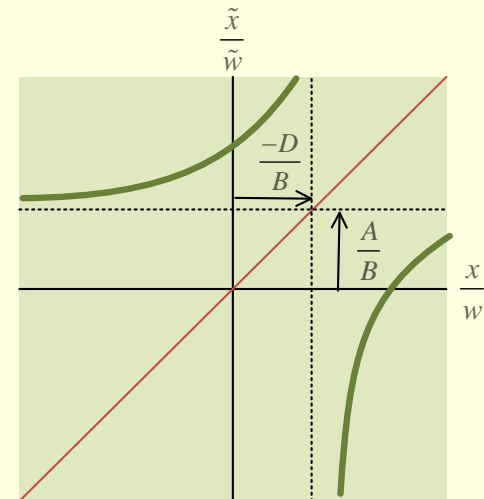
Examples of function

$$T\left(\frac{x}{w}\right) = \frac{A \frac{x}{w} + C}{B \frac{x}{w} + D}$$

$$T'\left(\frac{x}{w}\right) = \left(\frac{A \frac{x}{w} + C}{B \frac{x}{w} + D}\right)' = \frac{AD - BC}{\left(B \frac{x}{w} + D\right)^2}$$



$$\begin{bmatrix} x_e & w_e \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \lambda \begin{bmatrix} x_e & w_e \end{bmatrix}$$



$$A = -D \quad \text{trace} = 0$$

Three Invariants $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$

Determinant $\Delta = AD - BC$

trace $t = A + D$

Characteristic equation

$$\det \begin{bmatrix} A - \lambda & B \\ C & D - \lambda \end{bmatrix} = \lambda^2 + (-A - D)\lambda + (AD - BC) = 0$$

$$\begin{aligned} \delta &= (-A - D)^2 - 4(AD - BC) \\ &= (A - D)^2 + 4BC \end{aligned}$$

Three Invariants $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$

Determinant $\Delta = AD - BC$

trace $t = A + D$

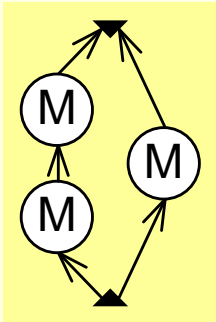
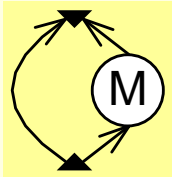
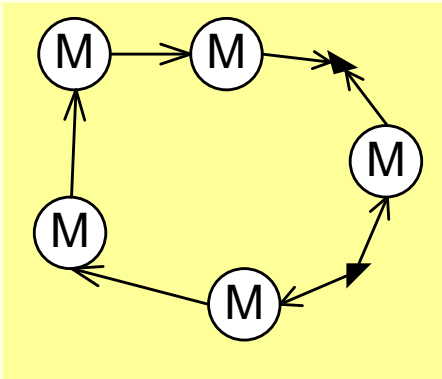
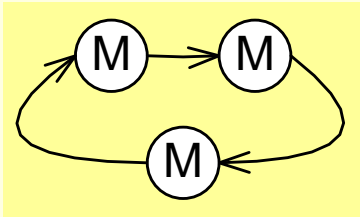
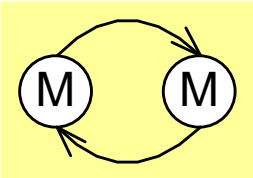
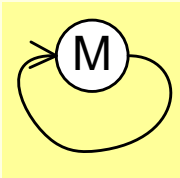
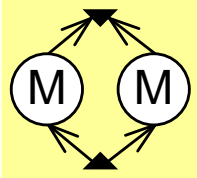
Characteristic equation discriminant

$$\delta = A^2 - 2AD + D^2 + 4BC$$

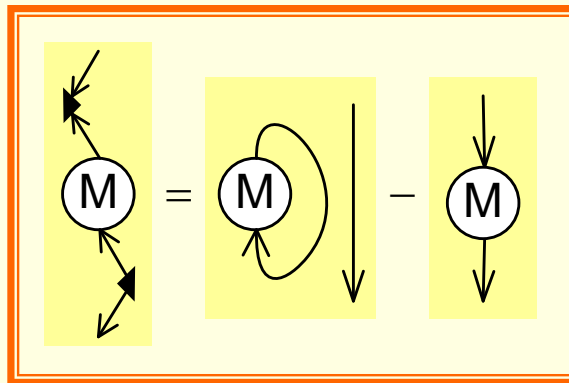
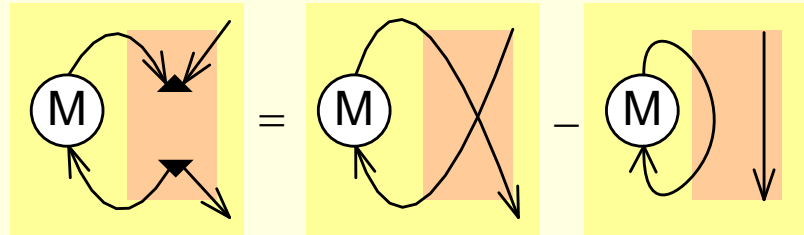
Relation between them:

$$4\Delta + \delta = t^2$$

Diagrams



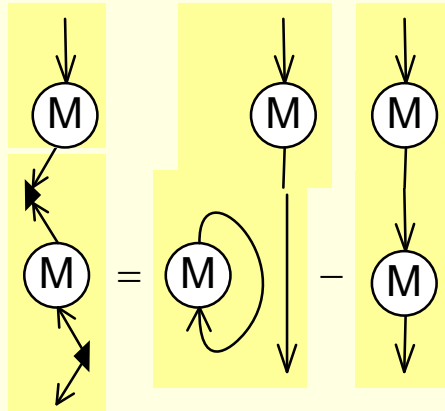
Identity 1



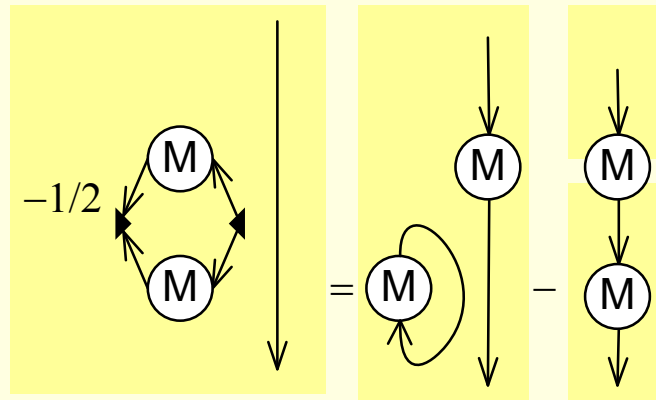
$$\text{adj}\mathbf{M} = (\text{trace}\mathbf{M})\mathbf{I} - \mathbf{M}$$

$$\begin{bmatrix} D & -B \\ -C & A \end{bmatrix} \begin{matrix} \swarrow \\ \searrow \end{matrix} \begin{bmatrix} A+D & 0 \\ 0 & A+D \end{bmatrix} - \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} D & -B \\ -C & A \end{bmatrix}$$

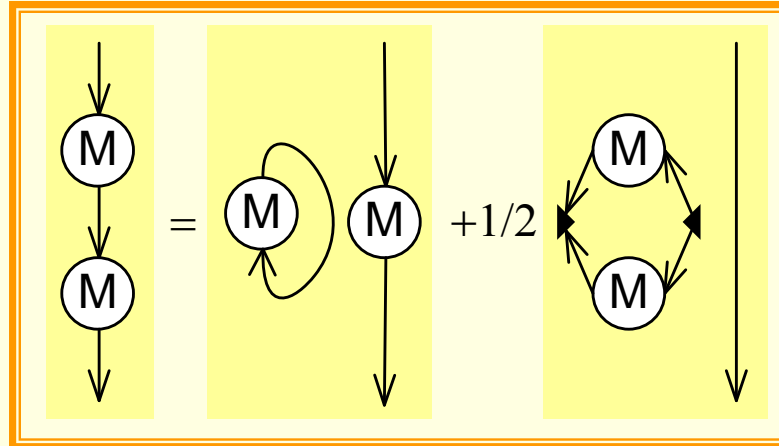
Identity 2



Identity 2



Identity 2



$$MM = (\text{trace } M)M - (\det M)I$$

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

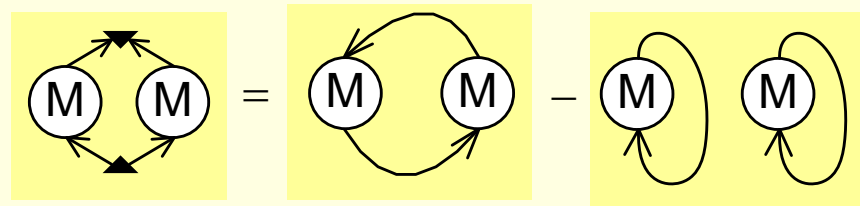
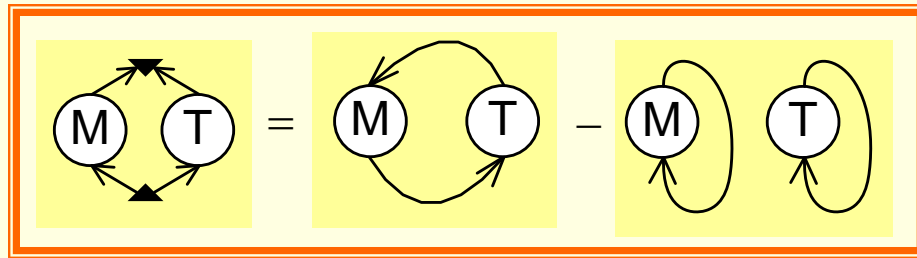
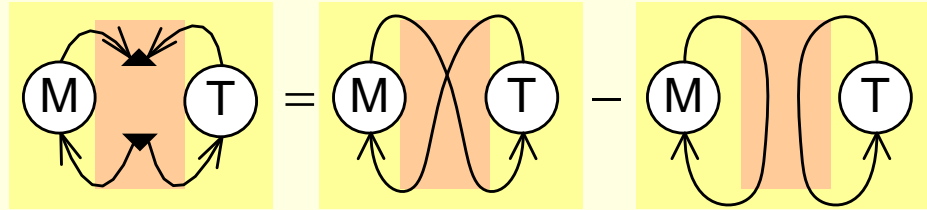
$$= \begin{bmatrix} AA + BC & AB + BD \\ AC + DC & BC + DD \end{bmatrix}$$

$$= (A + D) \begin{bmatrix} A & B \\ C & D \end{bmatrix} - (AD - CB) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} AA + AD & BA + BD \\ CA + CD & AD + DD \end{bmatrix} + \begin{bmatrix} BC - AD & 0 \\ 0 & BC - AD \end{bmatrix}$$

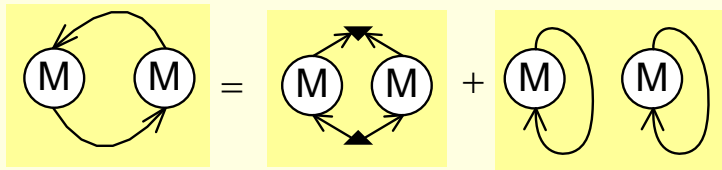
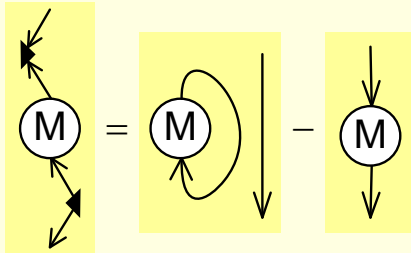
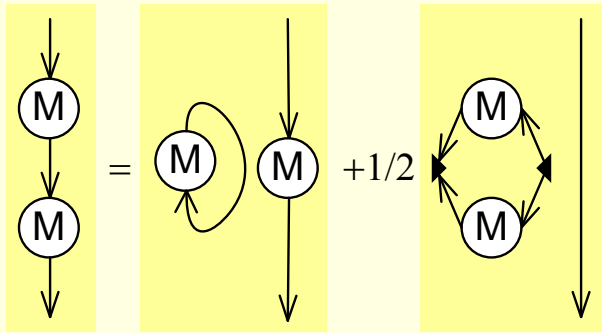
$$= \begin{bmatrix} AA + BC & AB + BD \\ AC + DC & BC + DD \end{bmatrix}$$

Identity 3

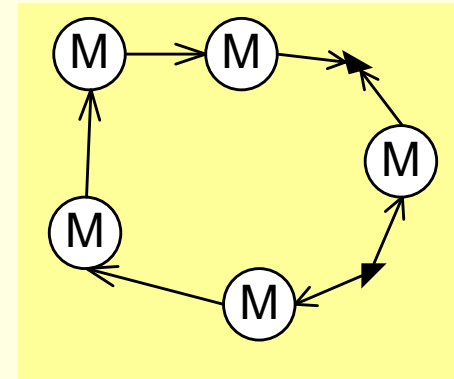


Reducing complex diagrams

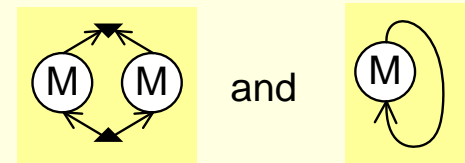
Tools:



To reduce this:

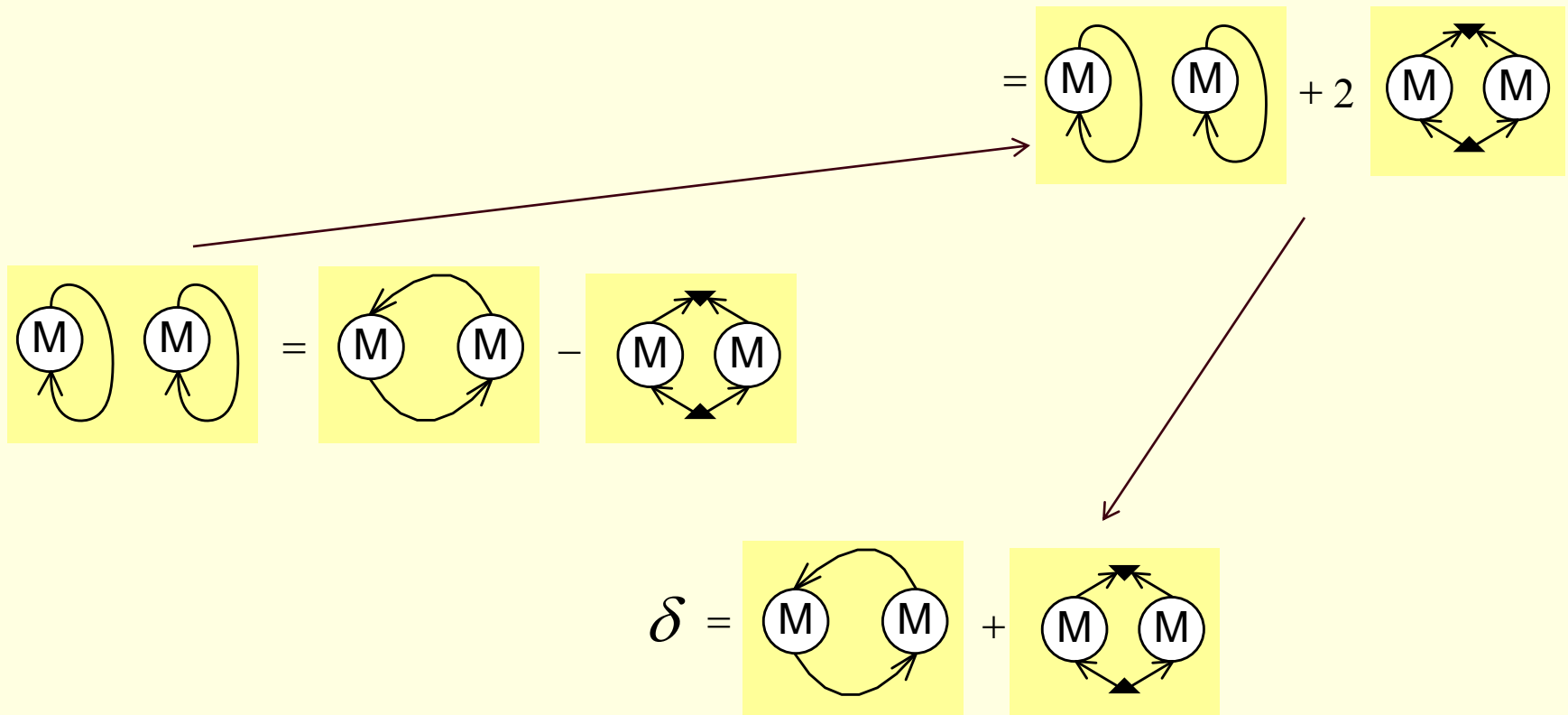


To simple combinations of:

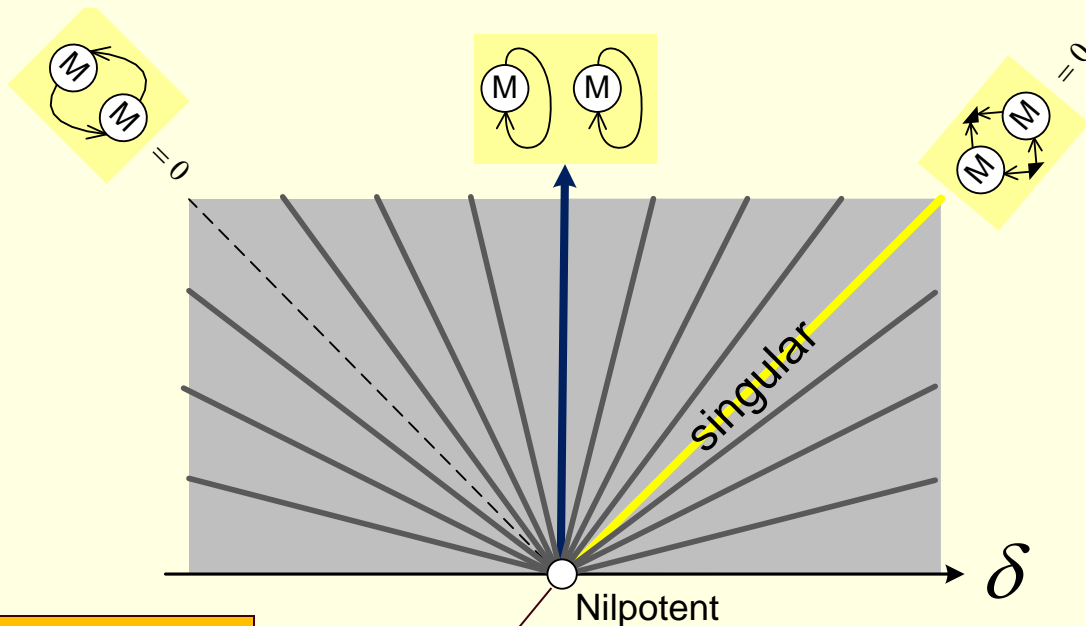
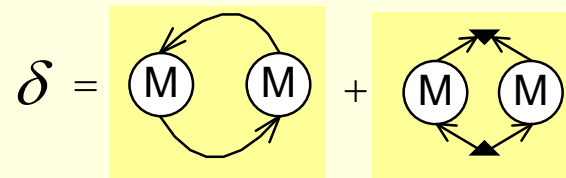
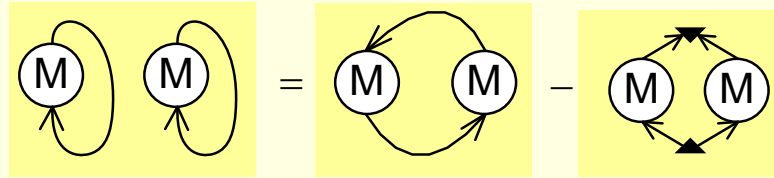


Discriminant of Characteristic Equation

$$\delta = t^2 - 4\Delta$$



The “Phase Space” of M



A valid matrix

Numeric “signature”

$$\frac{\delta}{t^2} = \chi$$

$$\tan^{-1}(\delta, t^2) = \phi$$

Plotting Invariants in ABCD Space

$$\Delta = AD - BC$$

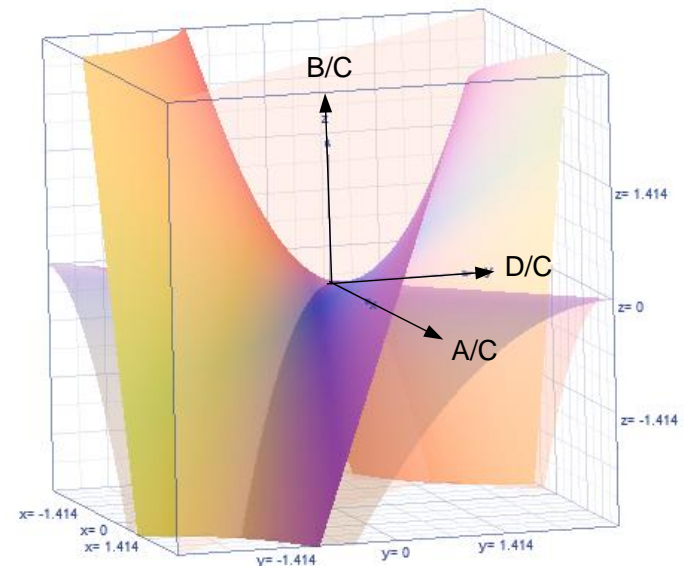
$$t = A + D$$

$$\delta = A^2 - 2AD + D^2 + 4BC$$

$$\Delta = 0 \Rightarrow \frac{B}{C} = \frac{A}{C} \frac{D}{C}$$

$$t = 0 \Rightarrow \frac{A}{C} + \frac{D}{C} = 0$$

$$\delta = 0 \Rightarrow \frac{B}{C} = -\frac{1}{4} \left(\frac{A}{C} - \frac{D}{C} \right)^2$$



New Coordinate System

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} E+F & G+H \\ G-H & E-F \end{bmatrix}$$

$$\begin{aligned} \Delta &= AD - BC \\ &= (E+F)(E-F) - (G+H)(G-H) \\ &= (E^2 + H^2) - (F^2 + G^2) \end{aligned}$$

$$t = A + D = 2E$$

$$\begin{aligned} \delta &= (-A - D)^2 - 4(AD - BC) \\ &= 4(F^2 + G^2 - H^2) \end{aligned}$$

$$\begin{aligned} \Delta &= E^2 + H^2 - F^2 - G^2 \\ \frac{1}{4}t^2 &= E^2 \\ \frac{1}{4}\delta &= -H^2 + F^2 + G^2 \end{aligned}$$

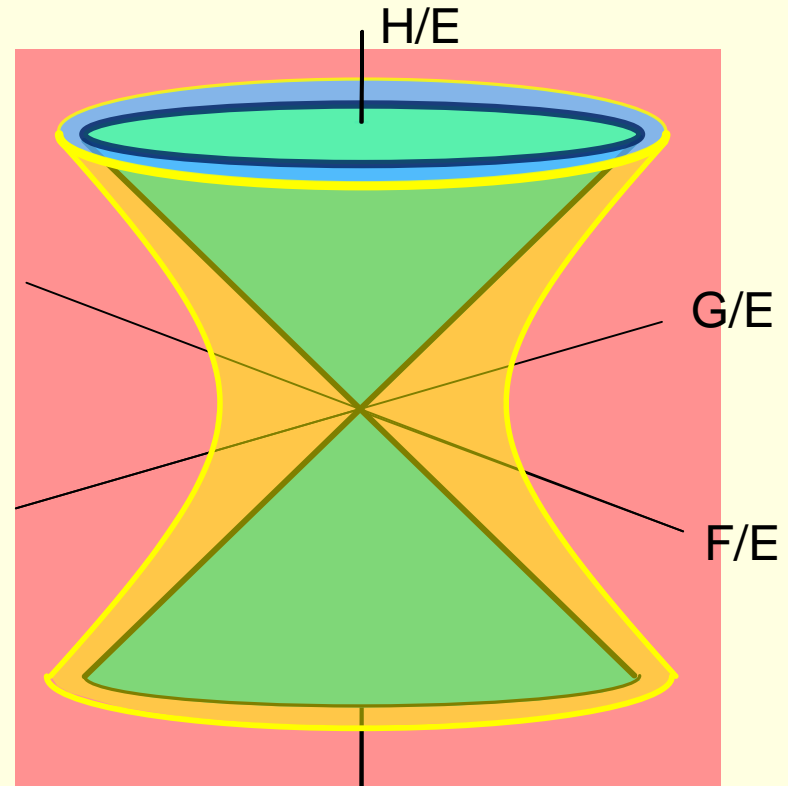
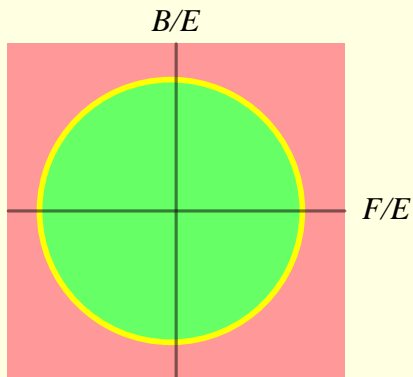
Plot in EFGH space

$$\Delta = 0 \quad \left(\frac{H}{E}\right)^2 = \left(\frac{F}{E}\right)^2 + \left(\frac{G}{E}\right)^2 - 1$$

$$\delta = 0 \quad \left(\frac{H}{E}\right)^2 = \left(\frac{F}{E}\right)^2 + \left(\frac{G}{E}\right)^2$$

$t = 0$ plane at infinity

Compare with Q version



Plot in EFGH space

$$\Delta = 0 \quad \left(\frac{H}{E}\right)^2 = \left(\frac{F}{E}\right)^2 + \left(\frac{G}{E}\right)^2 - 1$$

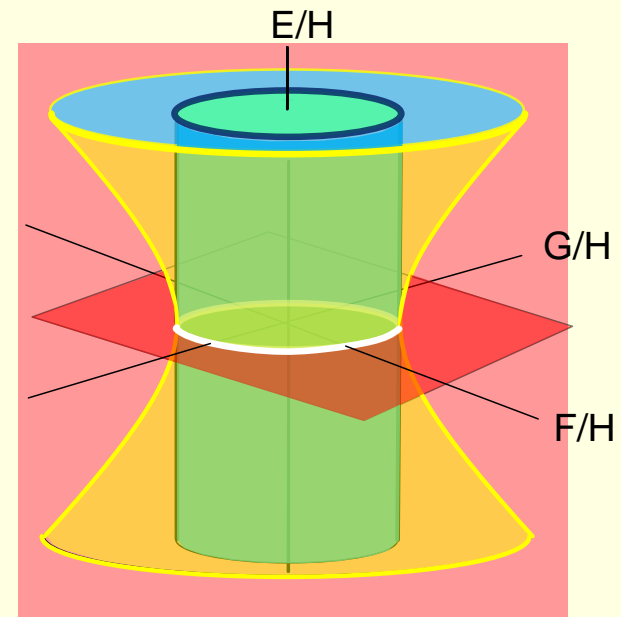
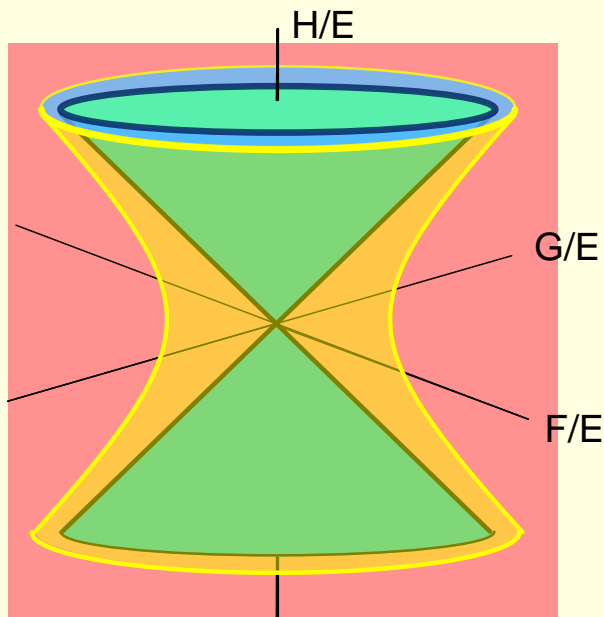
$t = 0$ plane at infinity

$$\delta = 0 \quad \left(\frac{F}{E}\right)^2 + \left(\frac{G}{E}\right)^2 = \left(\frac{H}{E}\right)^2$$

$$\Delta = 0 \quad \left(\frac{E}{H}\right)^2 = \left(\frac{F}{H}\right)^2 + \left(\frac{G}{H}\right)^2 - 1$$

$$t = 0 \quad \left(\frac{E}{H}\right) = 0$$

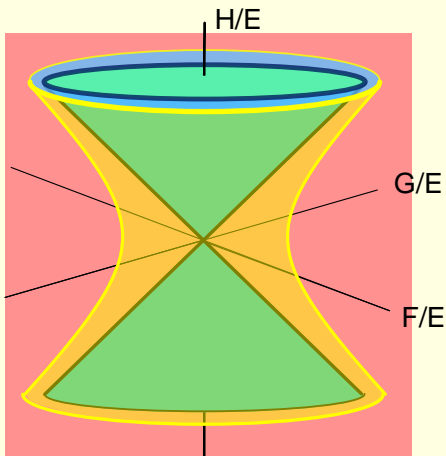
$$\delta = 0 \quad \left(\frac{F}{H}\right)^2 + \left(\frac{G}{H}\right)^2 = 1$$



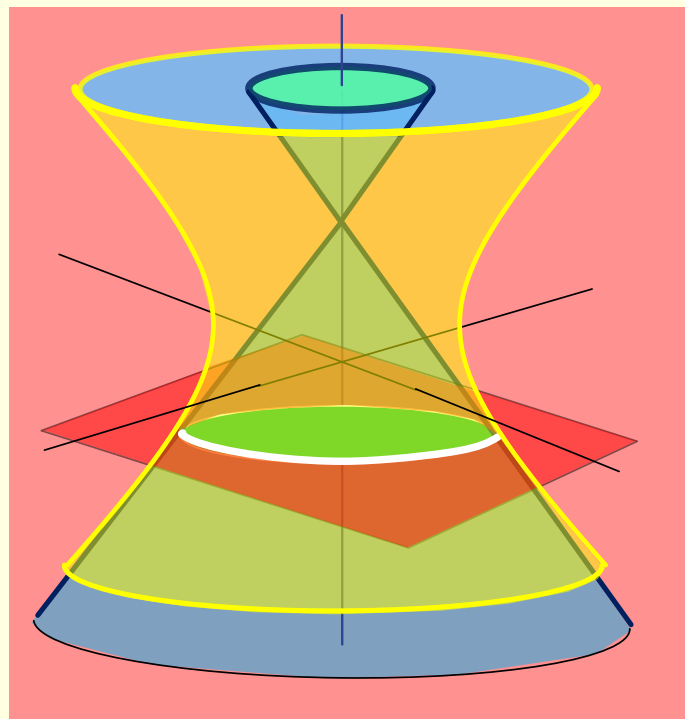
More generally, rotate along E,H axis

$$\begin{bmatrix} E \\ H \end{bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} \hat{E} \\ \hat{H} \end{bmatrix}$$

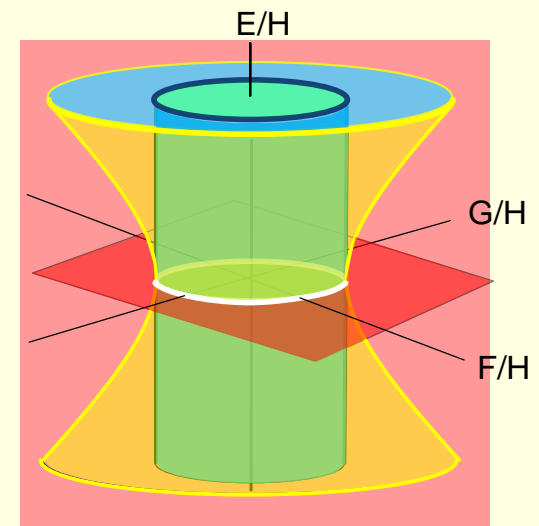
$\theta = 0^\circ$



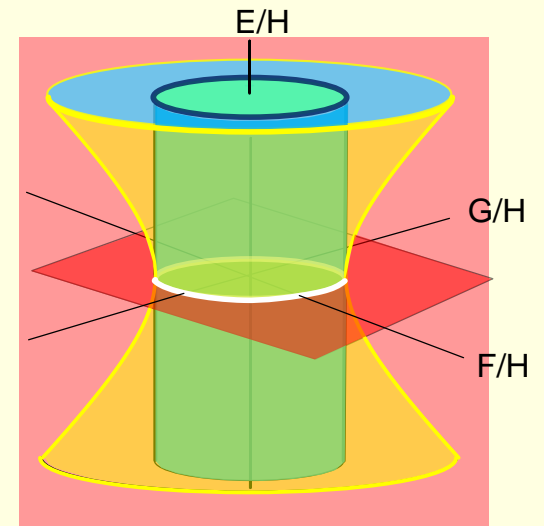
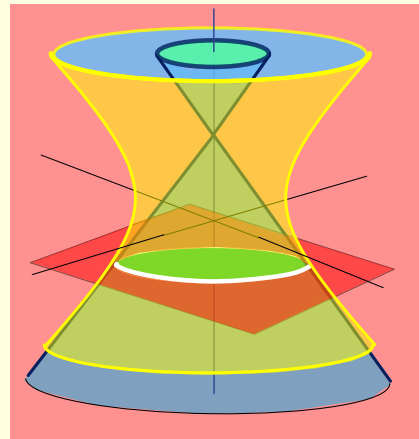
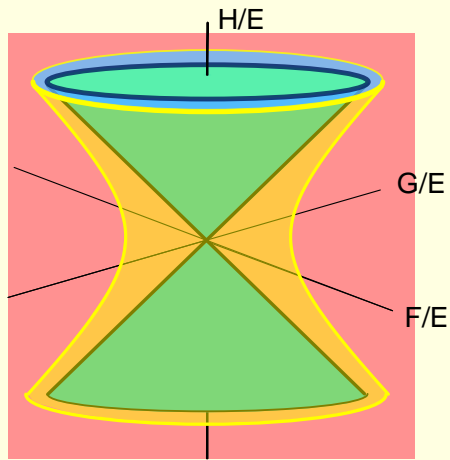
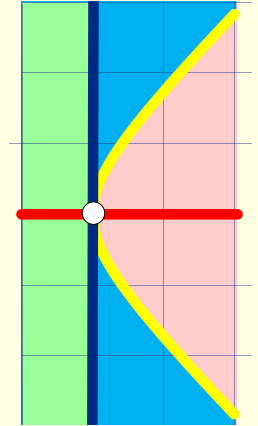
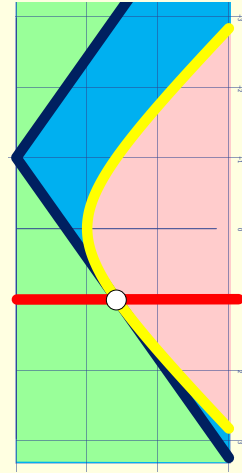
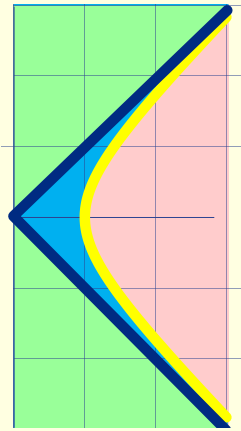
$\theta = 45^\circ$



$\theta = 90^\circ$



Cross section



Roadmap of M

