CSE590B Lecture 2
Introduction to P^1

More than you thought you wanted to know about 2x2 matrices

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http://courses.cs.washington.edu/courses/cse590b/13au/

The space of spaces

$$\text{Algebra} \qquad \rightarrow \text{Geometry}$$

$$\text{projective 1-space } P^1 : \begin{bmatrix} x & w \end{bmatrix} \rightarrow \begin{bmatrix} \frac{x}{w} \end{bmatrix}$$

$$\text{projective 2-space } P^2 : \begin{bmatrix} x & y & w \end{bmatrix} \rightarrow \begin{bmatrix} \frac{x}{w} & \frac{y}{w} \end{bmatrix}$$

$$\text{projective 3-space } P^3 : \begin{bmatrix} x & y & z & w \end{bmatrix} \rightarrow \begin{bmatrix} \frac{x}{w} & \frac{y}{w} & \frac{z}{w} \end{bmatrix}$$

Naming Conventions

"homogeneous" coordinate at end $\left[x \quad y \quad w\right]$ $\left[x \quad y \quad z\right]$

"homogeneous" coordinate at beginning $\left[p_0 \quad p_1 \quad p_2\right]$ $\left[w \quad x \quad y\right]$

$$\mathbf{P}=\left[P_x \quad P_y \quad P_w\right],\ \mathbf{S}=\left[S_x \quad S_y \quad S_w\right],\ \mathbf{T}=\left[T_x \quad T_y \quad T_w\right]$$

$$\mathbf{P}=\left[x_P \quad y_p \quad w_p\right],\ \mathbf{S}=\left[x_S \quad y_S \quad w_S\right],\ \mathbf{T}=\left[x_T \quad y_T \quad w_T\right]$$

$$\mathbf{P}_1=\left[x_1 \quad y_1 \quad w_1\right],\ \mathbf{P}_2=\left[x_2 \quad y_2 \quad w_2\right],\ \mathbf{P}_3=\left[x_3 \quad y_3 \quad w_3\right]$$

$$\mathbf{P}\times\mathbf{S}\cdot\mathbf{T}=\left[PST\right] \qquad \left[123\right]$$

P^1 Space

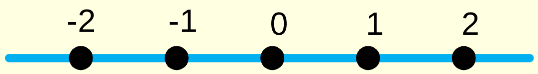
$$\text{Algebra} \rightarrow \text{Geometry}$$

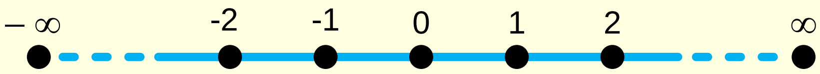
$$\text{projective 1-space } P^1: \begin{bmatrix} x & w \end{bmatrix} \rightarrow \begin{bmatrix} \dfrac{x}{w} \end{bmatrix}$$

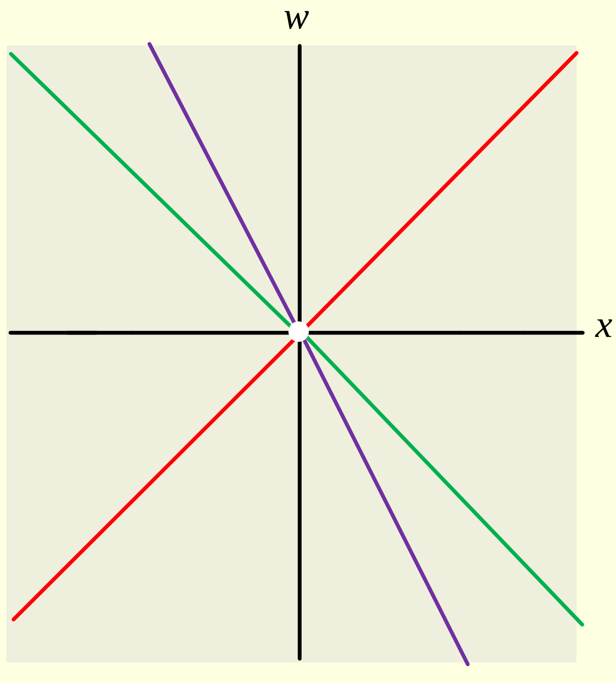
$$\text{matrices}: \begin{bmatrix} A & B \\ C & D \end{bmatrix} \rightarrow \begin{bmatrix} A/D \\ B/D \\ C/D \end{bmatrix}$$

Can still visualize entire matrix in 3 dimensional (projective) plots

P^1 Geometry

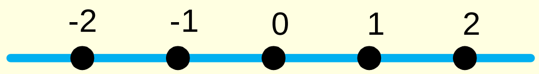


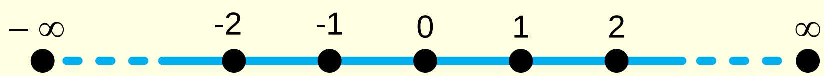


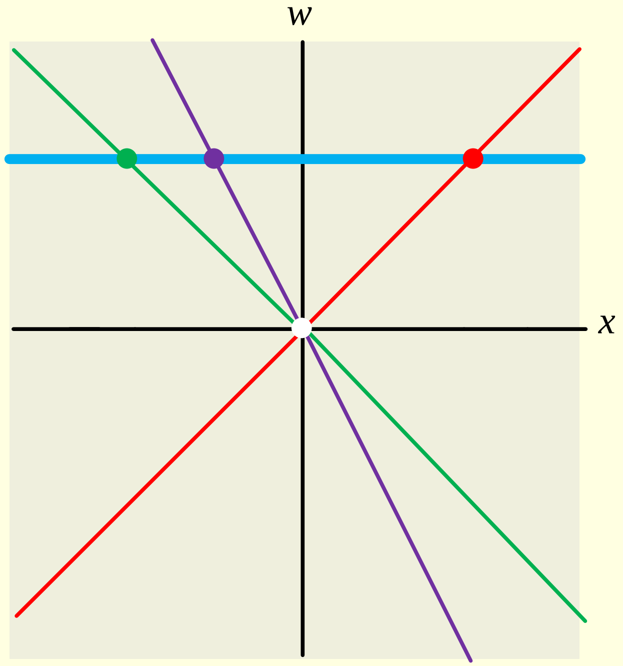


projective 1-space $P^1 : \begin{bmatrix} x & w \end{bmatrix}$

P1 Geometry

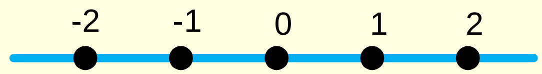


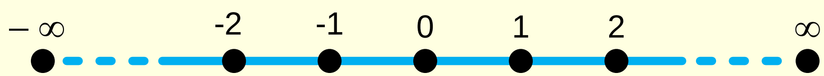


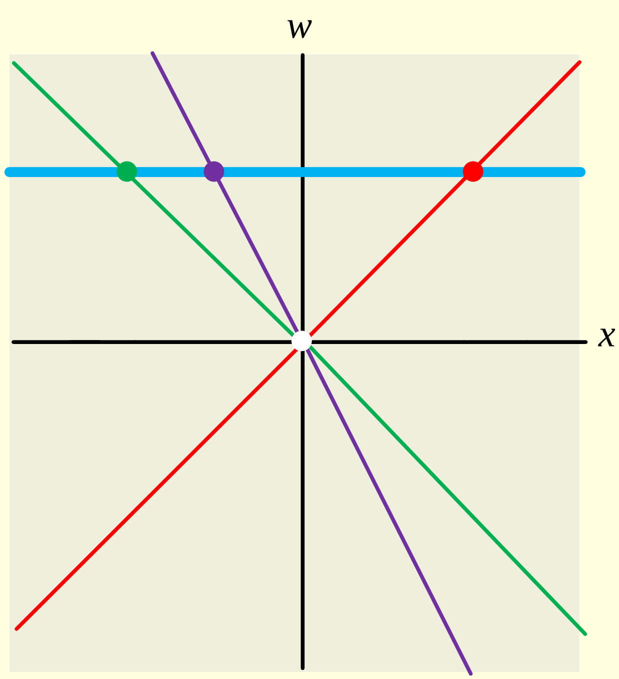


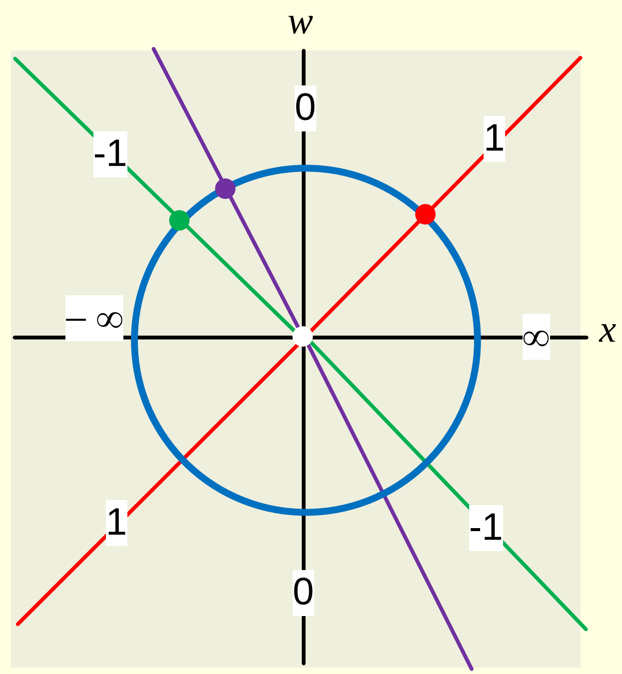
projective 1-space $P^1 : \begin{bmatrix} x & w \end{bmatrix} \rightarrow \begin{bmatrix} \dfrac{x}{w} \end{bmatrix}$

P^1 Geometry







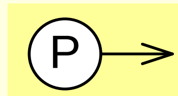


$$x = \sin\theta$$

$$w = \cos\theta$$

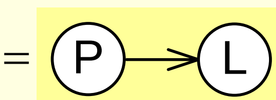
Basic Diagrams

points $[x \quad w] =$

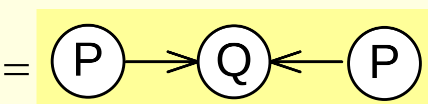


polynomials

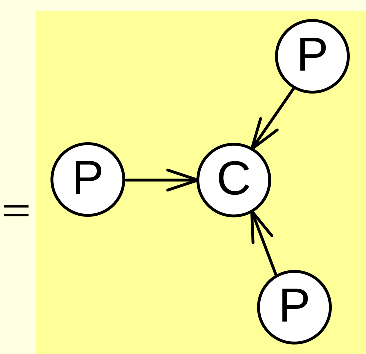
$$Ax + Bw = [x \quad w]\begin{bmatrix}A\\B\end{bmatrix} =$$



$$Ax^2 + 2Bxw + Cw^2 = [x \quad w]\begin{bmatrix}A & B\\B & C\end{bmatrix}\begin{bmatrix}x\\w\end{bmatrix} =$$

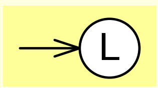


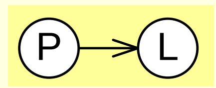
$$Ax^3 + 3Bx^2w + 3Cxw^2 + Dw^3 = [x \quad w]\left[\begin{bmatrix}A & B\\B & C\end{bmatrix} \quad \begin{bmatrix}B & C\\C & D\end{bmatrix}\right]\begin{bmatrix}x\\w\end{bmatrix}\begin{bmatrix}x\\w\end{bmatrix} =$$



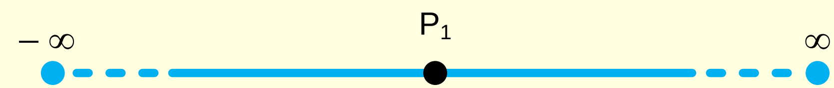
Point Sets (1d "curves")

$$\begin{bmatrix} A \\ B \end{bmatrix} =$$

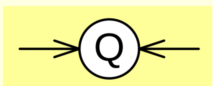


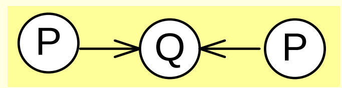


$= 0$

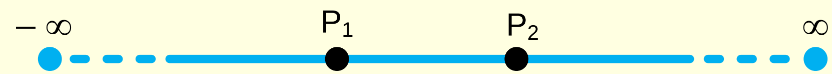


$$\begin{bmatrix} A & B \\ B & C \end{bmatrix} =$$

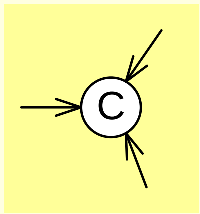


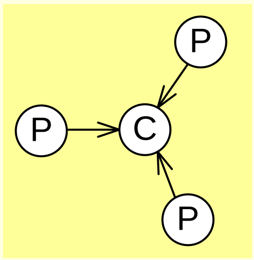


$= 0$

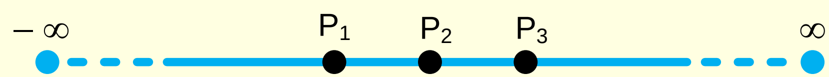


$$\begin{bmatrix} \begin{bmatrix} A & B \\ B & C \end{bmatrix} & \begin{bmatrix} B & C \\ C & D \end{bmatrix} \end{bmatrix} =$$





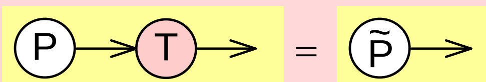
$= 0$



Transformations

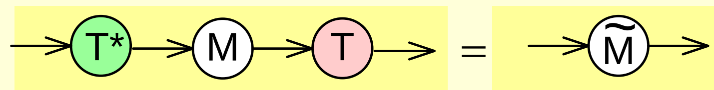
points

$$[x \quad w]\begin{bmatrix} t & u \\ s & v \end{bmatrix} = [\tilde{x} \quad \tilde{w}]$$



transformations

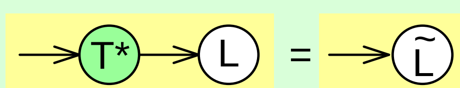
$$\mathbf{T}^* \begin{bmatrix} A & B \\ C & D \end{bmatrix} \mathbf{T} = \begin{bmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{bmatrix}$$



polynomials

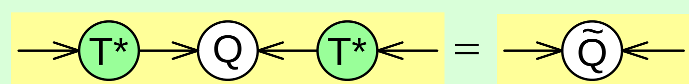
linear

$$\begin{bmatrix} t & u \\ s & v \end{bmatrix}^* \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} \tilde{A} \\ \tilde{B} \end{bmatrix}$$



quadratic

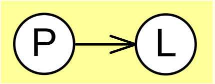
$$\mathbf{T}^* \begin{bmatrix} A & B \\ B & C \end{bmatrix} \mathbf{T}^{*t} = \begin{bmatrix} \tilde{A} & \tilde{B} \\ \tilde{B} & \tilde{C} \end{bmatrix}$$



Epsilon

Solving linear equation

$$Ax + Bw = 0$$



$= 0$

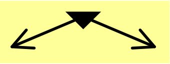
Given L find P

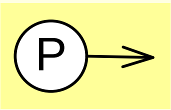
Answer

$$\begin{bmatrix} x & w \end{bmatrix} = \begin{bmatrix} B & -A \end{bmatrix} = \begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

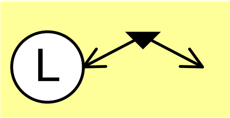
diagram

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} =$$

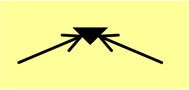




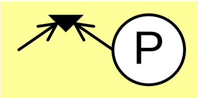
$=$



dual diagram



$$= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$



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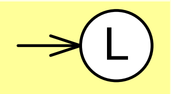

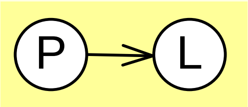

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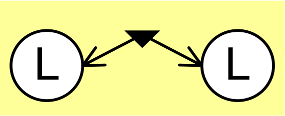
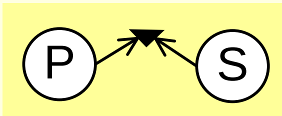


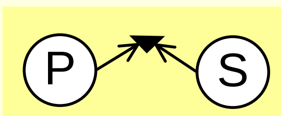
$= 0$

Basic Epsilon Identities

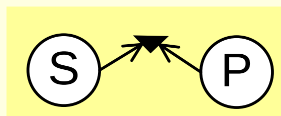


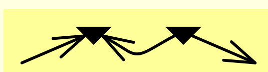
$$= \begin{bmatrix} x_P & w_P \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_S \\ w_S \end{bmatrix}$$

$$= x_P w_S - x_S w_P$$



$= -$

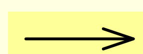


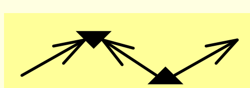


$$= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

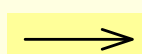
$$= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

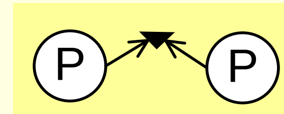
$= \; -1$



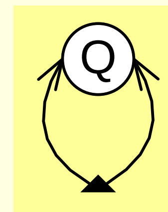


$=$

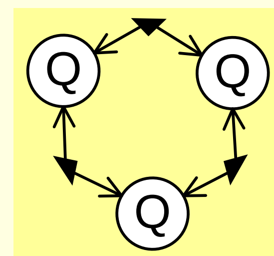




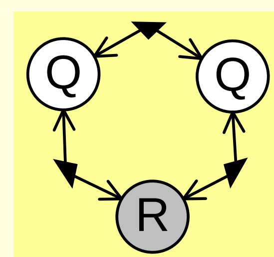
$= 0$



$= 0$



$= 0$

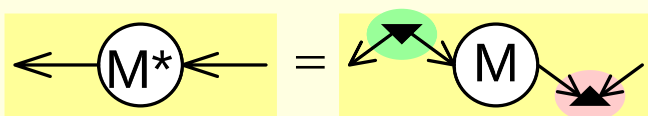


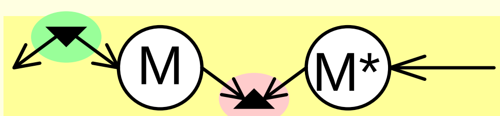
$= 0$

Adjugate, Determinant

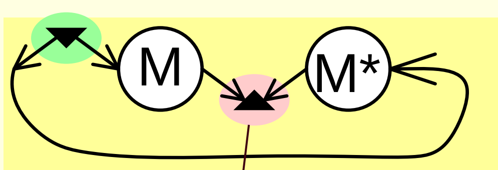
Mixed tensor

$$adj\mathbf{M} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}^{*} = \begin{bmatrix} D & -B \\ -C & A \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$



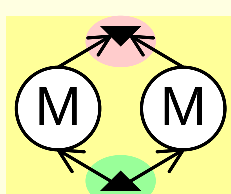


$$= \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} D & -B \\ -C & A \end{bmatrix} = \begin{bmatrix} AD-BC & 0 \\ 0 & AD-BC \end{bmatrix}$$



$$= 2\left(AD-BC\right) = 2\det\mathbf{M}$$

Sign flip
for aesthetics

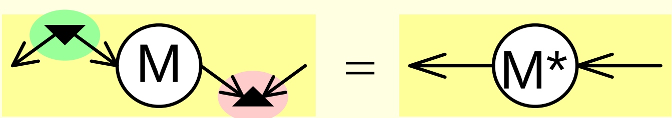


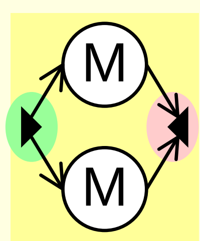
$$= -2\ \det\mathbf{M}$$

Adjugate, Determinant

Mixed tensor

$$adj\mathbf{M} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}^* = \begin{bmatrix} D & -B \\ -C & A \end{bmatrix}$$

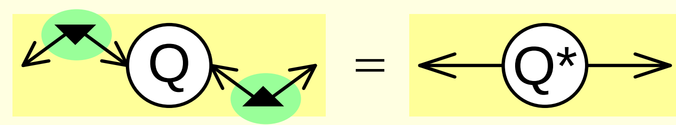


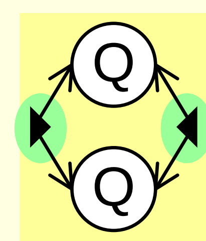


$= -2 \det\mathbf{M}$

Pure tensor

$$adj\mathbf{Q} = \begin{bmatrix} A & B \\ B & C \end{bmatrix}^* = \begin{bmatrix} C & -B \\ -B & A \end{bmatrix}$$



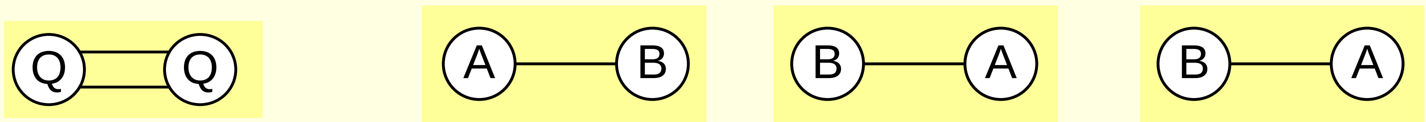


$= -2 \det\mathbf{Q}$

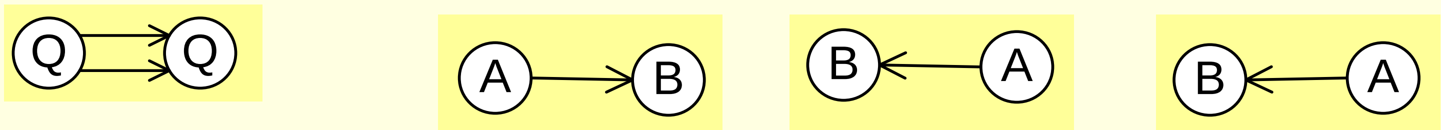
History Of Notation

Rotate 180^{o} Miror

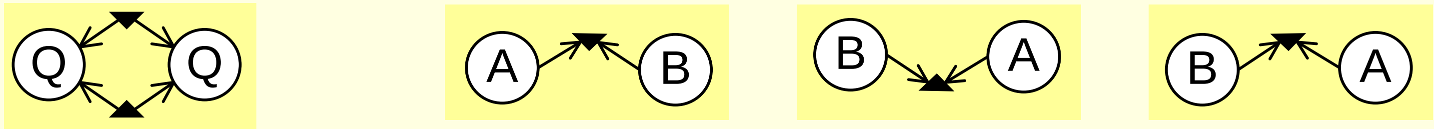
Sylvester 1878



Olver/Shakiban 1989



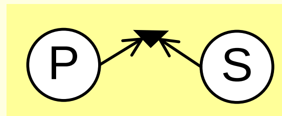
Blinn (from Stedman?)



Doesn't matter

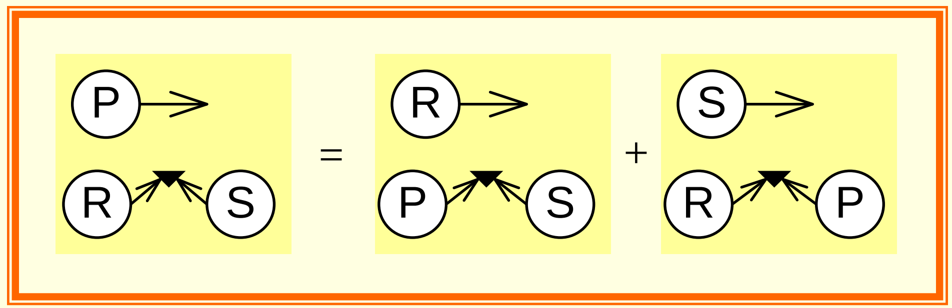
Changes sign

Grassman-Plucker Relation



$$= \begin{bmatrix} x_P & w_P \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_S \\ w_S \end{bmatrix}$$

$$= x_P w_S - x_S w_P$$

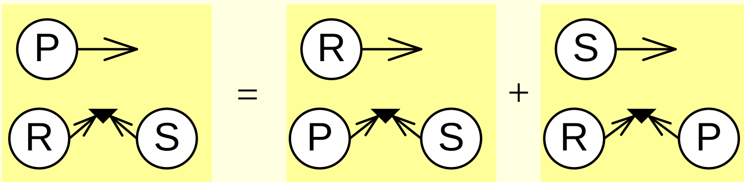


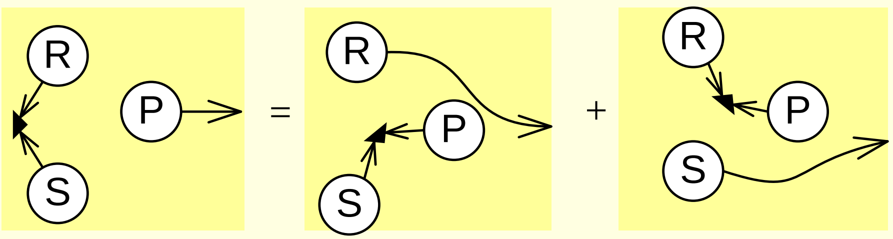
$$\left(x_R w_S - x_S w_R \right) \begin{bmatrix} x_P & w_P \end{bmatrix} = \left(x_P w_S - x_S w_P \right) \begin{bmatrix} x_R & w_R \end{bmatrix} + \left(x_R w_P - x_R w_P \right) \begin{bmatrix} x_S & w_S \end{bmatrix}$$

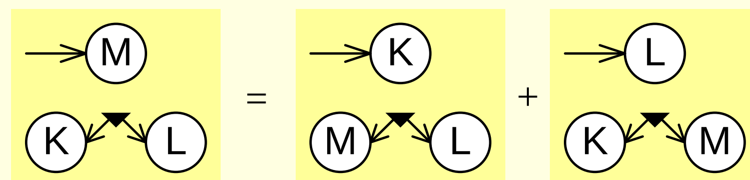
$$\left(x_R w_S - x_S w_R \right) x_P = \left(x_P w_S - x_S w_P \right) x_R + \left(x_R w_P - x_P w_R \right) x_S$$

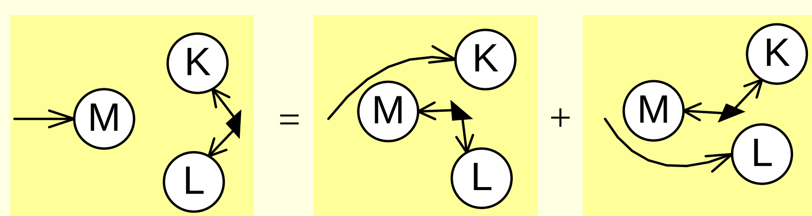
$$x_P x_R w_S - x_P w_R x_S = x_P x_R w_S - w_P x_R x_S + w_P x_R x_S - x_P w_R x_S$$

Arc Swap

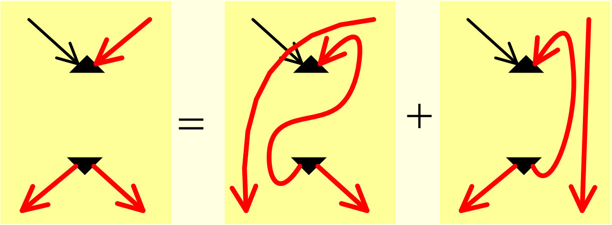


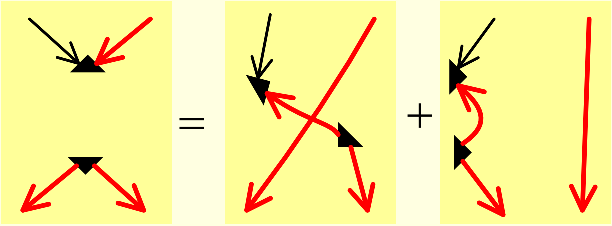


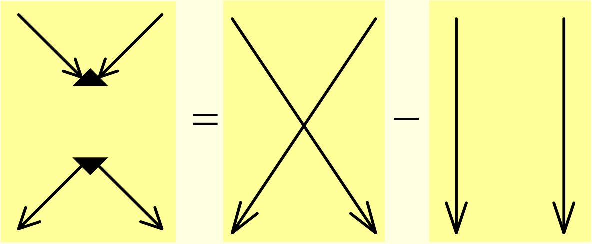




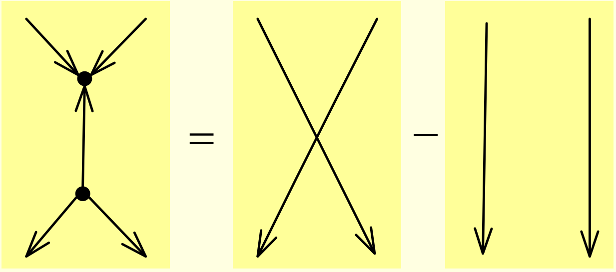
2D Epsilon Delta Rule



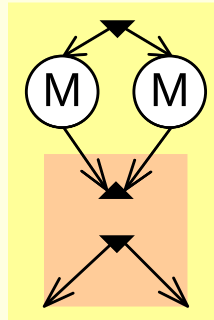




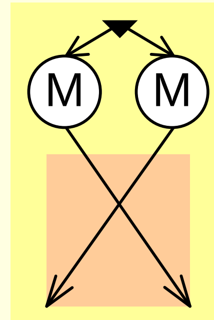
Compare 3D



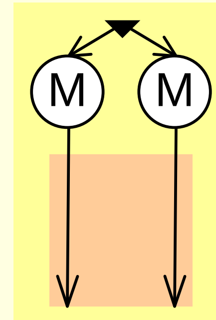
Application of ε/δ



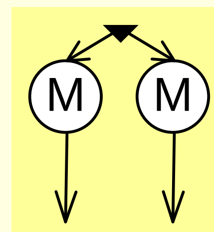
=

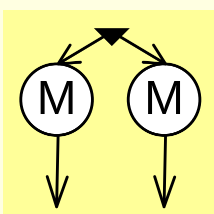


−

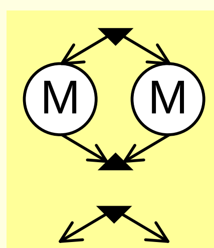


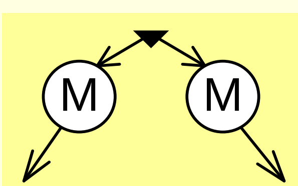
= − 2



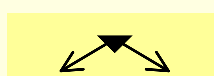


= − 1/2

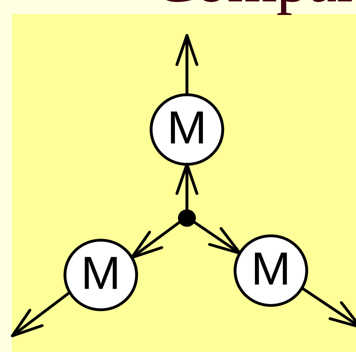




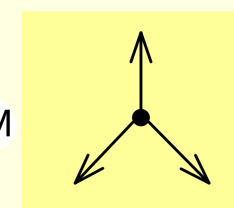
= det M



Compare 3D



= det M



What Does Equal Mean?

Equal

$$5 = 5$$

$$\begin{bmatrix}1 & 2 & 3\end{bmatrix} = \begin{bmatrix}1 & 2 & 3\end{bmatrix}$$

Equal up to a homogeneous scale

$$\begin{bmatrix}1 & 2 & 3\end{bmatrix} \cong \begin{bmatrix}2 & 4 & 6\end{bmatrix} = 2\begin{bmatrix}1 & 2 & 3\end{bmatrix}$$

$$\begin{bmatrix}x & y & w\end{bmatrix} \cong \begin{bmatrix}2x & 2y & 2w\end{bmatrix} = 2\begin{bmatrix}x & y & w\end{bmatrix}$$

$$\begin{bmatrix}p & q \\ r & s\end{bmatrix} \cong \begin{bmatrix}2p & 2q \\ 2r & 2s\end{bmatrix} = 2\begin{bmatrix}p & q \\ r & s\end{bmatrix}$$

What symbol to use?

$$= \quad \approx \quad \simeq \quad \cong \quad \equiv$$

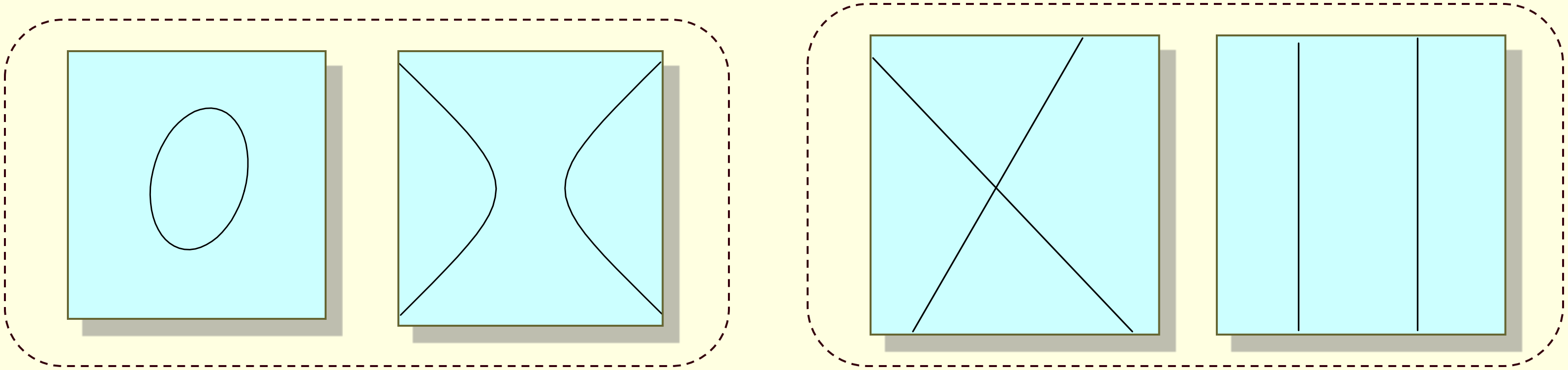
$$\triangleq \quad \triangleq \quad \mapsto \quad \doteq$$

Equivalent (under transformation) = congruent (?)

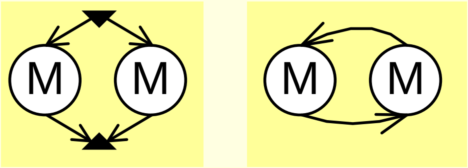
$$\begin{bmatrix}A & B \\ C & D\end{bmatrix} \cong \begin{bmatrix}A+B+C+D & -A+B-C+D \\ -A-B+C+D & A-B-C+D\end{bmatrix} = \begin{bmatrix}1 & 1 \\ -1 & 1\end{bmatrix}\begin{bmatrix}A & B \\ C & D\end{bmatrix}\begin{bmatrix}1 & -1 \\ 1 & 1\end{bmatrix}$$

Topics

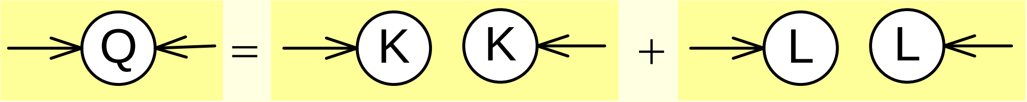
Equivalence Classes



Invariant Diagrams

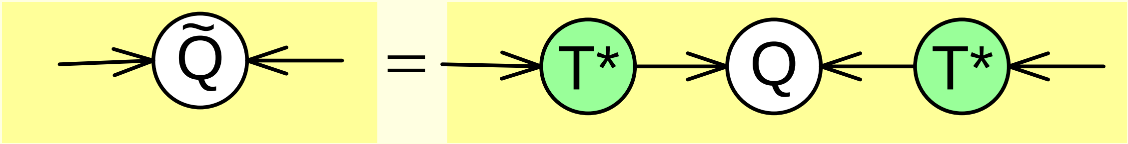


Internal Structure

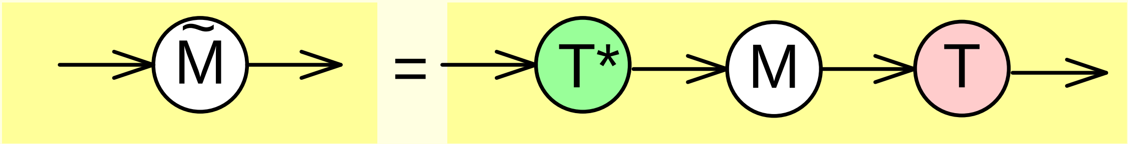


Two Types of Matrix

Quadratic polynomials (pure tensor)

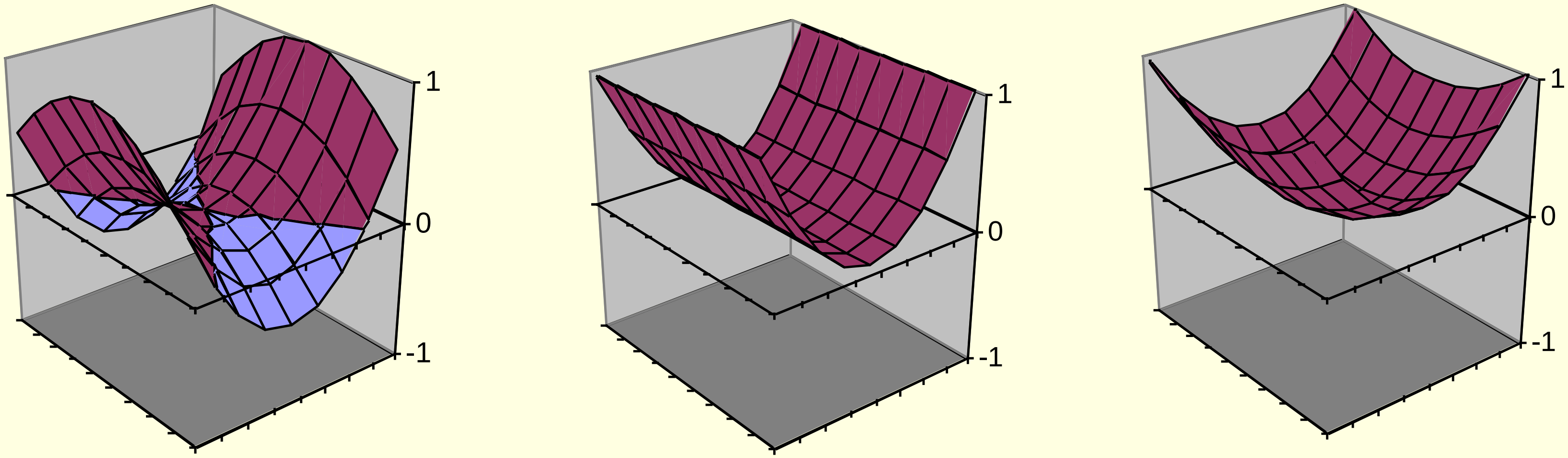


Transformations (mixed tensor)

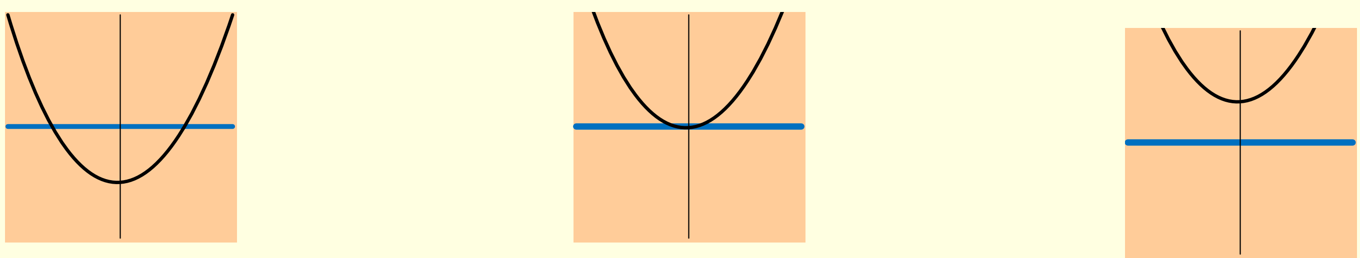


Quadratic Polynomials

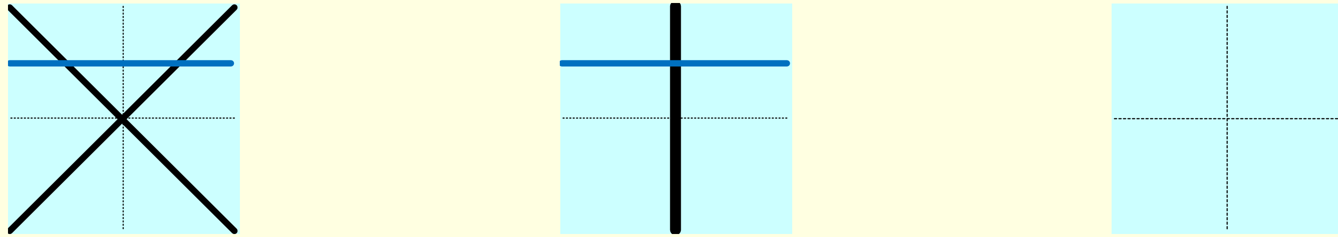
$$Ax^2 + 2Bxw + Cw^2 = 0$$



w=1 plane

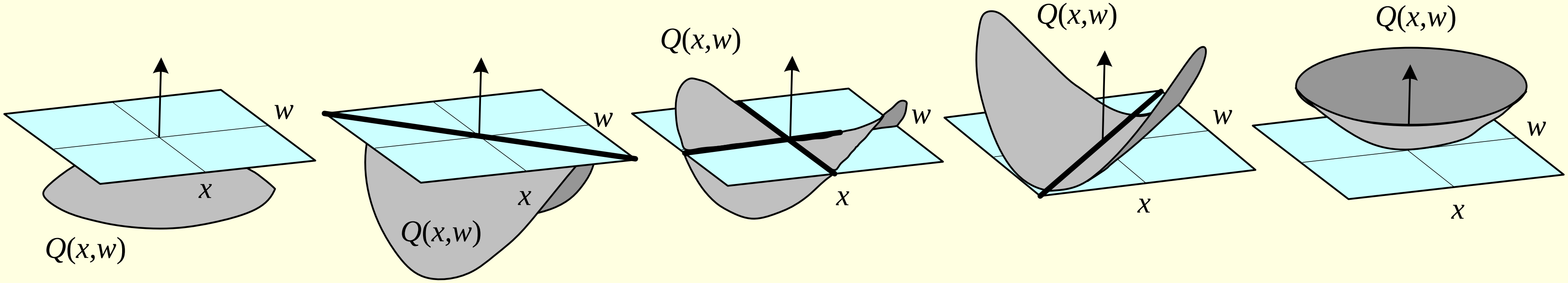


x,*w* plane



Quadratic Polynomials

$$Q(x,w) = Ax^2 + 2Bxw + Cw^2 = \begin{bmatrix} x & w \end{bmatrix} \begin{bmatrix} A & B \\ B & C \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix}$$



Type $\frac{1}{1}$ –	Type 2–	Type 11	Type 2+	Type $\frac{1}{1}$ +
No real roots Negative definite	*Two coincident roots*	*Two distinct roots*	*Two coincident roots*	*No real roots Positive definite*

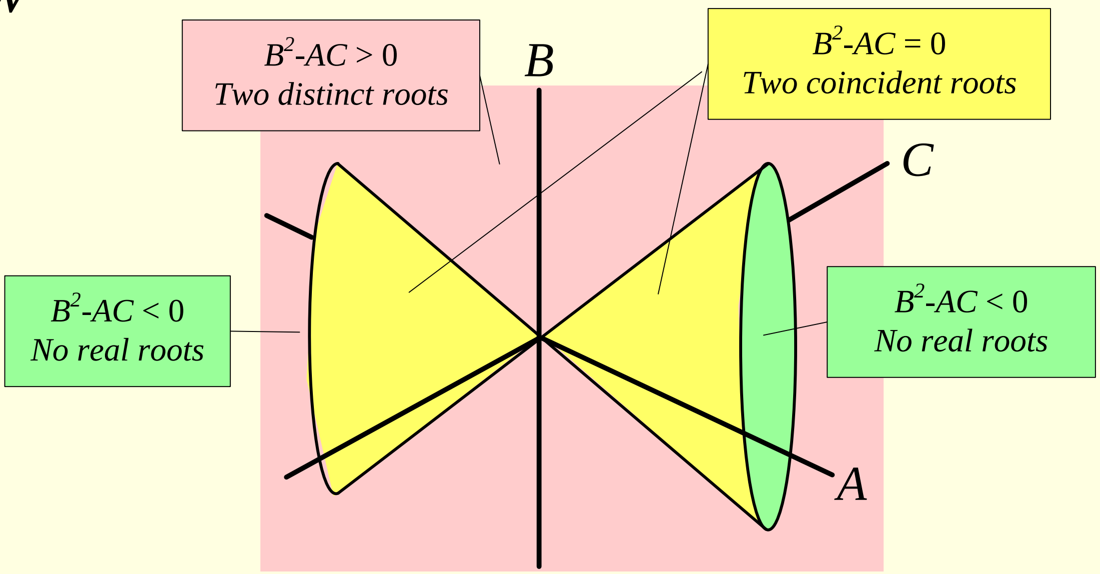
Notation: $\bar{1}$ vs. $\frac{1}{1}$

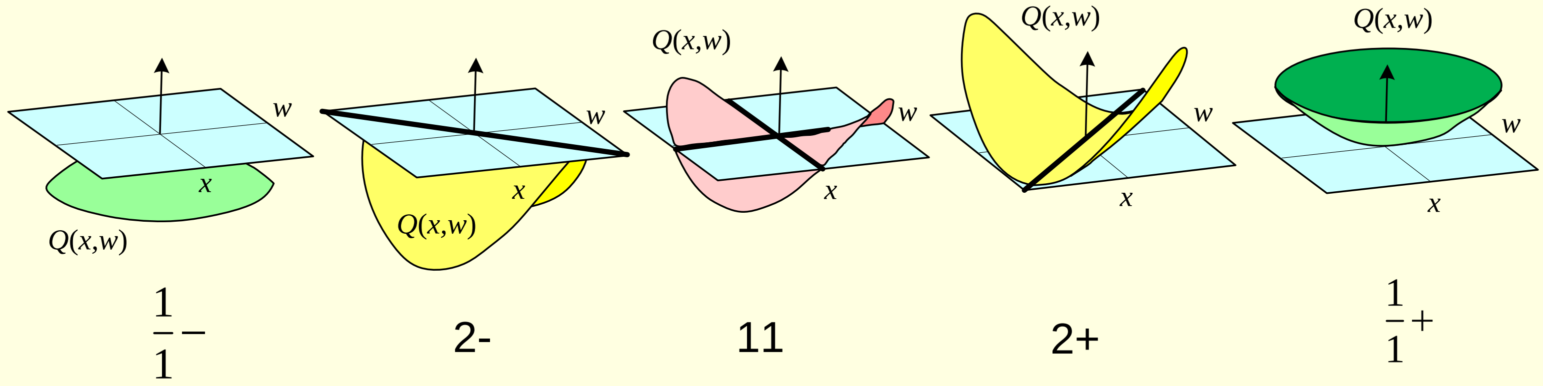
Discriminant

$$Q(x,w) = Ax^2 + 2Bxw + Cw^2$$

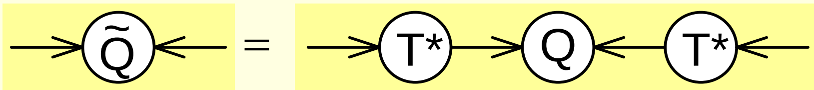
$$\operatorname{discr}(Q) = B^2 - AC$$

$$= -\det\begin{bmatrix} A & B \\ B & C \end{bmatrix}$$





Transformation of Q



$$\begin{bmatrix} \tilde{A} & \tilde{B} \\ \tilde{B} & \tilde{C} \end{bmatrix} = \mathbf{T}^{*} \begin{bmatrix} A & B \\ B & C \end{bmatrix} \mathbf{T}^{*T}$$

$$\mathbf{T}^{*} = \begin{bmatrix} t & u \\ s & v \end{bmatrix}$$

$$\begin{bmatrix} \tilde{A} & \tilde{B} \\ \tilde{B} & \tilde{C} \end{bmatrix} = \begin{bmatrix} t & u \\ s & v \end{bmatrix} \begin{bmatrix} A & B \\ B & C \end{bmatrix} \begin{bmatrix} t & s \\ u & v \end{bmatrix} = \begin{bmatrix} ttA + 2tuB + uuC & tsA + (su + tv)B + uvC \\ stA + (tv + su)B + uvC & ssA + 2svB + vvC \end{bmatrix}$$

$$\tilde{A} = ttA + 2tuB + uuC$$

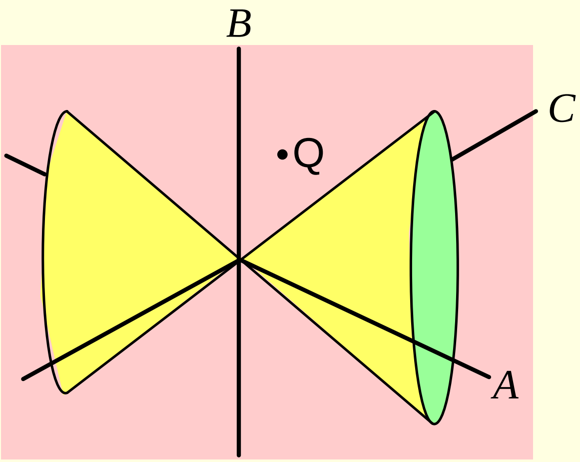
$$\tilde{B} = stA + (tv + su)B + uvC$$

$$\tilde{C} = ssA + 2svB + vvC$$

$$\begin{bmatrix} \tilde{A} \\ \tilde{B} \\ \tilde{C} \end{bmatrix} = \begin{bmatrix} tt & 2tu & uu \\ st & tv + su & uv \\ ss & 2sv & vv \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix}$$

What Does T do to Q?

$$\begin{bmatrix} \tilde{A} \\ \tilde{B} \\ \tilde{C} \end{bmatrix} = \begin{bmatrix} tt & 2tu & uu \\ st & tv+su & uv \\ ss & 2sv & vv \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix}$$



Expect: stays in same color

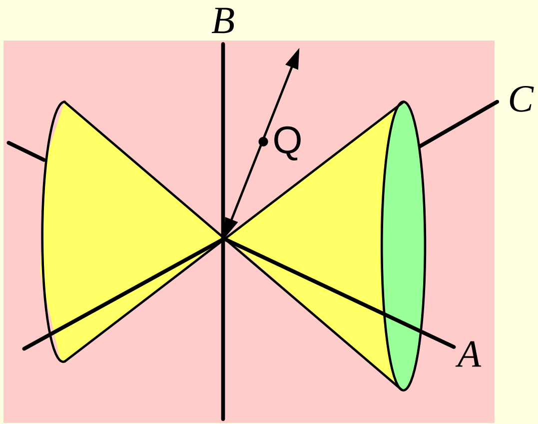
Only for Real T $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} ? \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} ? \end{bmatrix}^T$

If allow Complex T $\begin{bmatrix} t & u \\ s & v \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}$

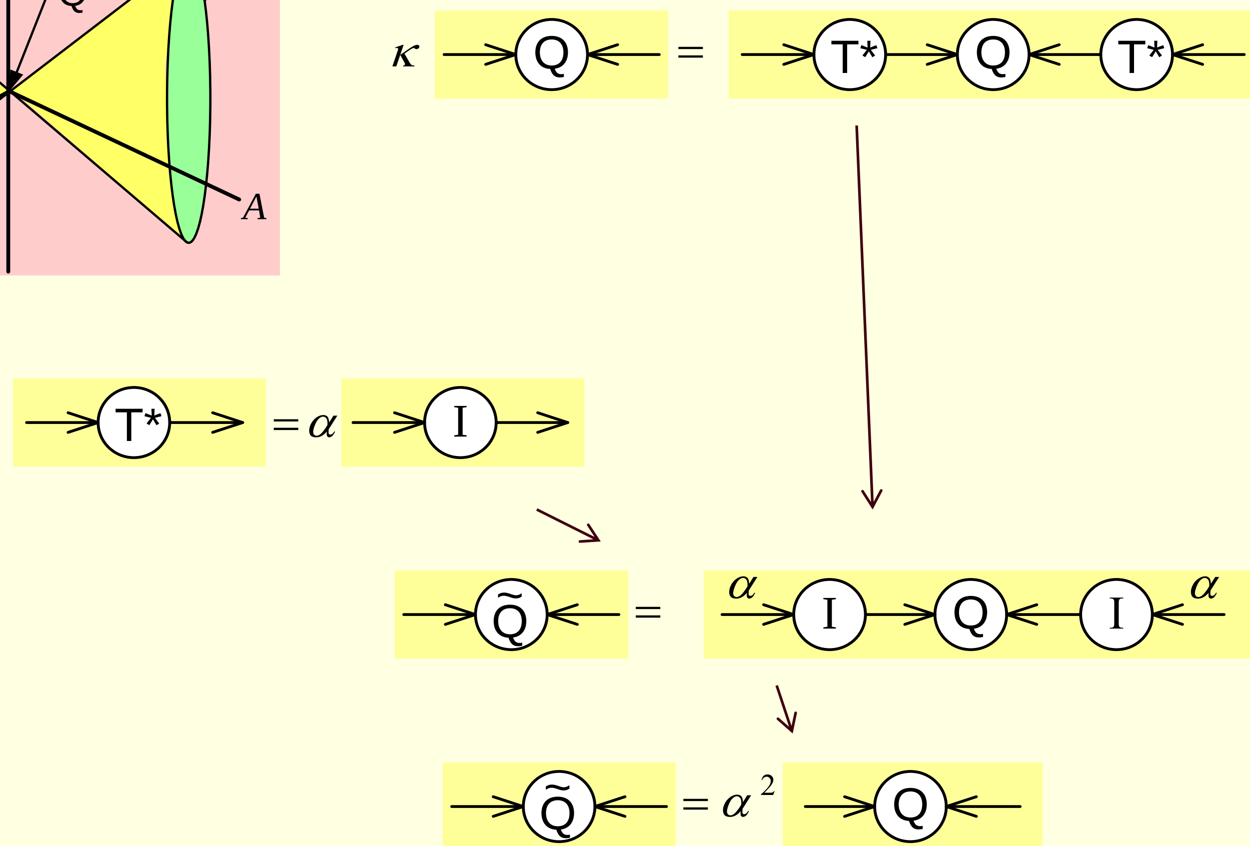
$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} t & u \\ s & v \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} t & s \\ u & v \end{bmatrix}$$

$$\begin{bmatrix} \tilde{A} \\ \tilde{B} \\ \tilde{C} \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ -i & 0 & i \\ -1 & 2 & -1 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix}$$

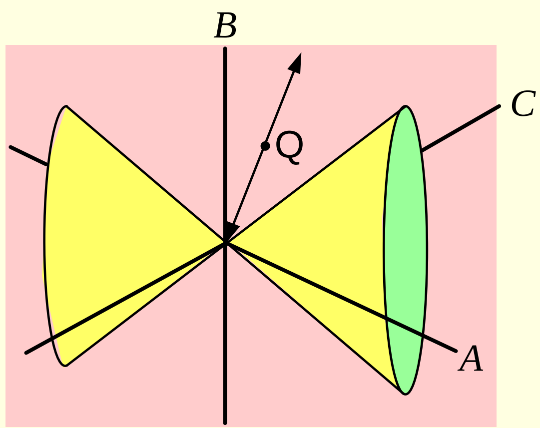
Stationary Transformation

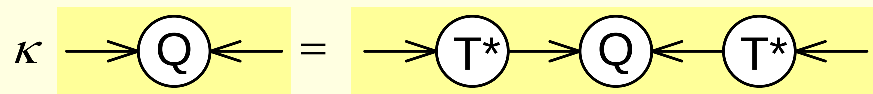


one choice

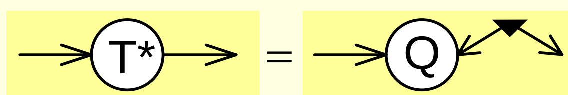


Stationary Transformation

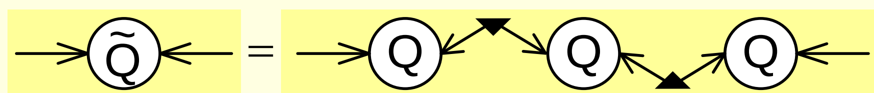


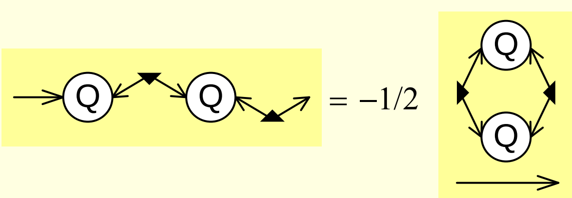


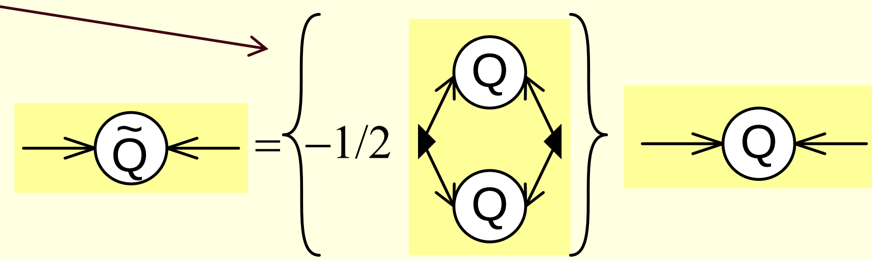
another choice



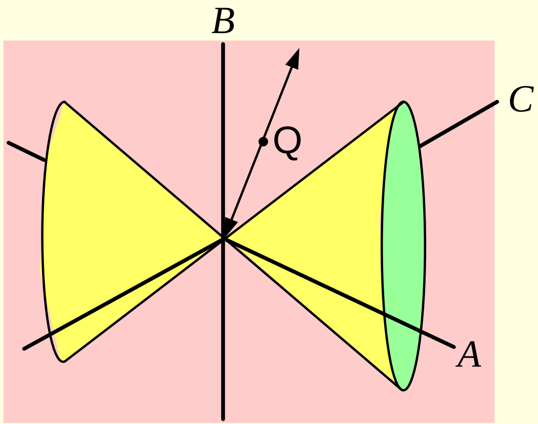
$$= \begin{bmatrix} A & B \\ B & C \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -B & A \\ -C & B \end{bmatrix}$$

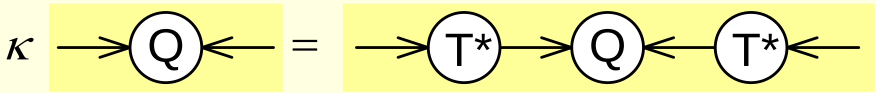




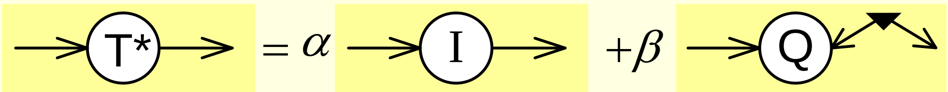


Stationary Transformation





General choice

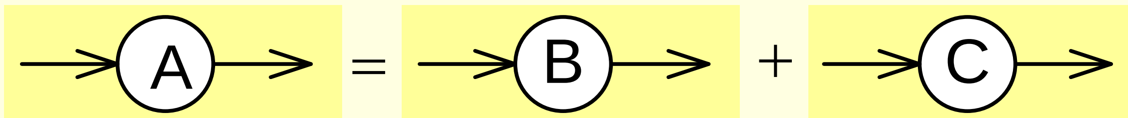


$$= \alpha \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \beta \begin{bmatrix} -B & A \\ -C & B \end{bmatrix}$$

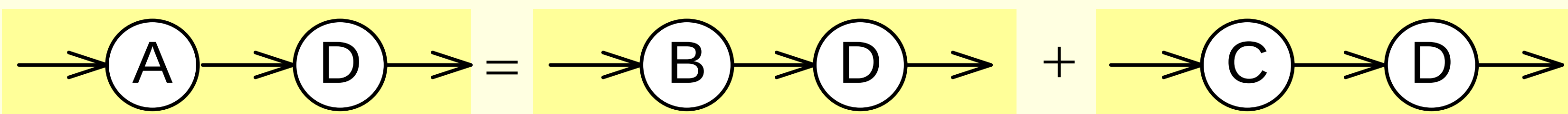
$$= \begin{bmatrix} \alpha - \beta B & \beta A \\ -\beta C & \alpha + \beta B \end{bmatrix}$$

Distributive Law

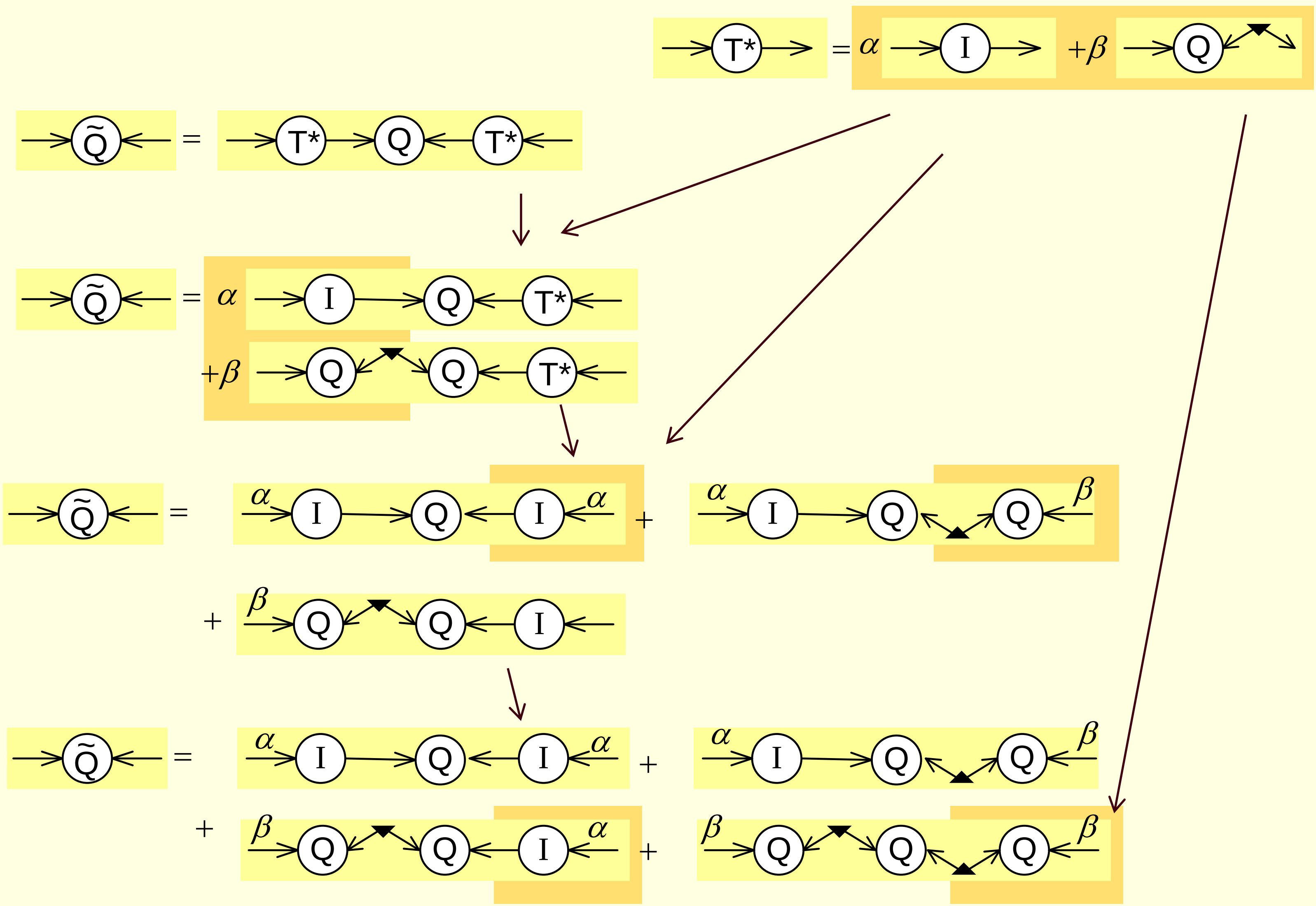
If



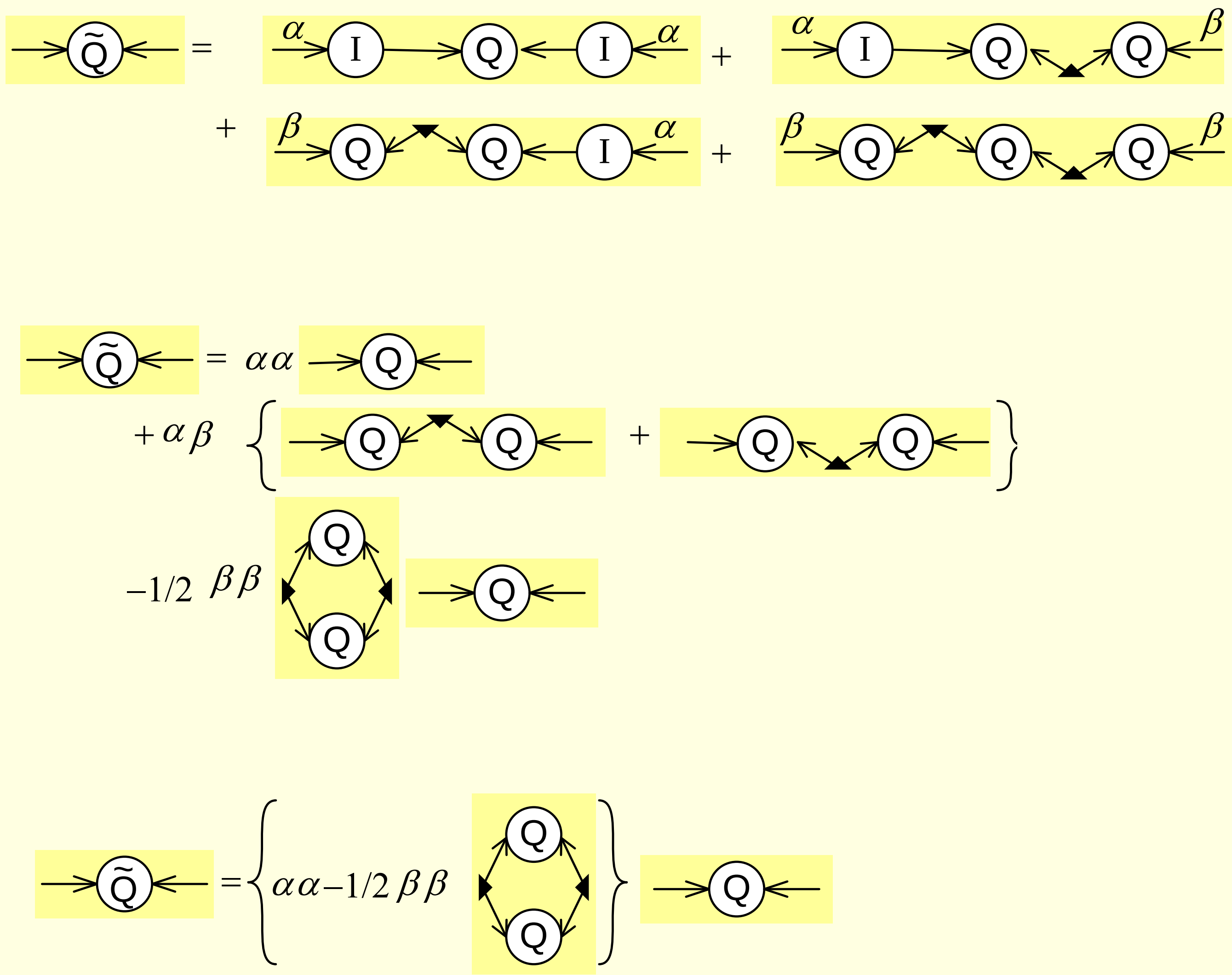
Then



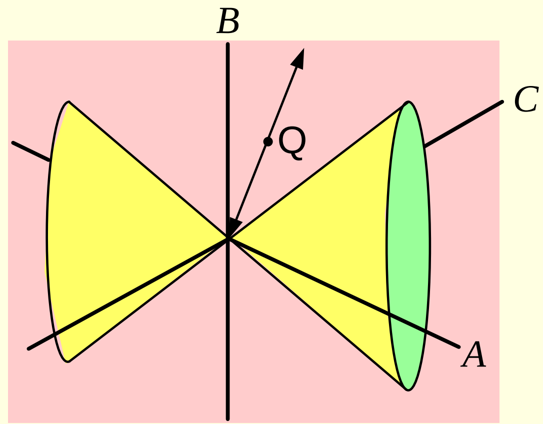
Distributive Law

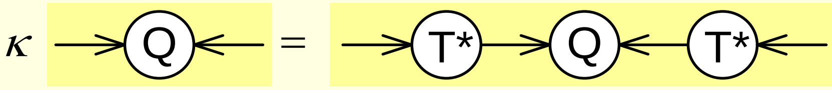


Clean Up

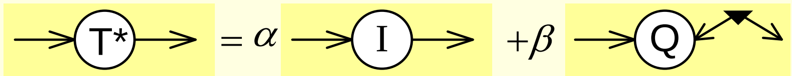


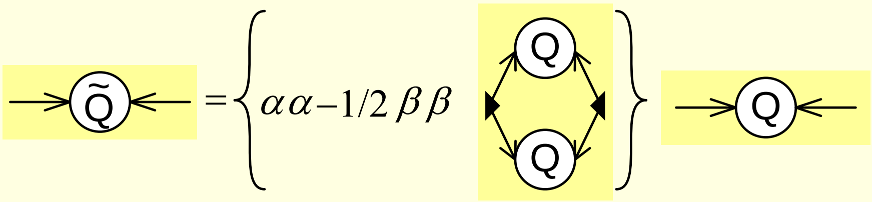
Stationary Transformation



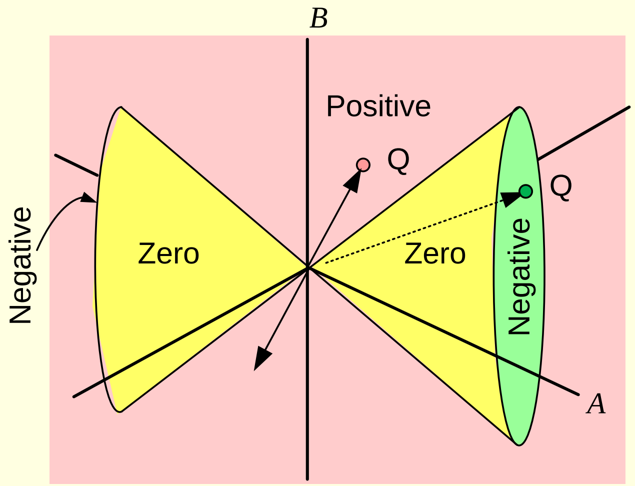


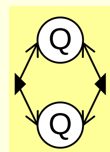
General choice

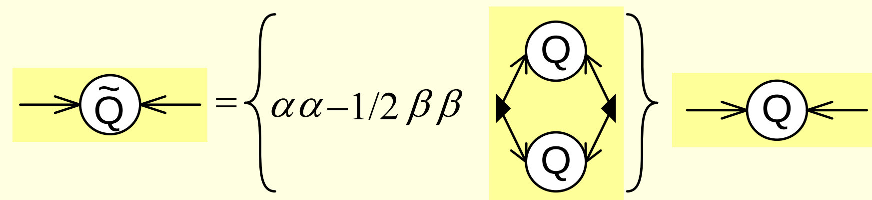


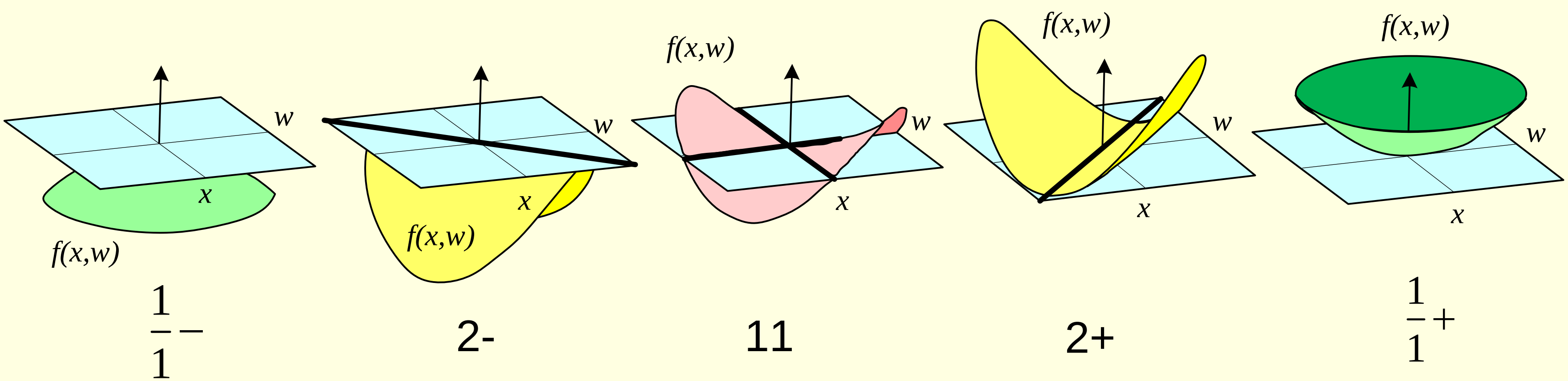


Look at scale factor



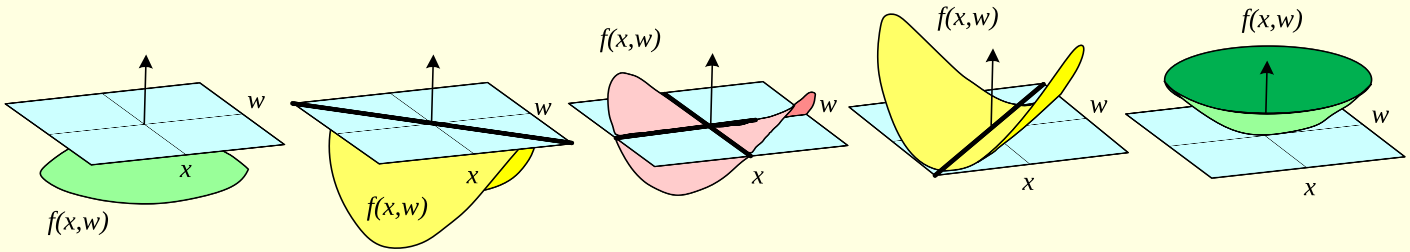






Can only flip sign of Q if in red region
(2 distinct real roots)

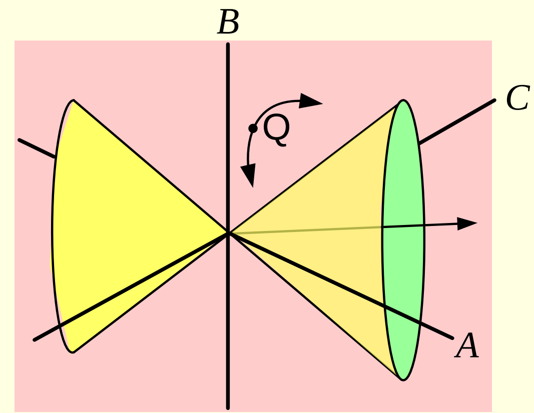
Rotation Transformation



Rotation in (x,w) implies rotation in (A,B,C)

$$\begin{bmatrix} t & u \\ s & v \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$\begin{bmatrix} \tilde{A} \\ \tilde{B} \\ \tilde{C} \end{bmatrix} = \begin{bmatrix} t^2 & 2tu & u^2 \\ ts & tv + us & uv \\ s^2 & 2sv & v^2 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix}$$

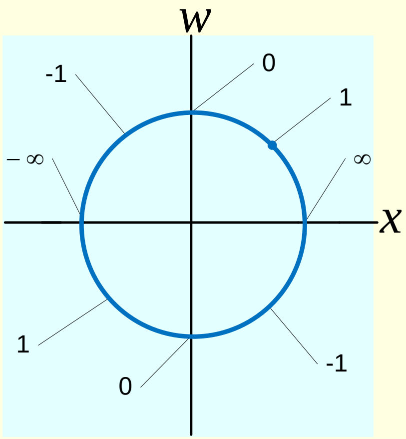


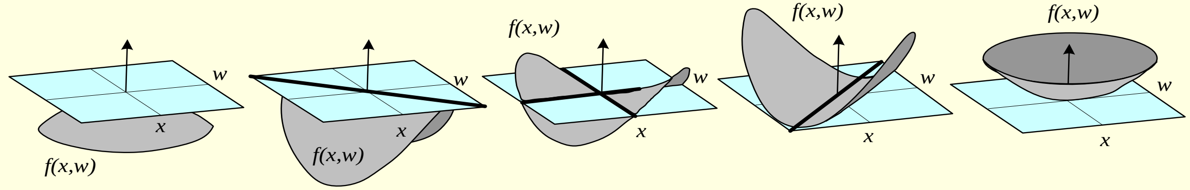
$$\begin{bmatrix} \tilde{A} \\ \tilde{B} \\ \tilde{C} \end{bmatrix} = \begin{bmatrix} \cos^2\theta & 2\cos\theta\sin\theta & \sin^2\theta \\ -\cos\theta\sin\theta & \cos^2\theta - \sin^2\theta & \sin\theta\cos\theta \\ \sin^2\theta & -2\sin\theta\cos\theta & \cos^2\theta \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix}$$

Axis:

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \cos^2\theta & 2\cos\theta\sin\theta & \sin^2\theta \\ -\cos\theta\sin\theta & \cos^2\theta - \sin^2\theta & \sin\theta\cos\theta \\ \sin^2\theta & -2\sin\theta\cos\theta & \cos^2\theta \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Make rotation more obvious





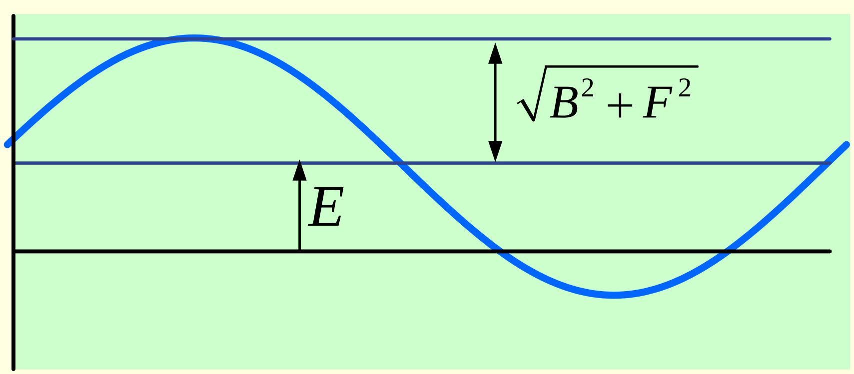
$$\begin{bmatrix} x & w \end{bmatrix} = \begin{bmatrix} r\cos\alpha & r\sin\alpha \end{bmatrix}$$

$$Q(x,w) = Ax^2 + 2Bxw + Cw^2$$

$$\begin{aligned} Q(r,\theta) &= A(r\cos\alpha)^2 + 2B\left(r^2\cos\alpha\sin\alpha\right) + C(r\sin\alpha)^2 \\ &= r^2\left(A\cos^2\alpha + B(2\cos\alpha\sin\alpha) + C\sin^2\alpha\right) \end{aligned}$$

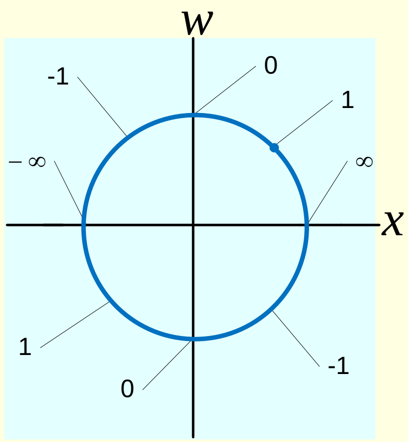
$$\begin{aligned} A &= E + F \\ C &= E - F \end{aligned}$$

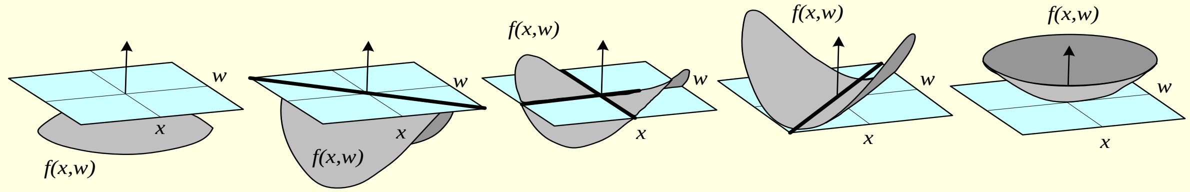
$$\begin{aligned} Q(r,\theta) &= r^2\left(E + F\left(\cos^2\alpha - \sin^2\alpha\right) + B(2\cos\alpha\sin\alpha)\right) \\ &= r^2\left(E + F\cos(2\alpha) + B\sin(2\alpha)\right) \end{aligned}$$

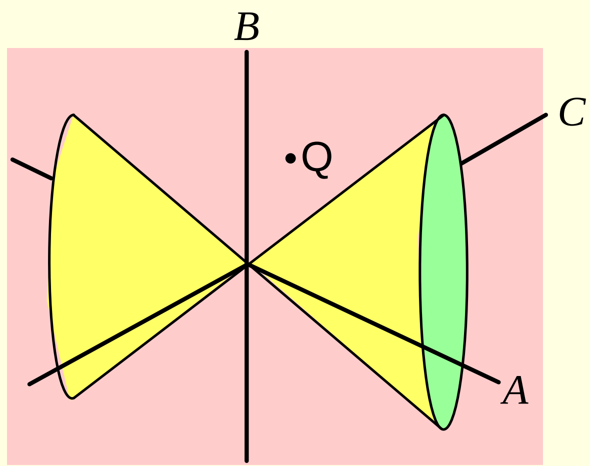


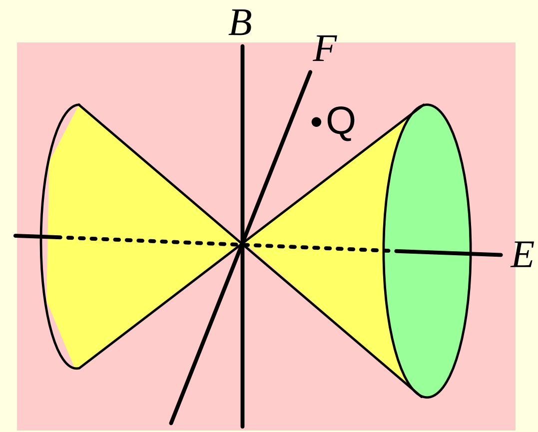
$$\operatorname{discr}(Q) = B^2 - AC = B^2 + F^2 - E^2$$

Make rotation more obvious









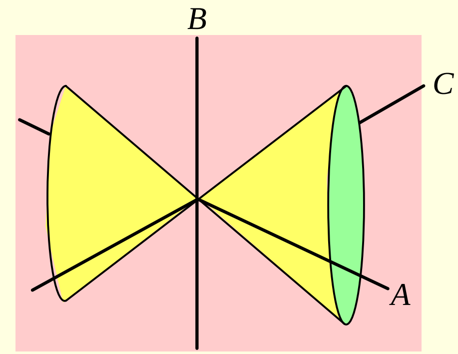
$$B^2 - AC = 0$$

$$B^2 + F^2 - E^2 = 0$$

Effect of E,F on Transformation

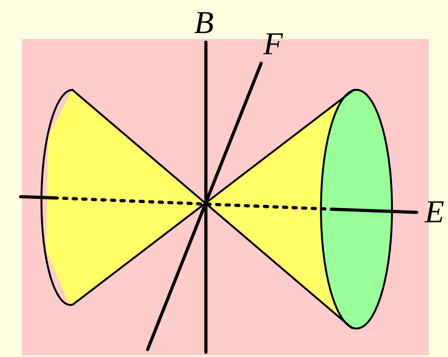
$$\begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} E \\ B \\ F \end{bmatrix}$$

$$\begin{bmatrix} \tilde{A} \\ \tilde{B} \\ \tilde{C} \end{bmatrix} = \begin{bmatrix} t^2 & 2tu & u^2 \\ ts & tv+us & uv \\ s^2 & 2sv & v^2 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix}$$



$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} \tilde{E} \\ \tilde{B} \\ \tilde{F} \end{bmatrix} = \begin{bmatrix} t^2 & 2tu & u^2 \\ ts & tv+us & uv \\ s^2 & 2sv & v^2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} E \\ B \\ F \end{bmatrix}$$

$$\begin{bmatrix} \tilde{E} \\ \tilde{B} \\ \tilde{F} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\left(ss+t^2+uu+vv\right) & sv+tu & \frac{1}{2}\left(ss+tt-u^2-vv\right) \\ st+uv & su+tv & st-uv \\ \frac{1}{2}\left(-s^2+tt+uu-vv\right) & tu-sv & \frac{1}{2}\left(-ss+tt-uu+vv\right) \end{bmatrix} \begin{bmatrix} E \\ B \\ F \end{bmatrix}$$



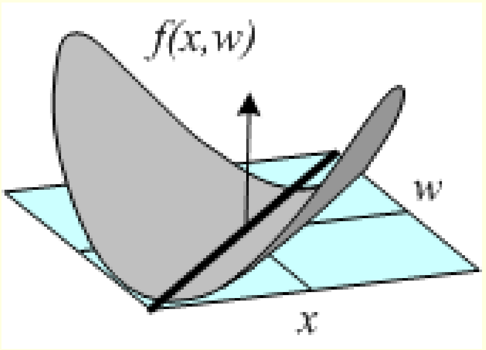
Effect of rotation matrix

$$\begin{bmatrix} t & u \\ s & v \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

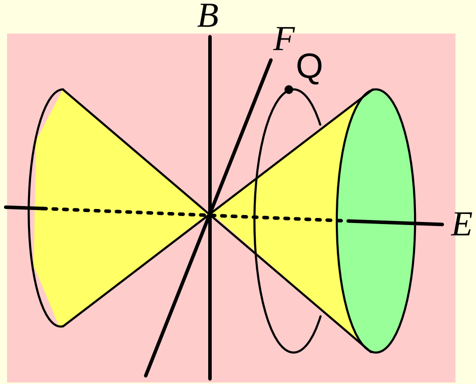
$$\begin{bmatrix} \tilde{E} \\ \tilde{B} \\ \tilde{F} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\left(ss+t^2+uu+vv\right) & sv+tu & \frac{1}{2}\left(ss+tt-u^2-vv\right) \\ st+uv & su+tv & st-uv \\ \frac{1}{2}\left(-s^2+tt+uu-vv\right) & tu-sv & \frac{1}{2}\left(-ss+tt-uu+vv\right) \end{bmatrix} \begin{bmatrix} E \\ B \\ F \end{bmatrix}$$

$$\begin{bmatrix} \tilde{E} \\ \tilde{B} \\ \tilde{F} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos^2\theta-\sin^2\theta & 2\cos\theta\sin\theta \\ 0 & -2\sin\theta\cos\theta & \cos^2\theta-\sin^2\theta \end{bmatrix} \begin{bmatrix} E \\ B \\ F \end{bmatrix}$$

$$\begin{bmatrix} \tilde{E} \\ \tilde{B} \\ \tilde{F} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 2\theta & \sin 2\theta \\ 0 & -\sin 2\theta & \cos 2\theta \end{bmatrix} \begin{bmatrix} E \\ B \\ F \end{bmatrix}$$

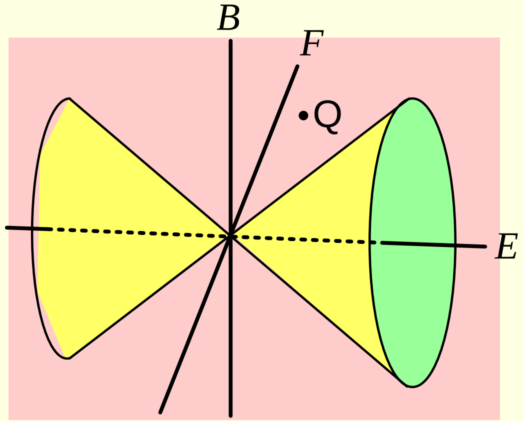


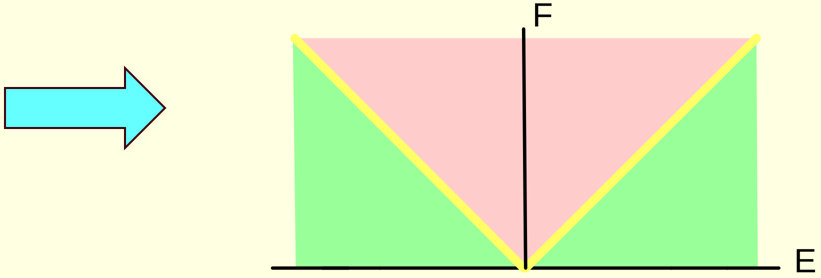
Rotation of 180^{o} for (x,w) gives same function
$\Rightarrow$ rotation by 360^{o} in (E,F,B)



Use rotation matrix to "distill" EBF space

$$\begin{bmatrix} \tilde{E} \\ \tilde{B} \\ \tilde{F} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 2\theta & \sin 2\theta \\ 0 & -\sin 2\theta & \cos 2\theta \end{bmatrix} \begin{bmatrix} E \\ B \\ F \end{bmatrix}$$





Further Distillation

Scale transformation

$$\begin{bmatrix} t & u \\ s & v \end{bmatrix} = \begin{bmatrix} t & 0 \\ 0 & v \end{bmatrix}$$

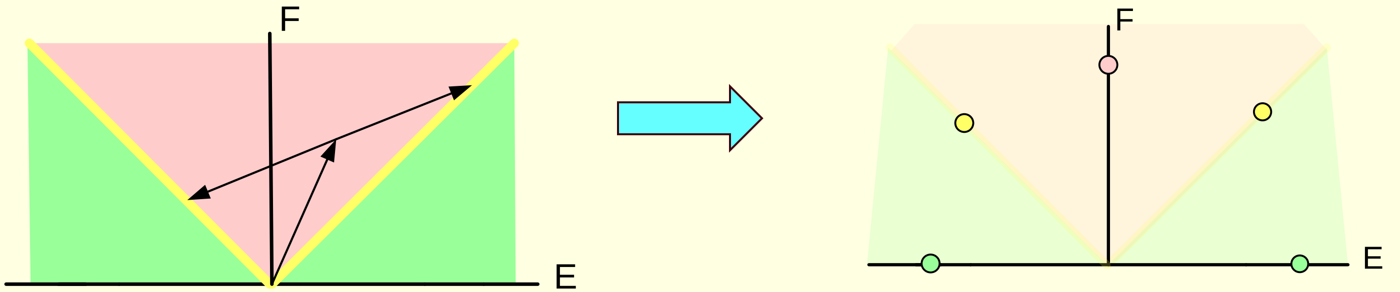
$$\begin{bmatrix} \tilde{E} \\ \tilde{B} \\ \tilde{F} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\left(s^2+t^2+u^2+v^2\right) & sv+tu & \frac{1}{2}\left(s^2+t^2-u^2-v^2\right) \\ st+uv & su+tv & st-uv \\ \frac{1}{2}\left(-s^2+t^2+u^2-v^2\right) & tu-sv & \frac{1}{2}\left(-s^2+t^2-u^2+v^2\right) \end{bmatrix} \begin{bmatrix} E \\ B \\ F \end{bmatrix}$$

$$\begin{bmatrix} \tilde{E} \\ 0 \\ \tilde{F} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\left(t^2+vv\right) & 0 & \frac{1}{2}\left(tt-vv\right) \\ 0 & tv & 0 \\ \frac{1}{2}\left(tt-vv\right) & 0 & \frac{1}{2}\left(tt+vv\right) \end{bmatrix} \begin{bmatrix} E \\ 0 \\ F \end{bmatrix}$$

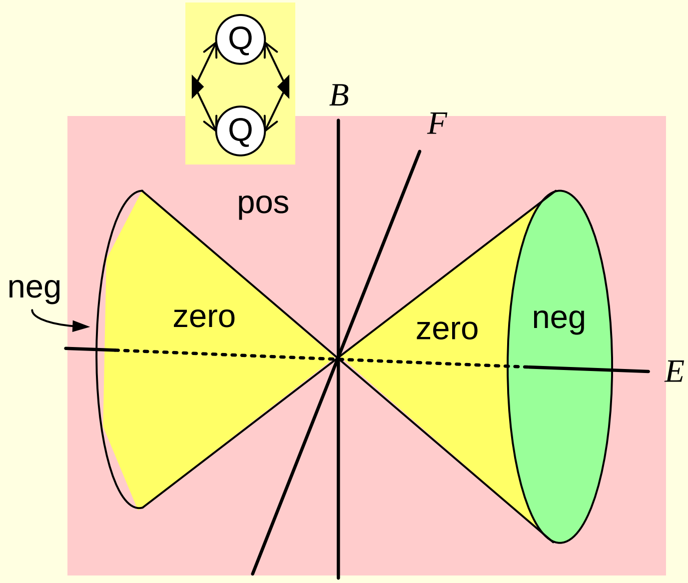
$$t = r\sin\theta$$
$$v = r\cos\theta$$

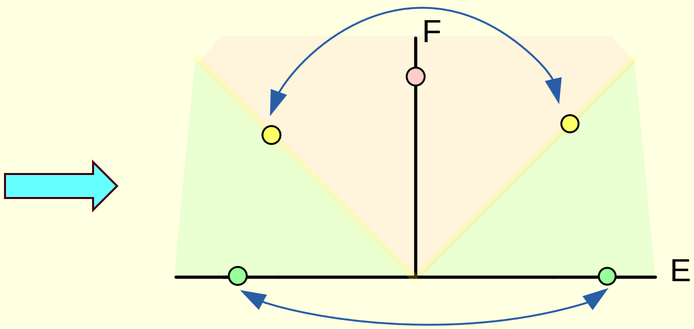
$$\begin{bmatrix} \tilde{E} \\ \tilde{F} \end{bmatrix} = \frac{1}{2} r^2 \left\{ \begin{bmatrix} E \\ F \end{bmatrix} + \cos 2\theta \begin{bmatrix} F \\ E \end{bmatrix} \right\}$$

Five Equivalence Classes

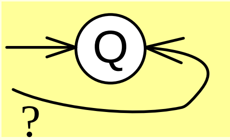


Distinguish Between Regions

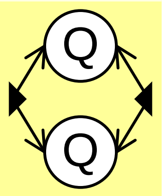




$$\text{trace}\,\mathbf{Q} = \sum_i Q_{ii} =$$

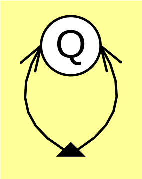


The only invariant

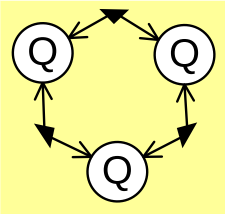


$= -2\ \det\mathbf{Q}$

These don't help



$= 0$

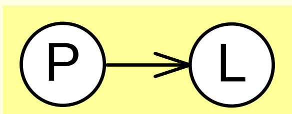


$= 0$

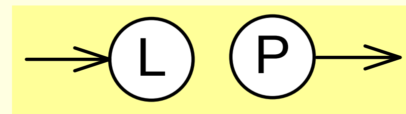
Want to use E=trace(Q)
But it's not invariant
But it is for the cases we want
What to do?

Internal Structure

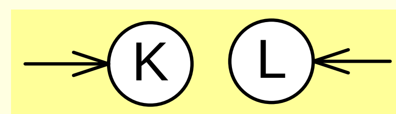
Inner and Outer Products



$$= \begin{bmatrix} x & w \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = Ax + Bw$$



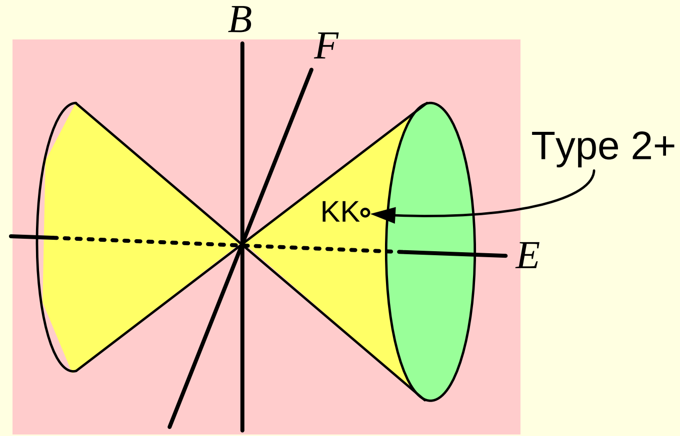
$$= \begin{bmatrix} A \\ B \end{bmatrix} \begin{bmatrix} x & w \end{bmatrix} = \begin{bmatrix} Ax & Aw \\ Bx & Bw \end{bmatrix}$$



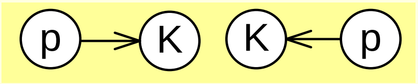
$$= \begin{bmatrix} K_1 \\ K_2 \end{bmatrix} \begin{bmatrix} L_1 & L_2 \end{bmatrix} = \begin{bmatrix} K_1 L_1 & K_1 L_2 \\ K_2 L_1 & K_2 L_2 \end{bmatrix}$$

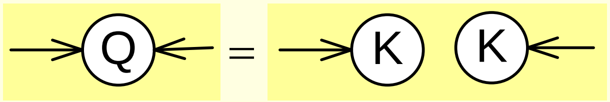
Subatomic Physics

singular quadratics



$$Q\left(x,w\right) = Ax^2 + 2Bxw + Cw^2 = \left(xK_1 + wK_2\right)^2$$

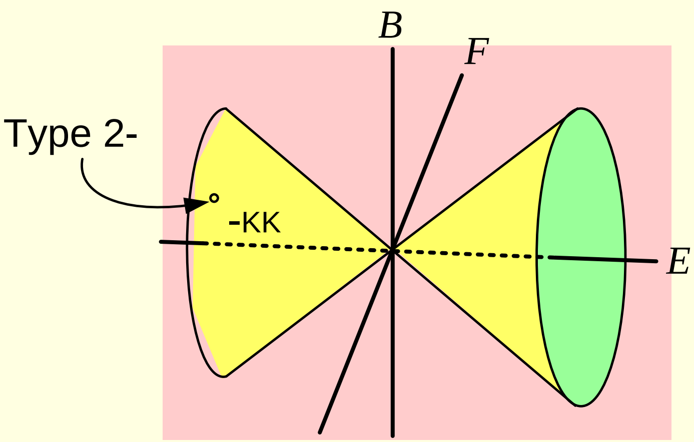


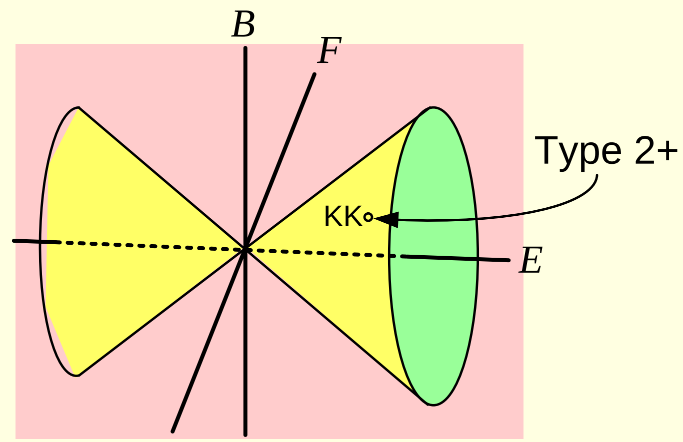


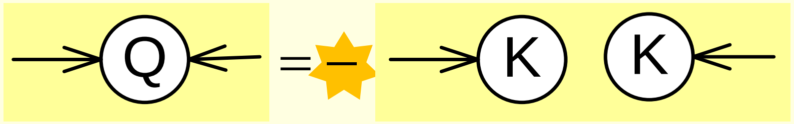
$$= \begin{bmatrix} K_1 \\ K_2 \end{bmatrix} \begin{bmatrix} K_1 & K_2 \end{bmatrix} = \begin{bmatrix} K_1{}^2 & K_1 K_2 \\ K_1 K_2 & K_2{}^2 \end{bmatrix}$$

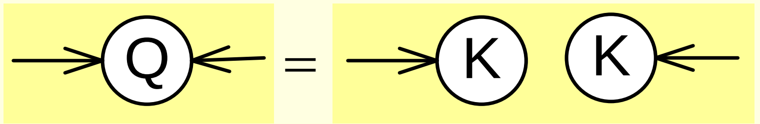
Subatomic Physics

singular quadratics







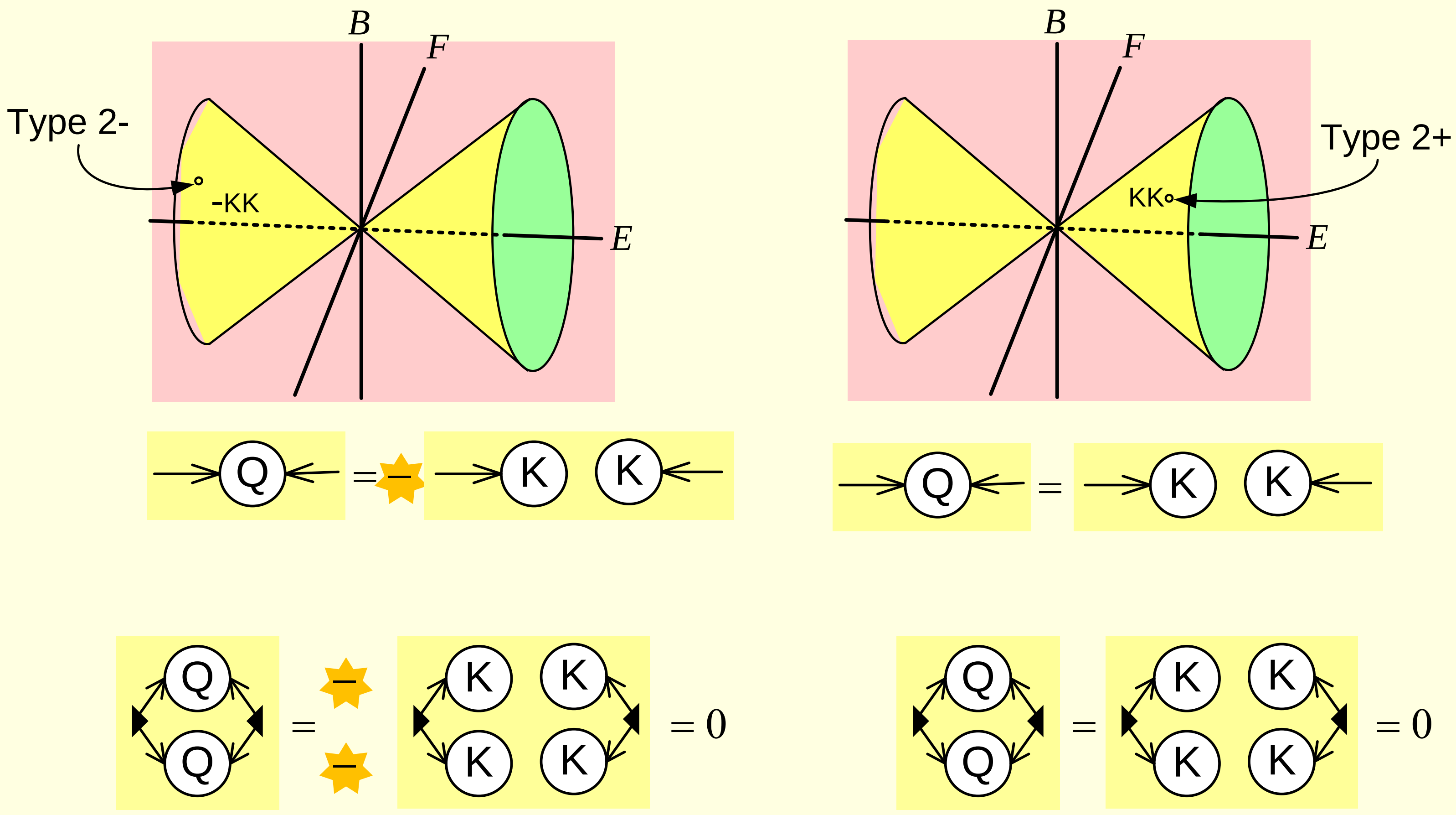


$$= \begin{bmatrix} -K_1{}^2 & -K_1K_2 \\ -K_1K_2 & -K_2{}^2 \end{bmatrix}$$

$$= \begin{bmatrix} K_1{}^2 & K_1K_2 \\ K_1K_2 & K_2{}^2 \end{bmatrix}$$

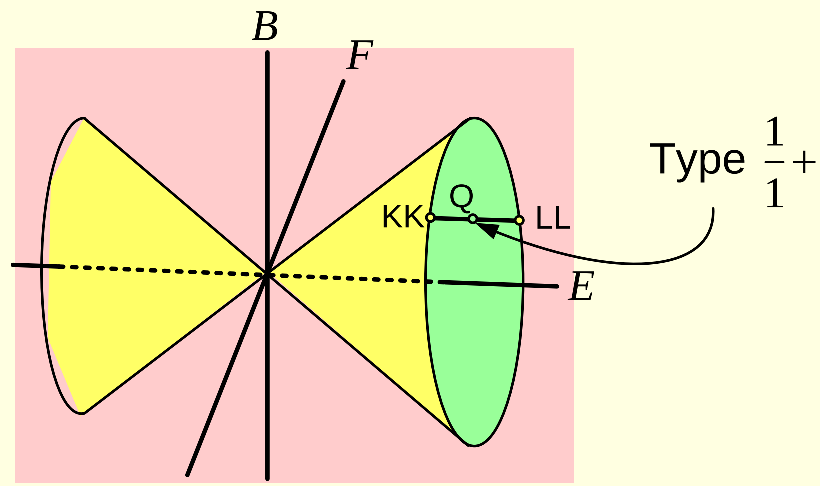
Subatomic Physics

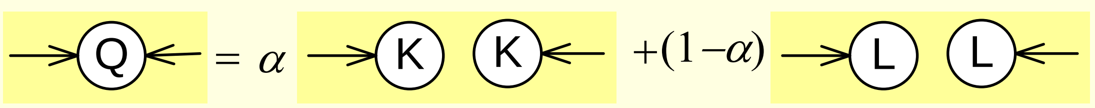
singular quadratics [invariant test]



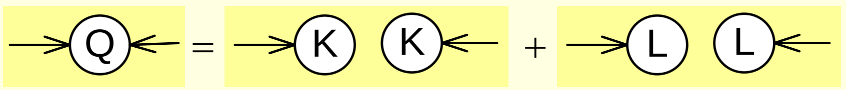
Subatomic physics

The no-real-root quadratic (+)

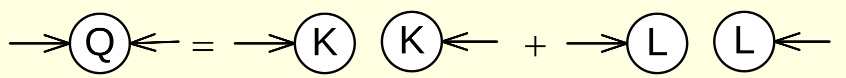




Can absorb α and (1-α) into K and L

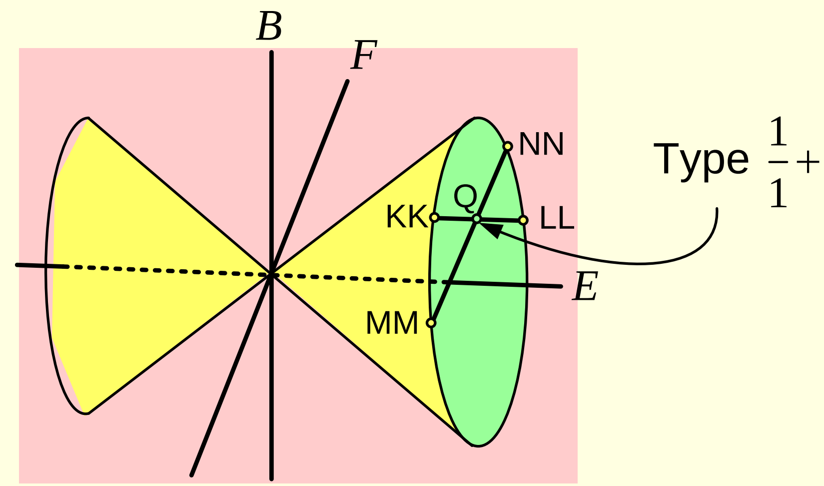


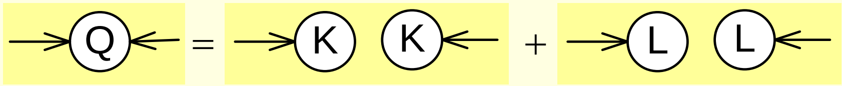
Why we need yellow boxes



Subatomic physics

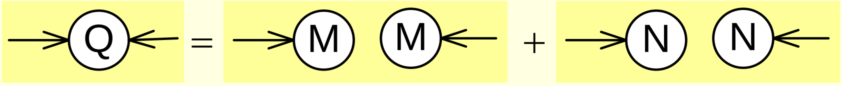
The no-real-root quadratic (+)





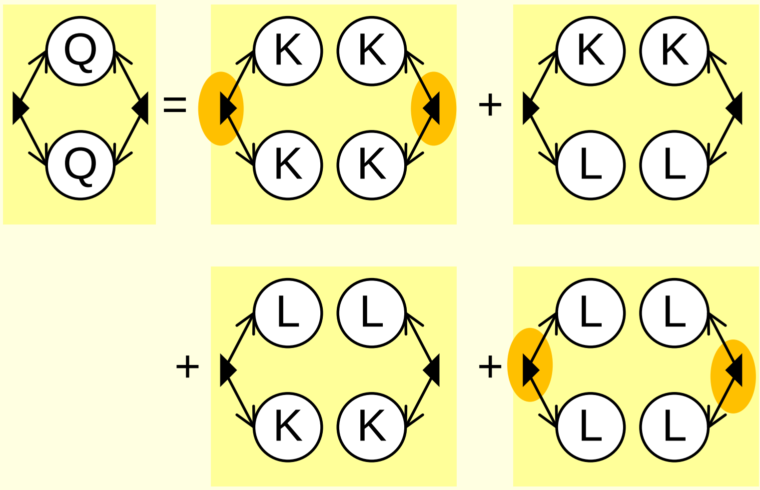
Not unique

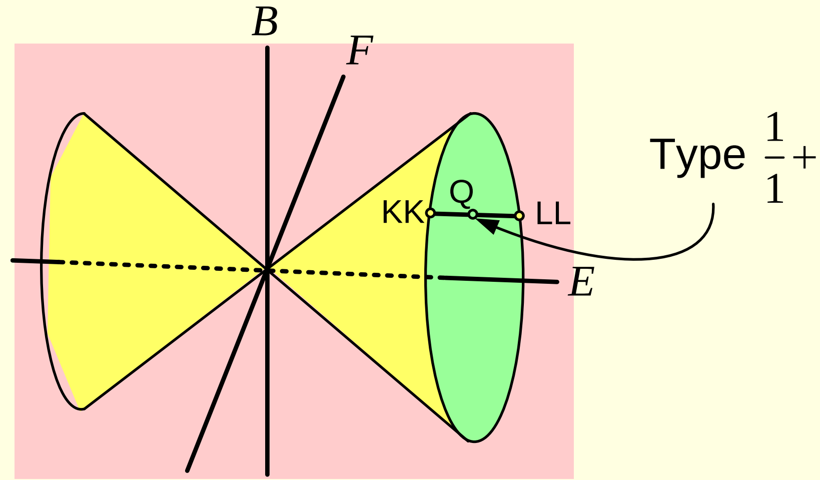
No one is preferred

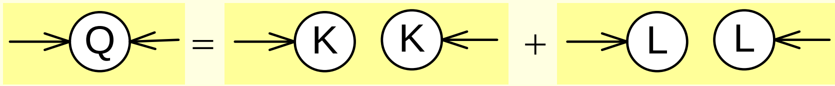


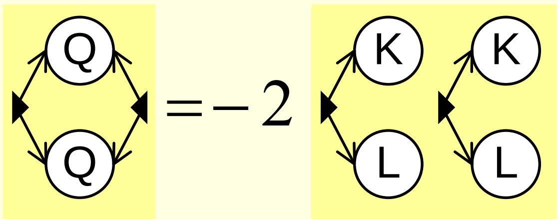
Subatomic physics

The no-real-root quadratic (+)   [invariant test]



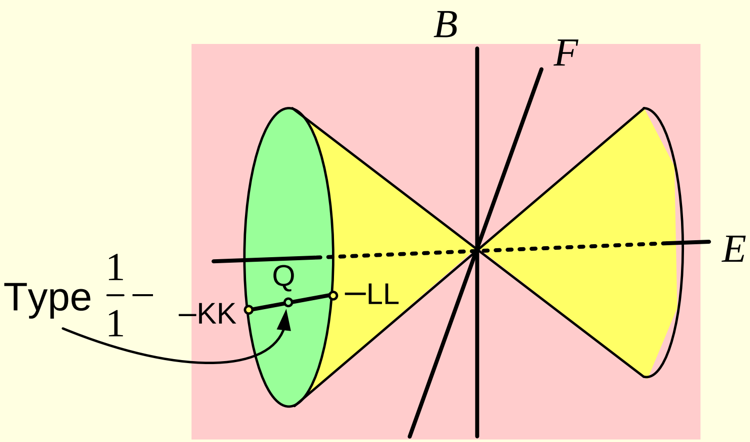


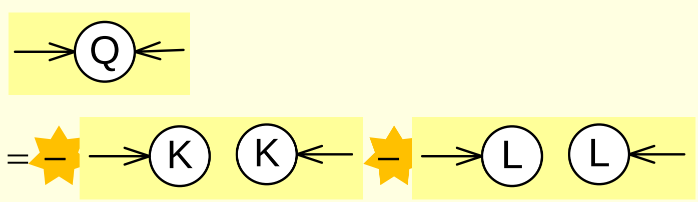




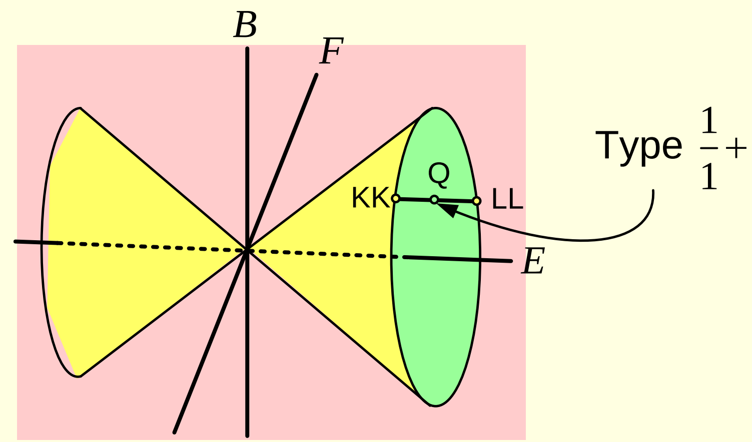
Subatomic physics

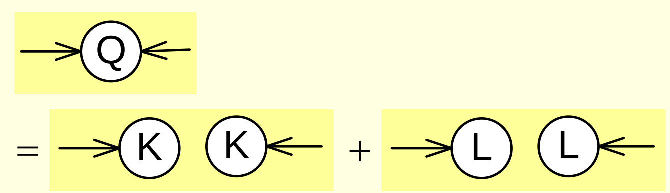
The no-real-root quadratic (-)





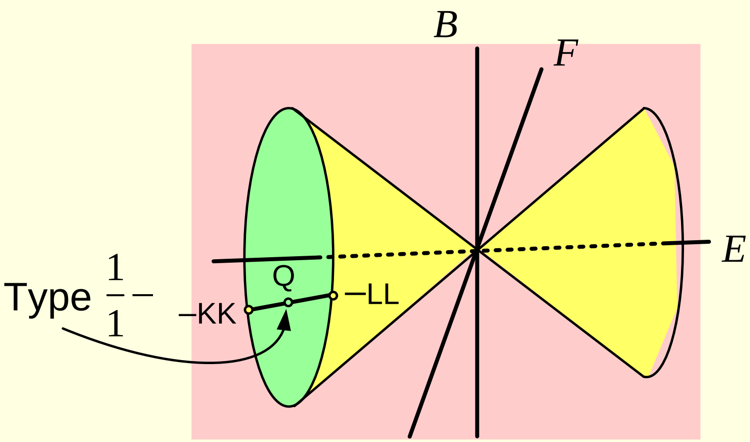
The no-real-root quadratic (+)

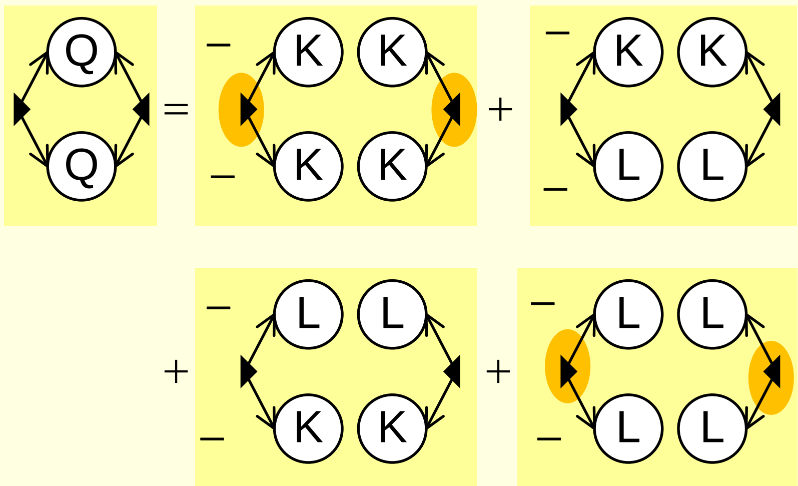


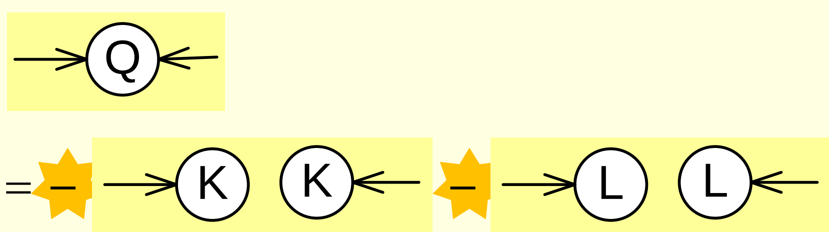


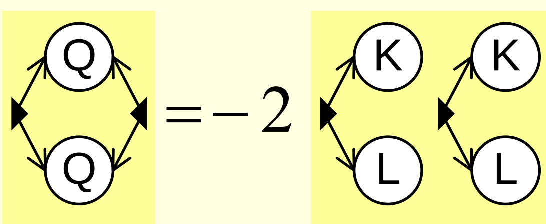
Subatomic physics **[invariant test]**

The no-real-root quadratic (-)



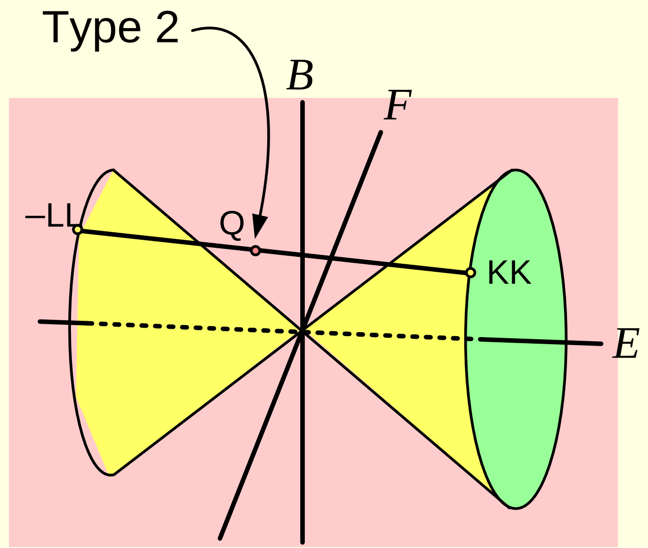


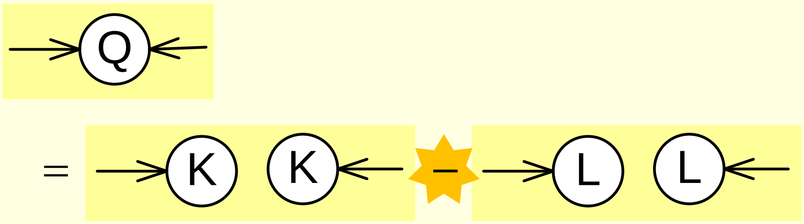




Subatomic physics

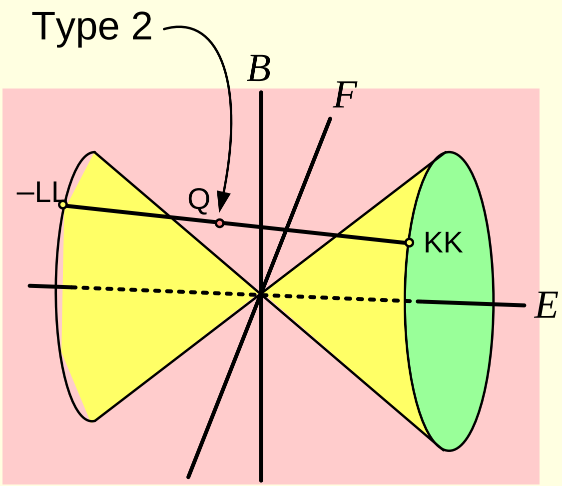
The two-real-roots quadratic

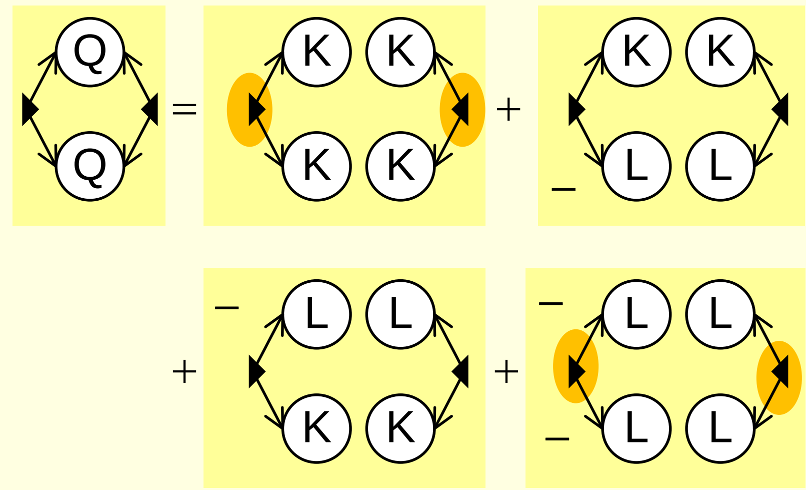


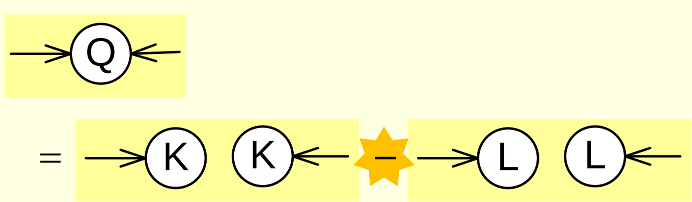


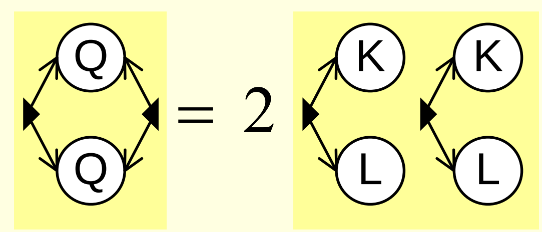
Subatomic physics

The two-real-roots quadratic [invariant test]





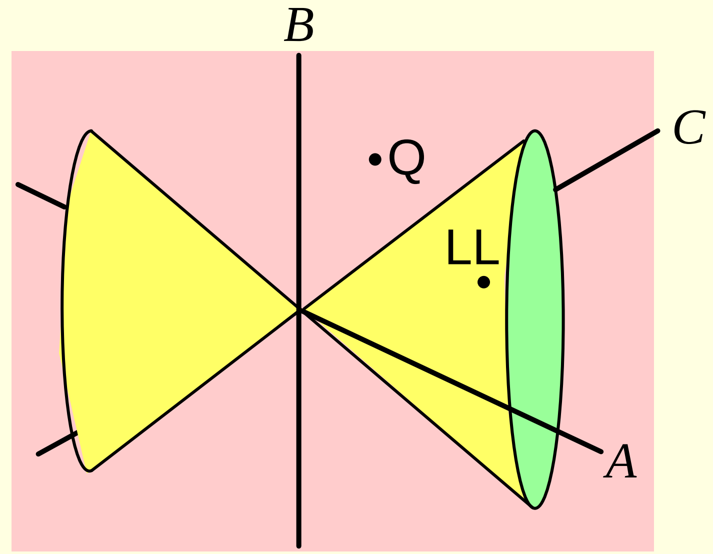




Locus of Q's containing L as factor

$$Q = (K_1 x + K_2 w)(L_1 x + L_2 w)$$
$$= K_1 L_1 x^2 + (K_1 L_2 + K_2 L_1) xw + K_2 L_2 w^2$$

$$\begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} K_1 L_1 \\ \frac{1}{2}(K_1 L_2 + K_2 L_1) \\ K_2 L_2 \end{bmatrix} = K_1 \begin{bmatrix} L_1 \\ \frac{1}{2} L_2 \\ 0 \end{bmatrix} + K_2 \begin{bmatrix} 0 \\ \frac{1}{2} L_1 \\ L_2 \end{bmatrix}$$



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fix

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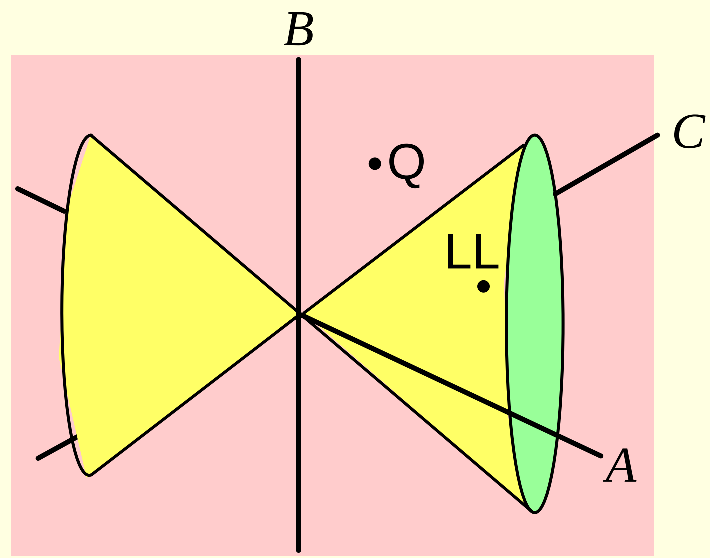
vary

Spans a plane



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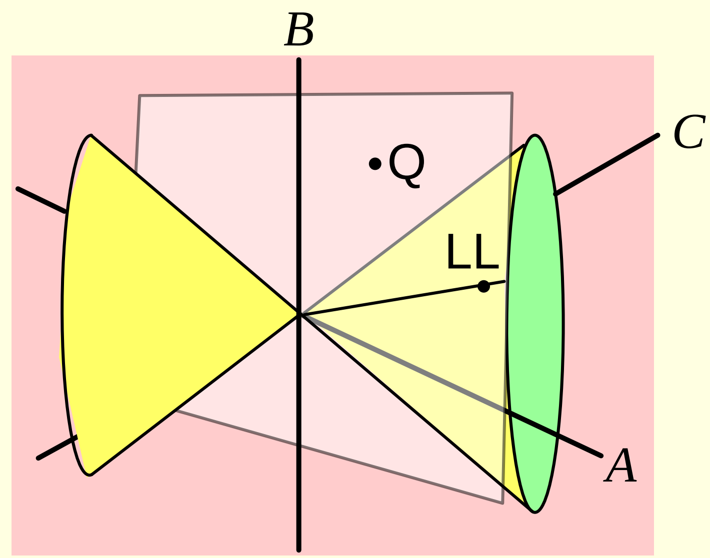
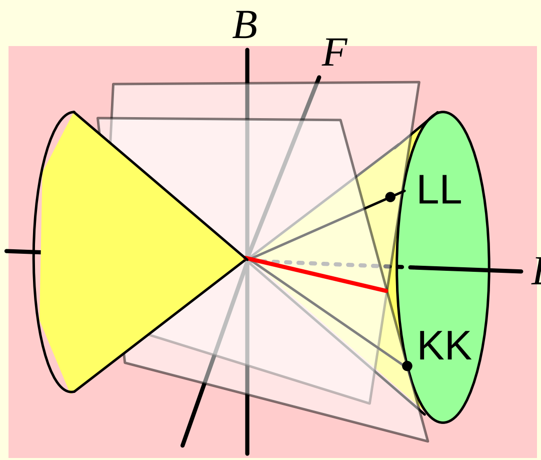
vary

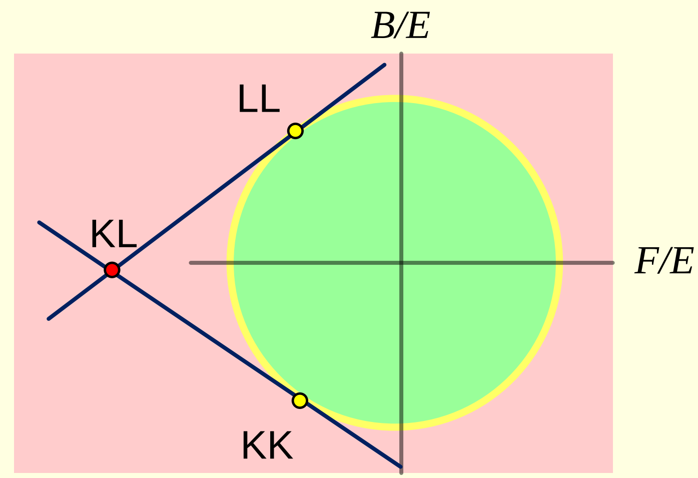
Spans a plane

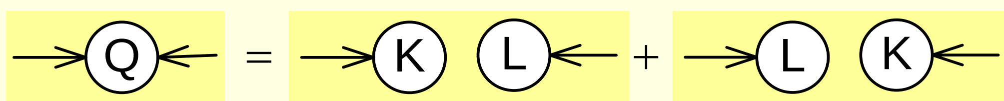


Plane tangent to cone along line LL

Intersection of 2 loci



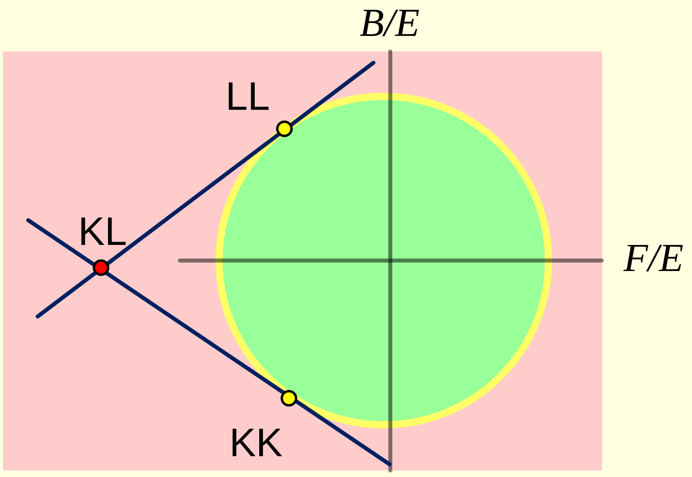


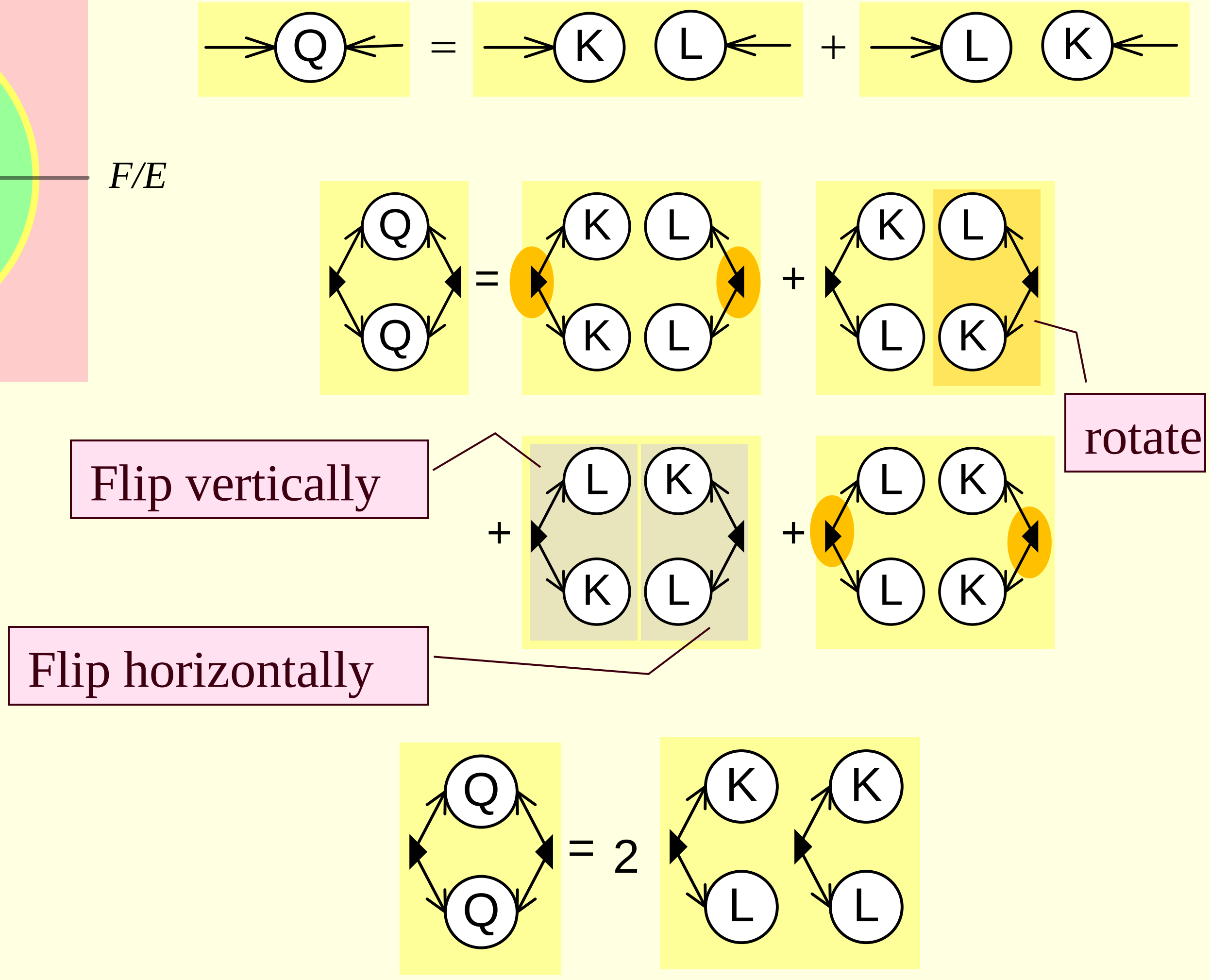


Given Q, find K,L

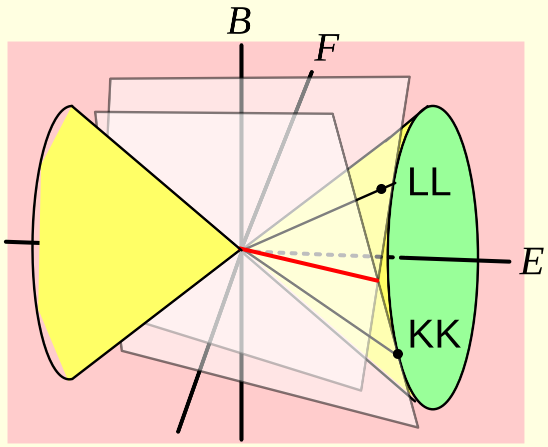
→Draw tangents from Q to cone

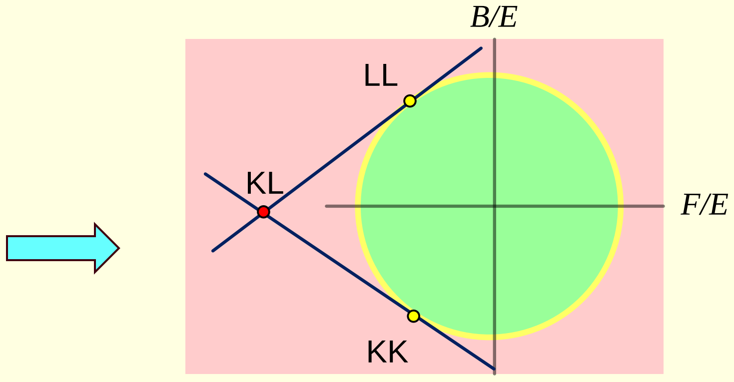
Another Invariant Test



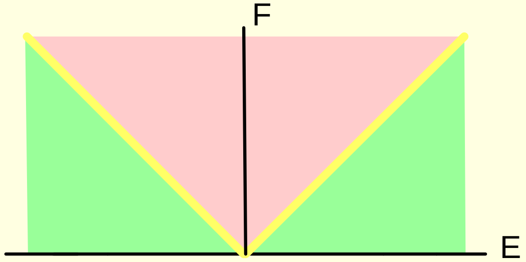


Two possible mappings 3D->2D





☺ Preserves lines
☹ Maps + and – cones together



☺ + and – cones distinct
☹ Lines not preserved

To be continued