The EM Algorithm for Image Segmentation

1 Notation

- The image contains r pixels.
- The feature vector for pixel j is called x_j.
- There are going to be K segments; K is given.
- The ith segment has a Gaussian distribution with parameters $\theta_i = (\mu_i, \Sigma_i)$.
- There is a support map S_i for segment i to represent the probabilistic assignment of a pixel to a segment.
- $S_i[j]$ is the probability that pixel j belongs to segment i.
- The K Gaussians are used in a mixture model. The form of the probability density function is:

  $$f(x|\Theta) = \sum_{i=1}^{K} \alpha_i f_i(x|\theta_i)$$

  where x is a feature vector, the α_i's are the mixing weights (which sum to 1), Θ is the collection of parameters $(\alpha_1, \ldots, \alpha_K, \theta_1, \ldots, \theta_K)$, and f_i is the multivariate Gaussian density function

  $$f_i(x|\theta_i) = \frac{1}{(2\pi)^{d/2} det\Sigma_i^{1/2}} e^{-\frac{1}{2}(x-u_i)^T \Sigma_i^{-1}(x-u_i)}$$

  where d is the dimension of the feature space.

2 Initialization

- Each of the K Gaussians will have parameters $\theta_i = (\mu_i, \Sigma_i)$ where
  - μ_i is the mean of the ith Gaussian.
  - Σ_i is the covariance matrix of the ith Gaussian.
- The covariance matrices are initialed to be the identity matrix.
- The means can be initialized by finding the average feature vectors in each of K windows in the image; this is data-driven initialization.

3 Iteration

3.1 E-step

- Calculate $S_i[j]$ based on the current estimation of Θ (expectation):

$$S_i[j] = p(i|x_j, \Theta) = \frac{\alpha_i f_i(x_j|\theta_i)}{\sum_{k=1}^{K} \alpha_k f_k(x_j|\theta_k)}$$

3.2 M-step

- Re-estimate the parameters by maximize the likelihood:

$$\mu_i^{new} = \frac{\sum_{j=1}^{N} x_j p(i|x_j, \Theta^{old})}{\sum_{j=1}^{N} p(i|x_j, \Theta^{old})}$$

$$\Sigma_i^{new} = \frac{\sum_{j=1}^{N} p(i|x_j, \Theta^{old})(x_j - \mu_i^{new})(x_j - \mu_i^{new})^T}{\sum_{j=1}^{N} p(i|x_j, \Theta^{old})}$$

$$\alpha_i^{new} = \frac{1}{N} \sum_{j=1}^{N} p(i|x_j, \Theta^{old})$$

Or, using the symbol of $S_i[j]$:

$$\mu_i^{new} = \frac{\sum_{j=1}^{N} x_j S_i[j]}{\sum_{j=1}^{N} S_i[j]}$$

$$\Sigma_i^{new} = \frac{\sum_{j=1}^{N} (x_j - \mu_i^{new})(x_j - \mu_i^{new})^T S_i[j]}{\sum_{j=1}^{N} S_i[j]}$$

$$\alpha_i^{new} = \frac{1}{N} \sum_{j=1}^{N} S_i[j]$$