
CSE 573: Artificial Intelligence

Hanna Hajishirzi
Markov Decision Processes

slides adapted from
Dan Klein, Pieter Abbeel ai.berkeley.edu
And Dan Weld, Luke Zettlemoyer

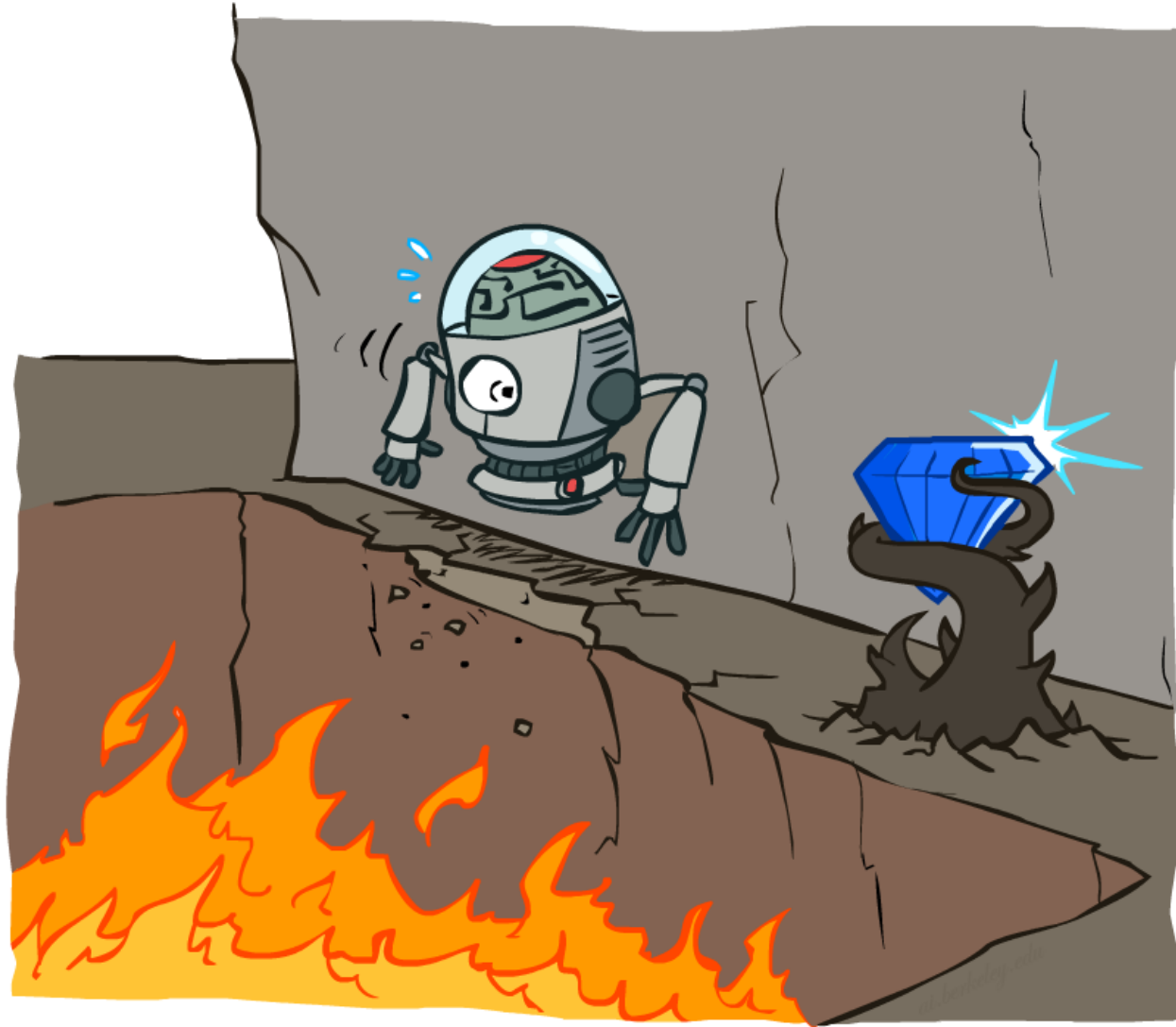


Review and Outline

- Adversarial Games
 - Minimax search
 - α - β search
 - Evaluation functions
 - Multi-player, non-0-sum
- Stochastic Games
 - Expectimax
- Markov Decision Processes
- Reinforcement Learning

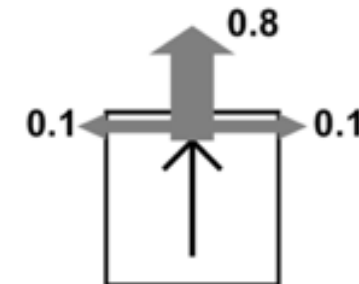
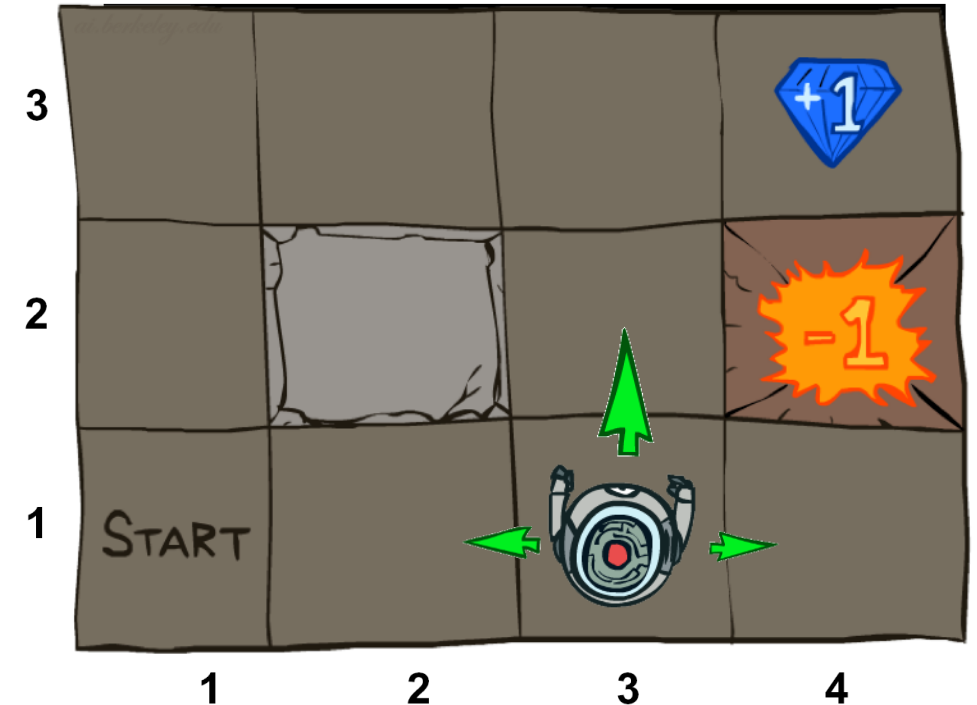


Non-Deterministic Search



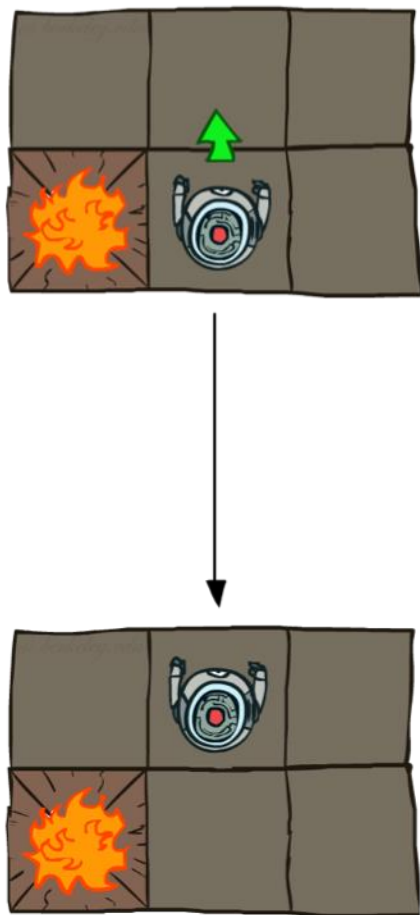
Example: Grid World

- A maze-like problem
 - The agent lives in a grid
 - Walls block the agent's path
- Noisy movement: actions do not always go as planned
 - 80% of the time, the action North takes the agent North (if there is no wall there)
 - 10% of the time, North takes the agent West; 10% East
 - If there is a wall in the direction the agent would have been taken, the agent stays put
- The agent receives rewards each time step
 - Small “living” reward each step (can be negative)
 - Big rewards come at the end (good or bad)
- Goal: maximize sum of rewards

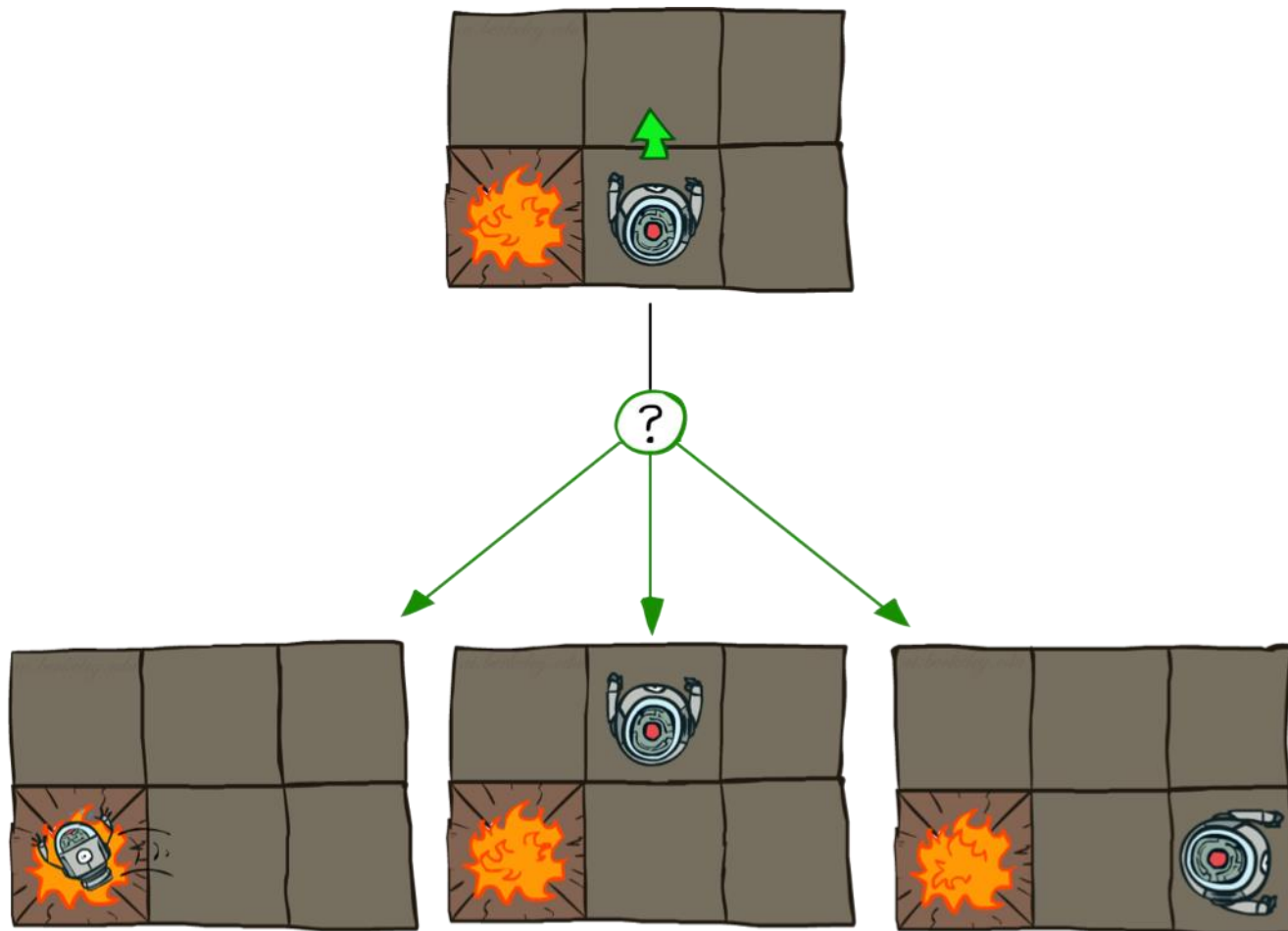


Grid World Actions

Deterministic Grid World



Stochastic Grid World



Markov Decision Processes

- An MDP is defined by:
 - A **set of states** $s \in S$
 - A **set of actions** $a \in A$
 - A **transition function** $T(s, a, s')$
 - Probability that a from s leads to s' , i.e., $P(s' | s, a)$
 - Also called the model or the dynamics

$T(s_{11}, E, s_{11}) = 1$

$T(s_{31}, N, s_{11}) = 0$

$T(s_{31}, N, s_{32}) = 0.8$

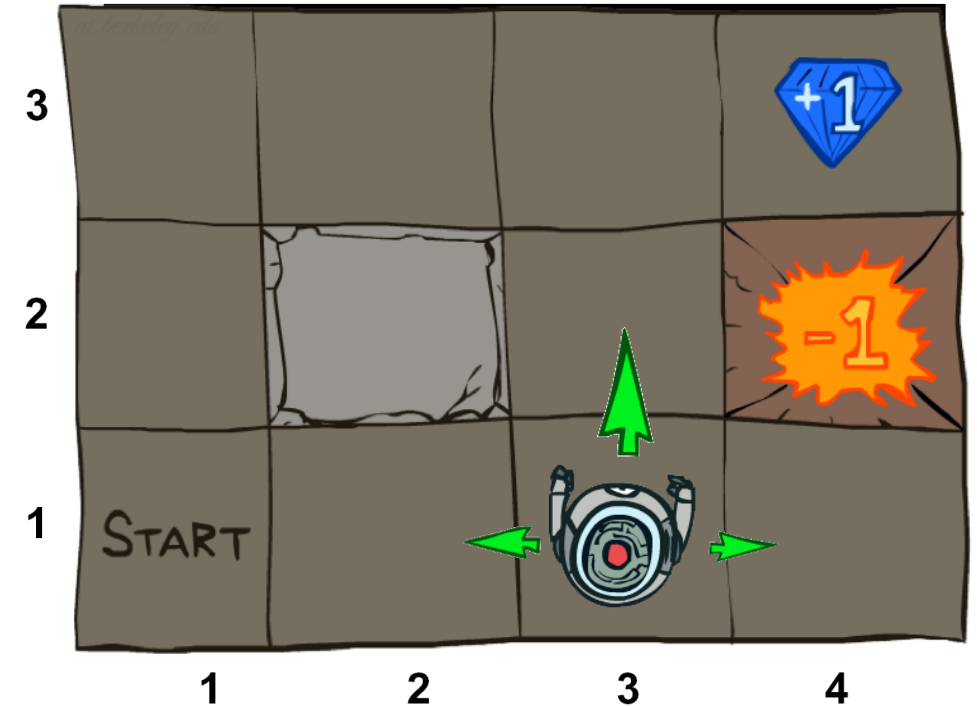
$T(s_{31}, N, s_{21}) = 0.1$

$T(s_{31}, N, s_{41}) = 0.1$

...

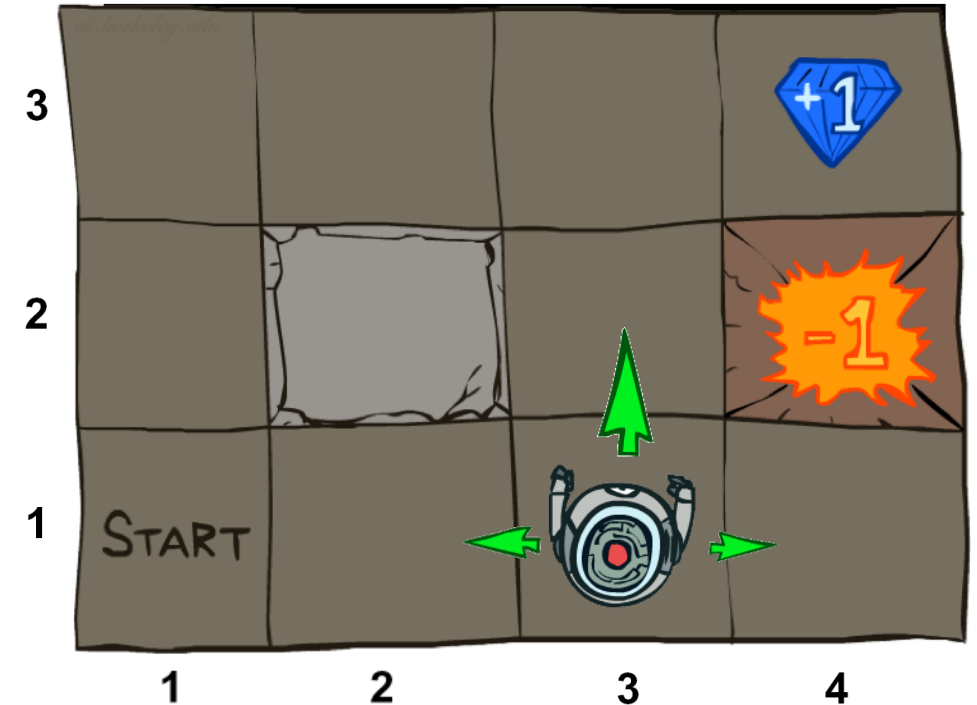
This is a Big Table!
 $11 \times 4 \times 11 = 484$ entries

For now, we give this as input to the agent



Markov Decision Processes

- An MDP is defined by:
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 - A **set of actions** $a \in A$
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 - Probability that a from s leads to s' , i.e., $P(s' | s, a)$
 - Also called the model or the dynamics
 - A **reward function** $R(s, a, s')$
 - Sometimes just $R(s)$ or $R(s')$



$$R(s_{32}, N, s_{33}) = 0.01$$

$$R(s_{32}, N, s_{42}) = 1.01$$

$$R(s_{33}, E, s_{43}) = 0.99$$

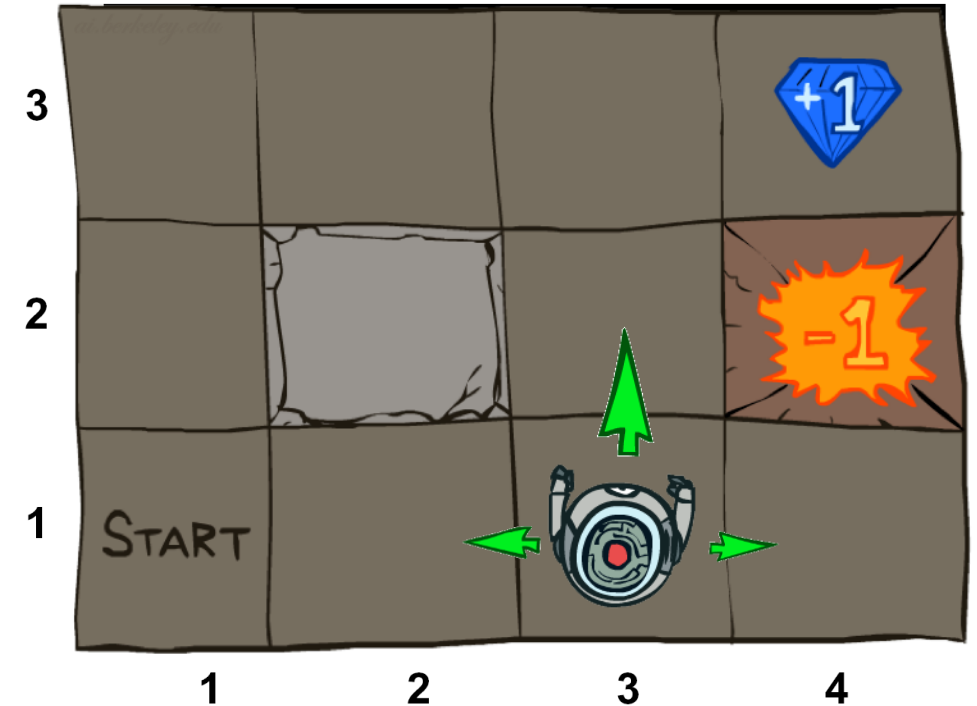
Cost of breathing

R is also a Big Table!

For now, we also give this to the agent

Markov Decision Processes

- An MDP is defined by:
 - A **set of states** $s \in S$
 - A **set of actions** $a \in A$
 - A **transition function** $T(s, a, s')$
 - Probability that a from s leads to s' , i.e., $P(s' | s, a)$
 - Also called the model or the dynamics
 - A **reward function** $R(s, a, s')$
 - Sometimes just $R(s)$ or $R(s')$
 - A **start state**
 - Maybe a **terminal state**
- MDPs are non-deterministic search problems
 - One way to solve them is with expectimax search
 - We'll have a new tool soon



What is Markov about MDPs?

- “Markov” generally means that given the present state, the future and the past are independent
- For Markov decision processes, “Markov” means action outcomes depend only on the current state

$$\begin{aligned} &P(S_{t+1} = s' | S_t = s_t, A_t = a_t, S_{t-1} = s_{t-1}, A_{t-1}, \dots, S_0 = s_0) \\ &= \\ &P(S_{t+1} = s' | S_t = s_t, A_t = a_t) \end{aligned}$$

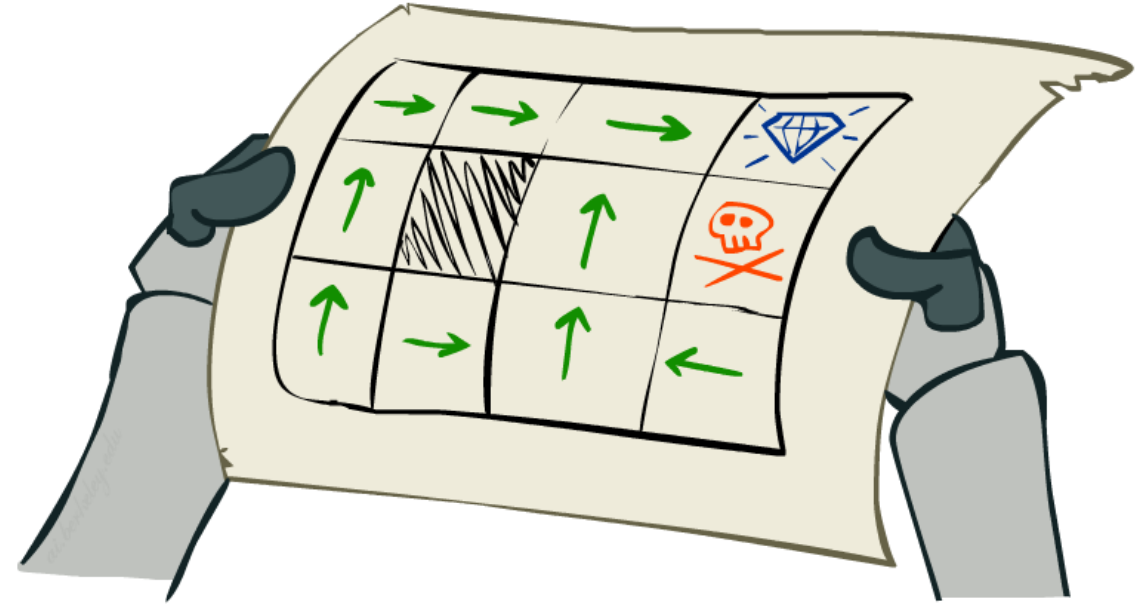
- This is just like search, where the successor function could only depend on the current state (not the history)



Andrey Markov
(1856-1922)

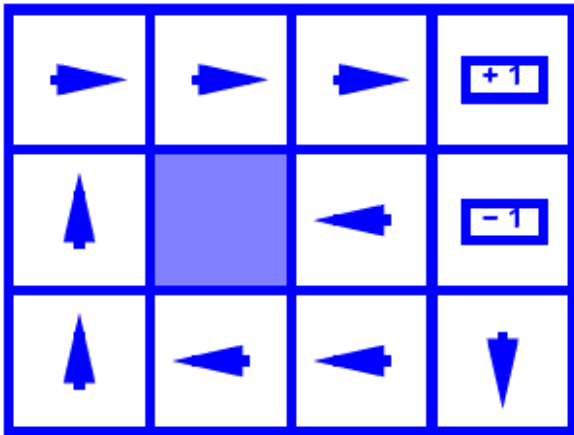
Policies

- In deterministic single-agent search problems, we wanted an optimal **plan**, or sequence of actions, from start to a goal
- For MDPs, we want an optimal **policy** $\pi^*: S \rightarrow A$
 - A policy π gives an action for each state
 - An optimal policy is one that maximizes expected utility if followed
 - An explicit policy defines a reflex agent

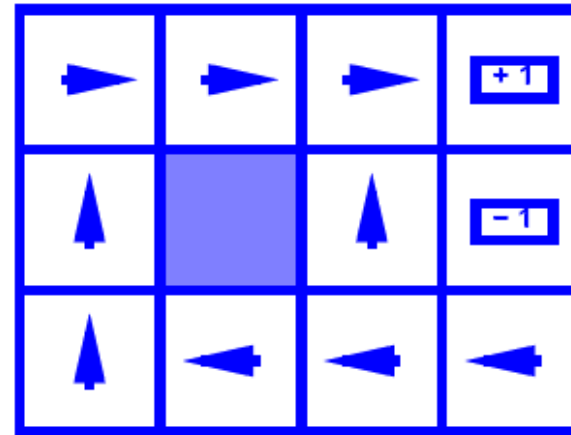


Optimal policy when $R(s, a, s') = -0.4$ for all non-terminals s

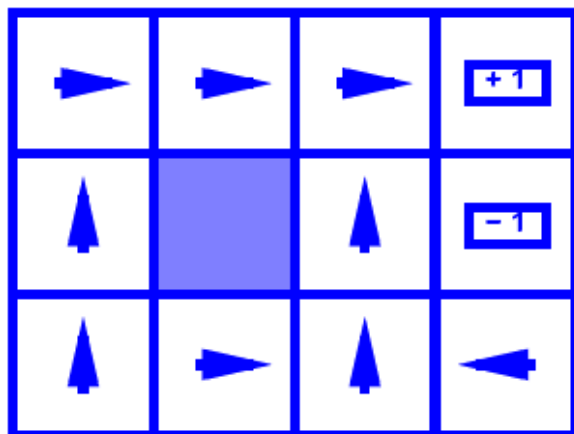
Optimal Policies



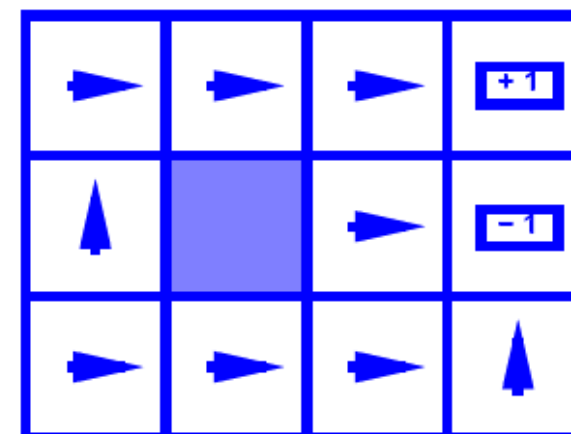
$R(s) = -0.01$



$R(s) = -0.03$

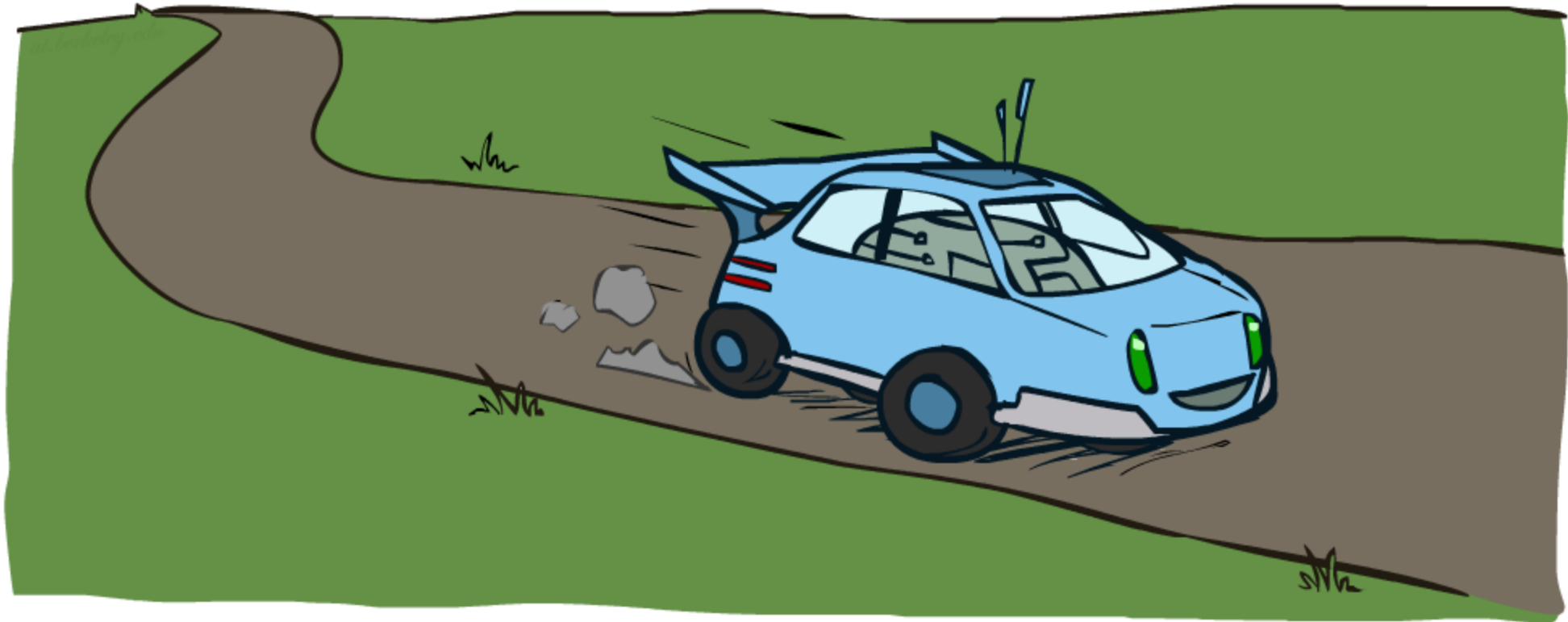


$R(s) = -0.4$



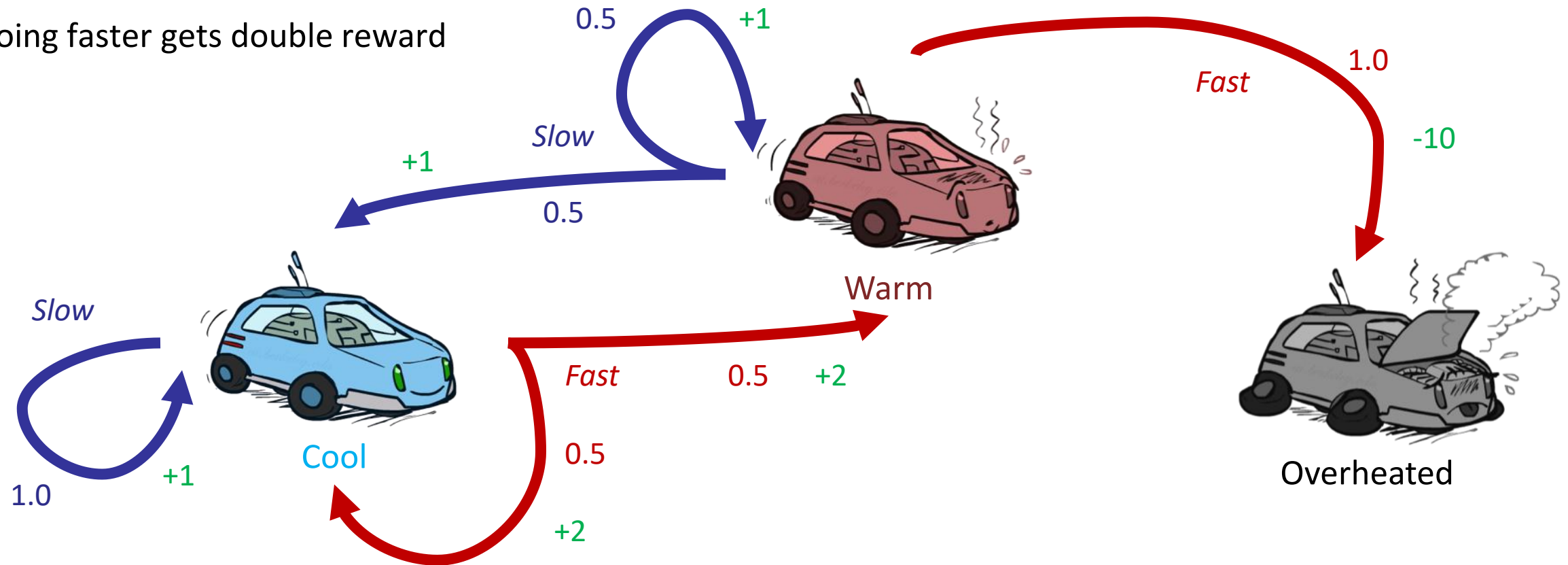
$R(s) = -2.0$

Example: Racing

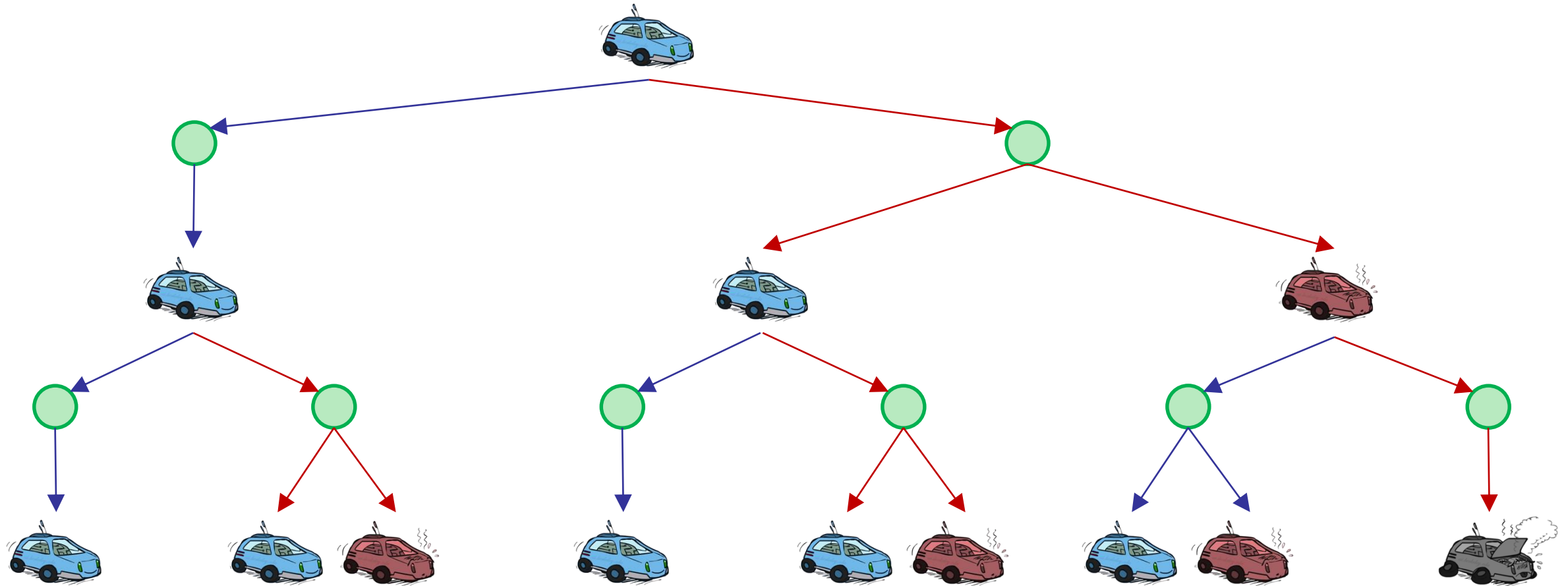


Example: Racing

- A robot car wants to travel far, quickly
- Three states: **Cool**, **Warm**, Overheated
- Two actions: *Slow*, *Fast*
- Going faster gets double reward

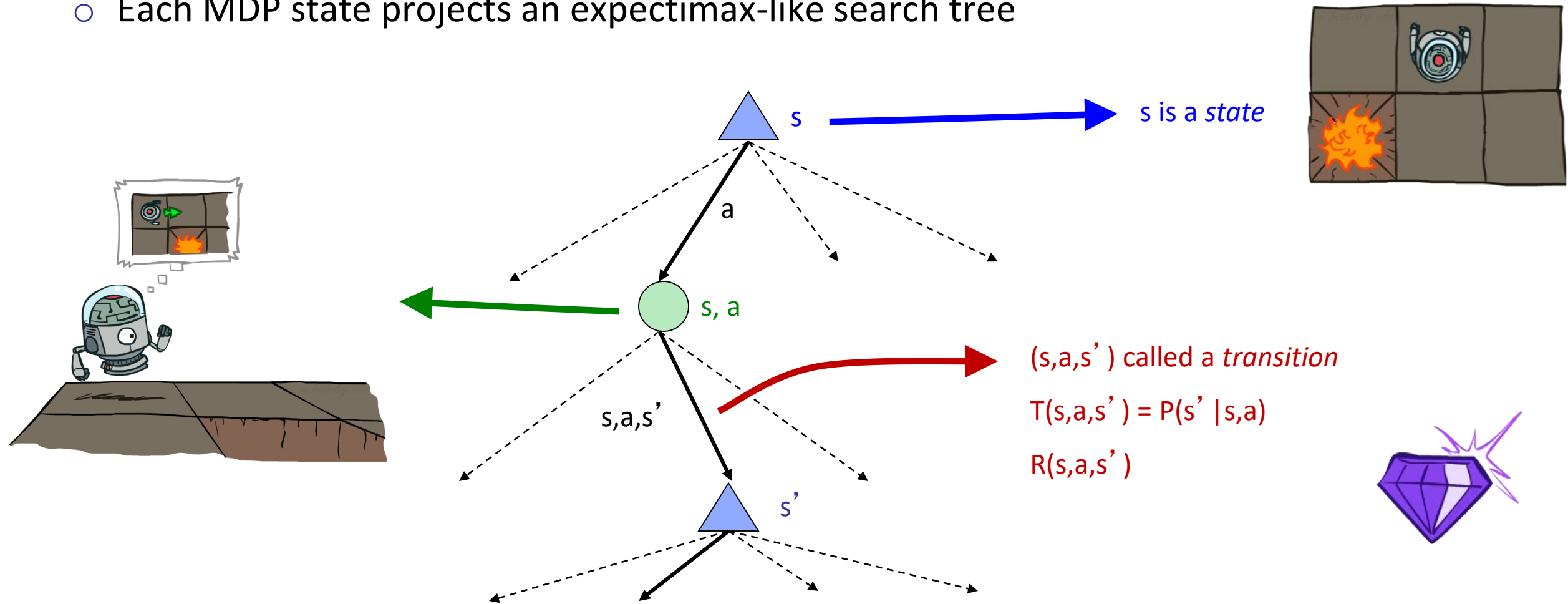


Racing Search Tree

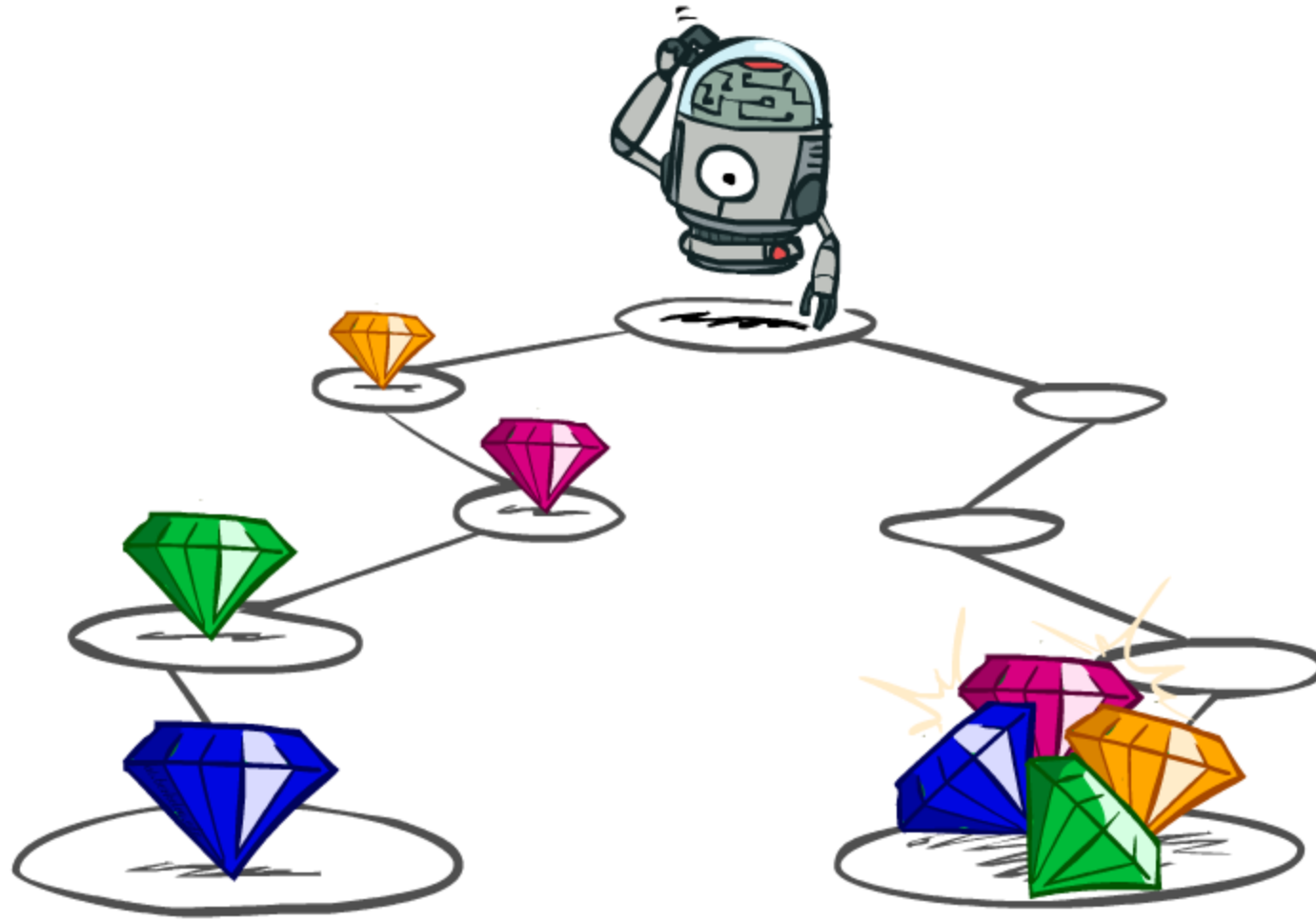


MDP Search Trees

- Each MDP state projects an expectimax-like search tree

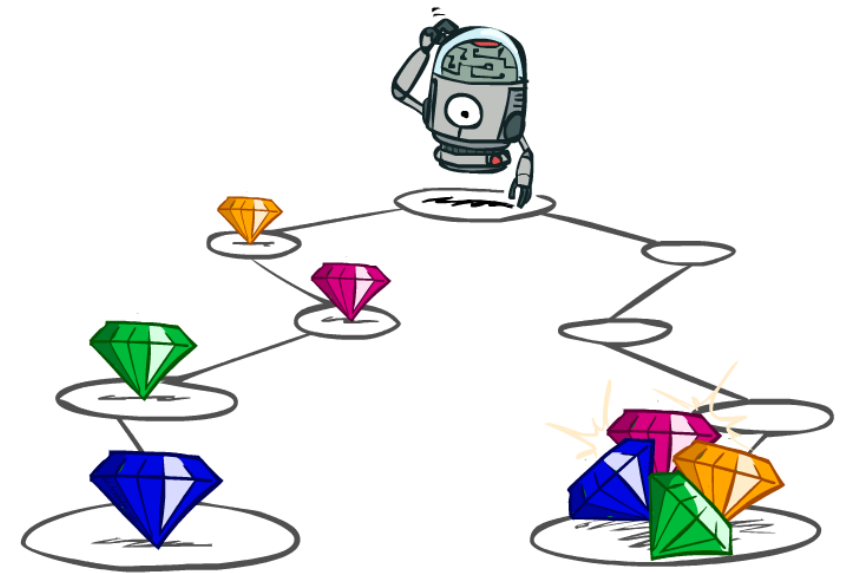


Utilities of Sequences



Utilities of Sequences

- What preferences should an agent have over reward sequences?
- More or less? $[1, 2, 2]$ or $[2, 3, 4]$
- Now or later? $[0, 0, 1]$ or $[1, 0, 0]$



Discounting

- It's reasonable to maximize the sum of rewards
- It's also reasonable to prefer rewards now to rewards later
- One solution: values of rewards decay exponentially



1

Worth Now



γ

Worth Next Step

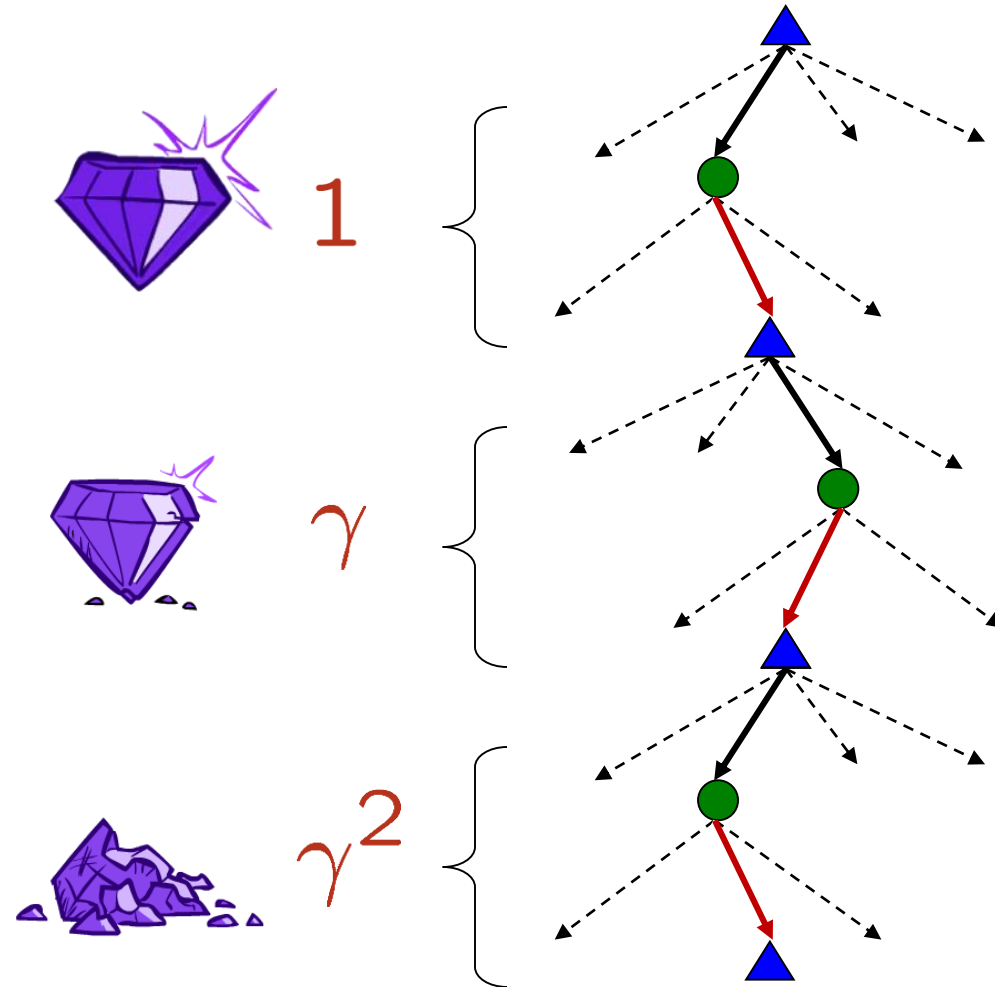


γ^2

Worth In Two Steps

Discounting

- How to discount?
 - Each time we descend a level, we multiply in the discount once
- Why discount?
 - Think of it as a gamma chance of ending the process at every step
 - Also helps our algorithms converge
- Example: discount of 0.5
 - $U([1,2,3]) = 1*1 + 0.5*2 + 0.25*3$
 - $U([1,2,3]) < U([3,2,1])$



Quiz: Discounting

○ Given:

10				1
a	b	c	d	e

- Actions: East, West, and Exit (only available in exit states a, e)
- Transitions: deterministic

○ Quiz 1: For $\gamma = 1$, what is the optimal policy?

10	<-	<-	<-	1
----	----	----	----	---

○ Quiz 2: For $\gamma = 0.1$, what is the optimal policy?

10	<-	<-	->	1
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○ Quiz 3: For which γ are West and East equally good when in state d?

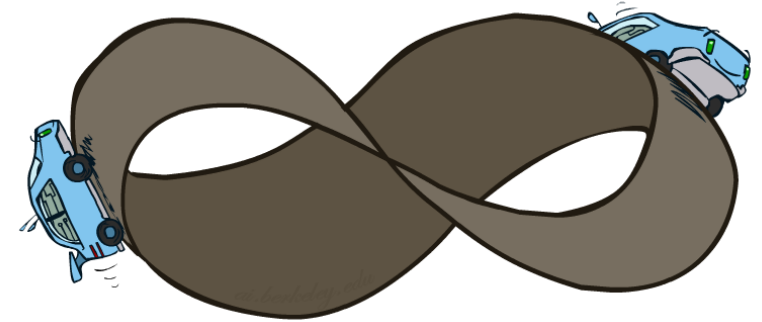
$$10\gamma = \gamma^3$$

Infinite Utilities?!

- Problem: What if the game lasts forever? Do we get infinite rewards?

- Solutions:

- Finite horizon: (similar to depth-limited search)
 - Terminate episodes after a fixed T steps (e.g. life)
 - Policy π depends on time left



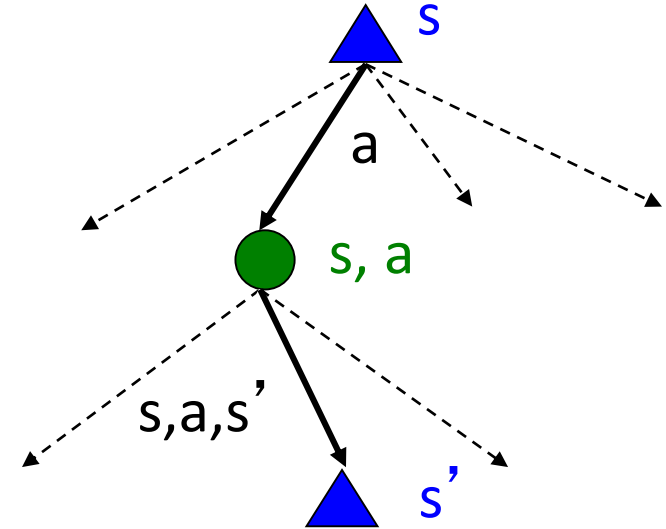
- Discounting: use $0 < \gamma < 1$

$$U([r_0, \dots, r_\infty]) = \sum_{t=0}^{\infty} \gamma^t r_t \leq R_{\max}/(1 - \gamma)$$

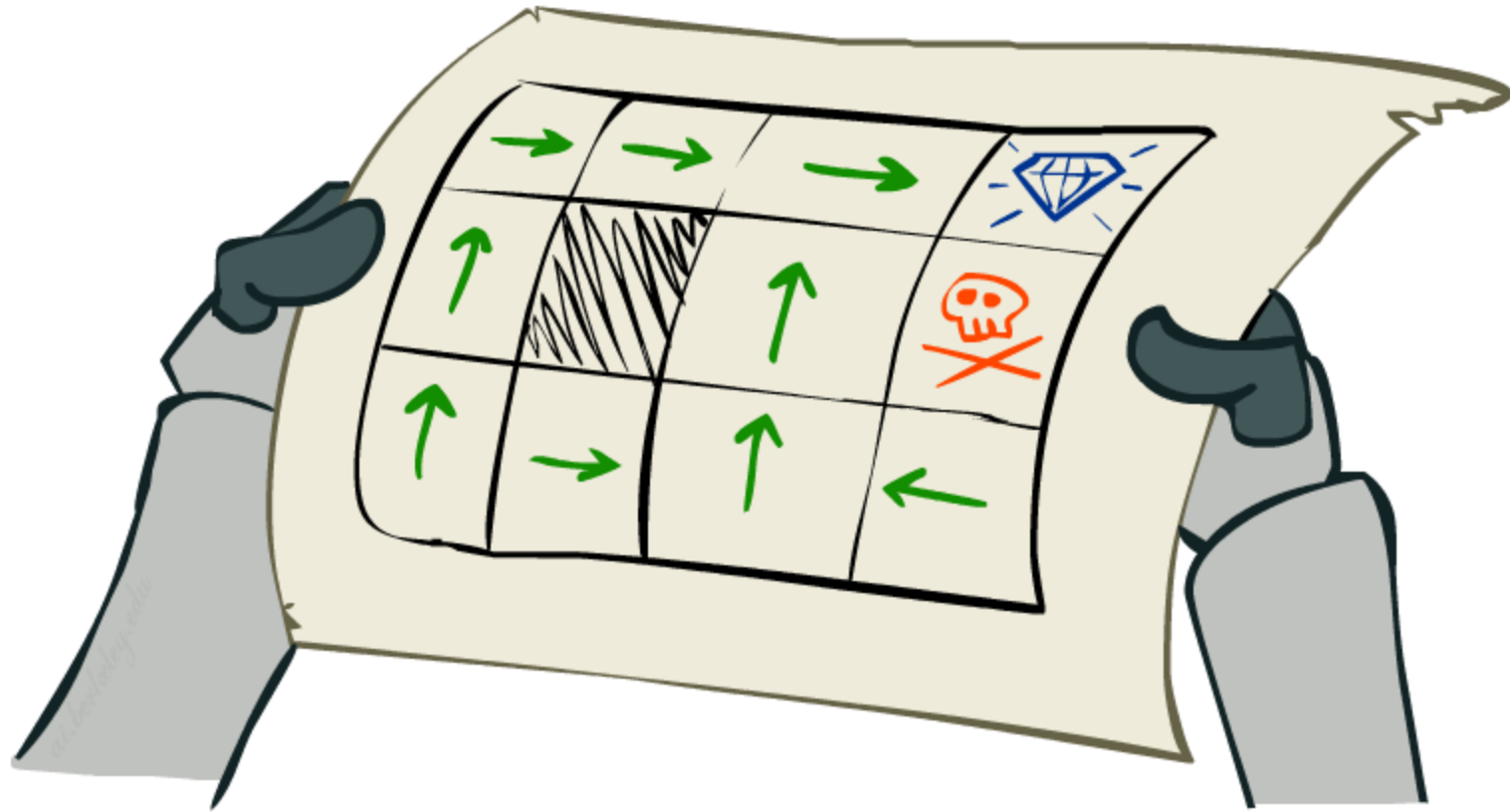
- Smaller γ means smaller “horizon” – shorter term focus
- Absorbing state: guarantee that for every policy, a terminal state will eventually be reached (like “overheated” for racing)

Recap: Defining MDPs

- Markov decision processes:
 - Set of states S
 - Start state s_0
 - Set of actions A
 - Transitions $P(s'|s,a)$ (or $T(s,a,s')$)
 - Rewards $R(s,a,s')$ (and discount γ)
- MDP quantities so far:
 - Policy = Choice of action for each state
 - Utility = sum of (discounted) rewards

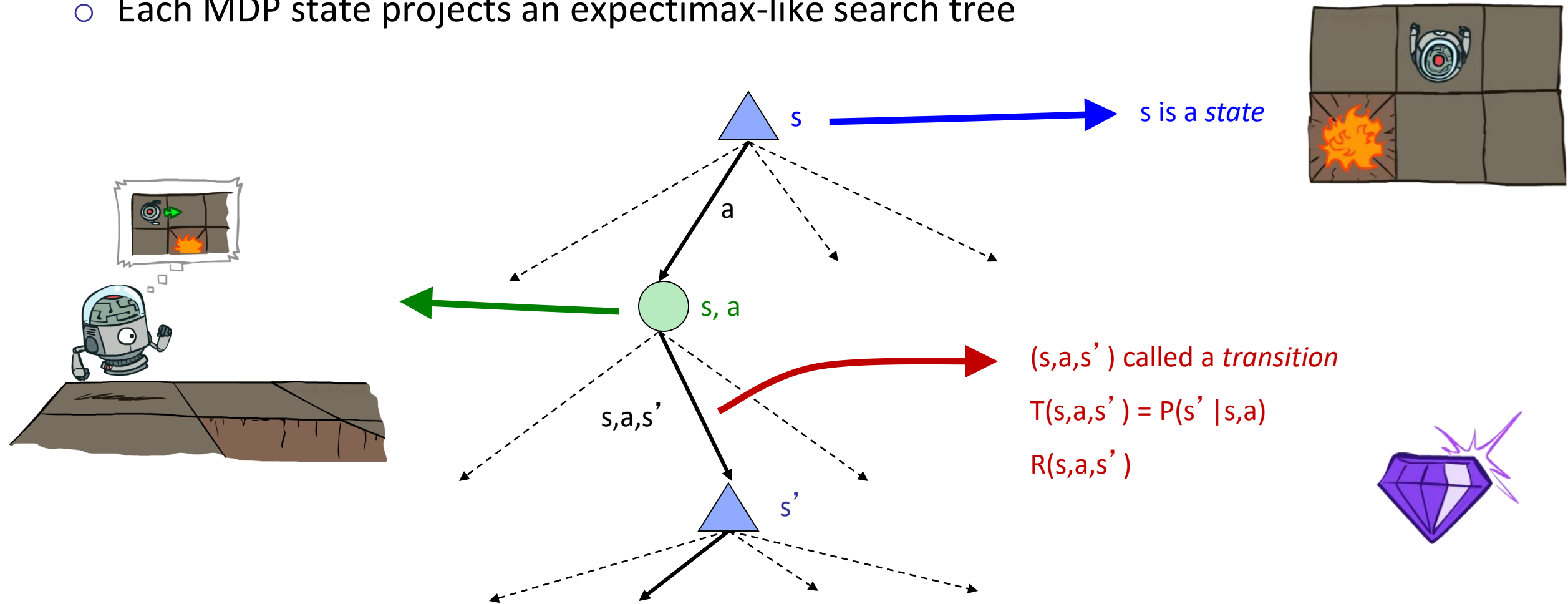


Solving MDPs



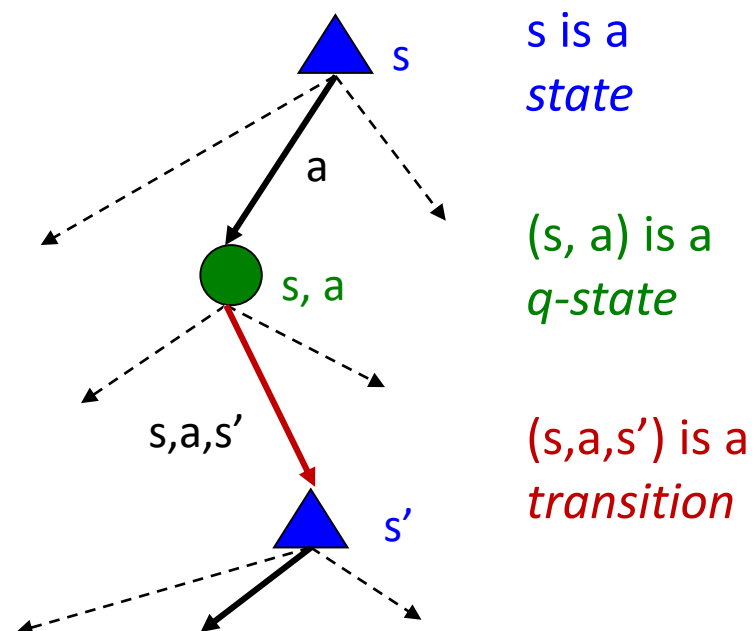
MDP Search Trees

- Each MDP state projects an expectimax-like search tree



Optimal Quantities

- The value (utility) of a state s :
 $V^*(s)$ = expected utility starting in s and acting optimally
- The value (utility) of a q-state (s,a) :
 $Q^*(s,a)$ = expected utility starting out having taken action a from state s and (thereafter) acting optimally
- The optimal policy:
 $\pi^*(s)$ = optimal action from state s

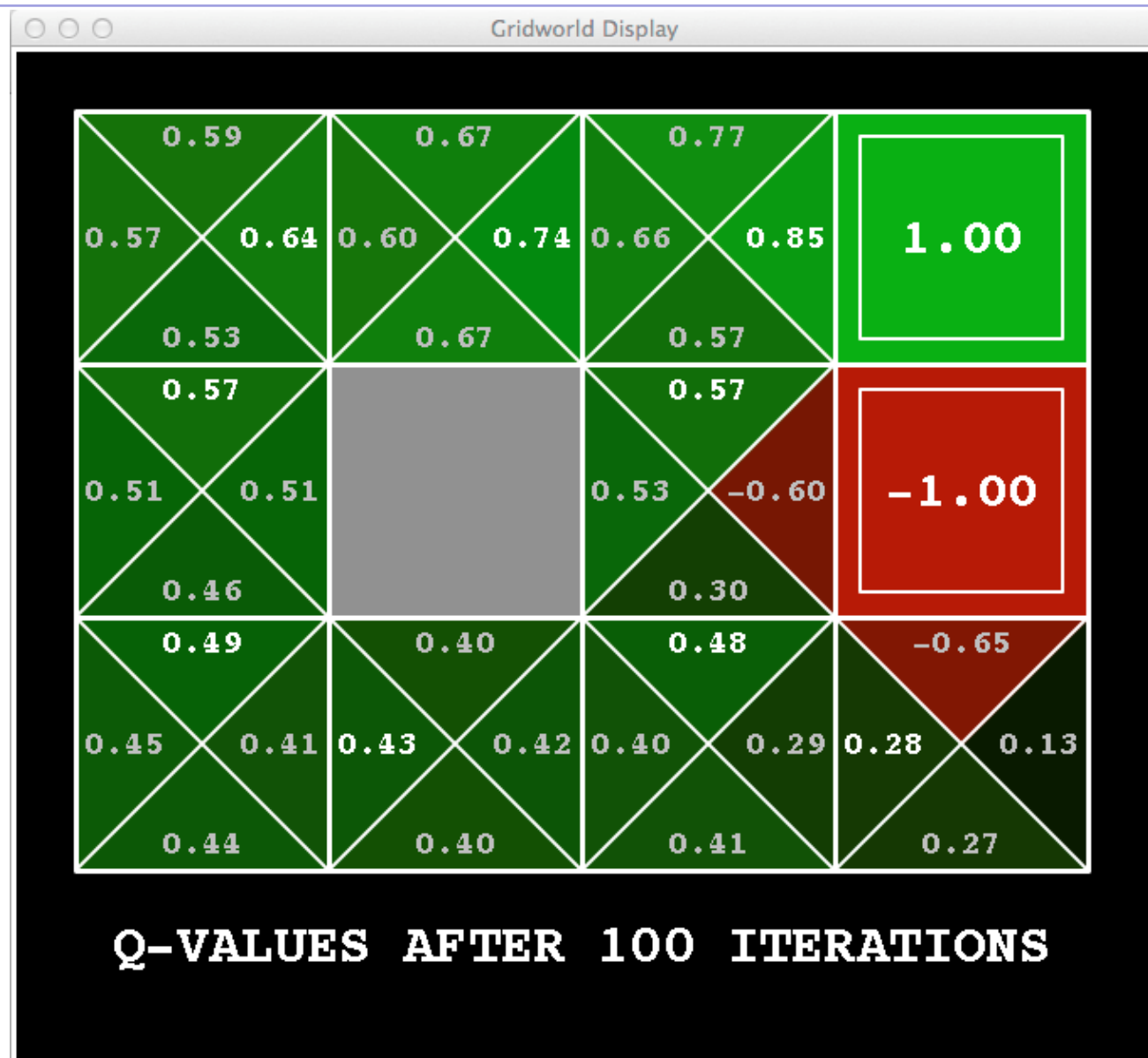


Snapshot Gridworld V Values



Noise = 0.2
Discount = 0.9
Living reward = 0

Snapshot of Gridworld Q Values



Noise = 0.2
Discount = 0.9
Living reward = 0

Values of States (Bellman Equations)

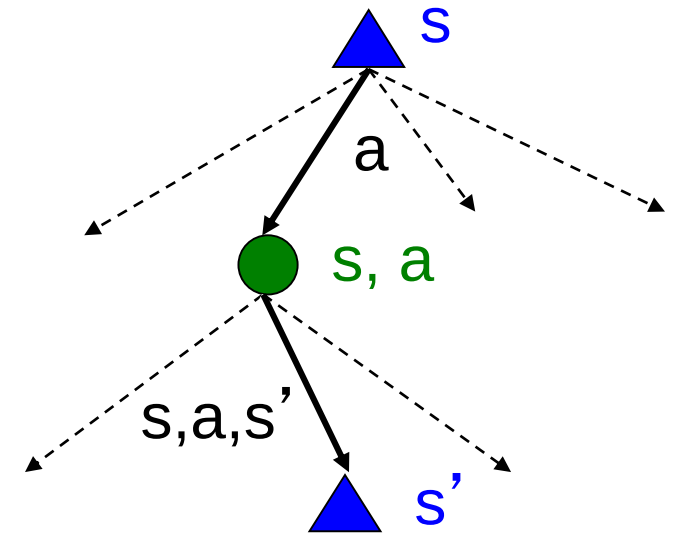
- Fundamental operation: compute the (expectimax) value of a state
 - Expected utility under optimal action
 - Average sum of (discounted) rewards
 - This is just what expectimax computed!

- Recursive definition of value:

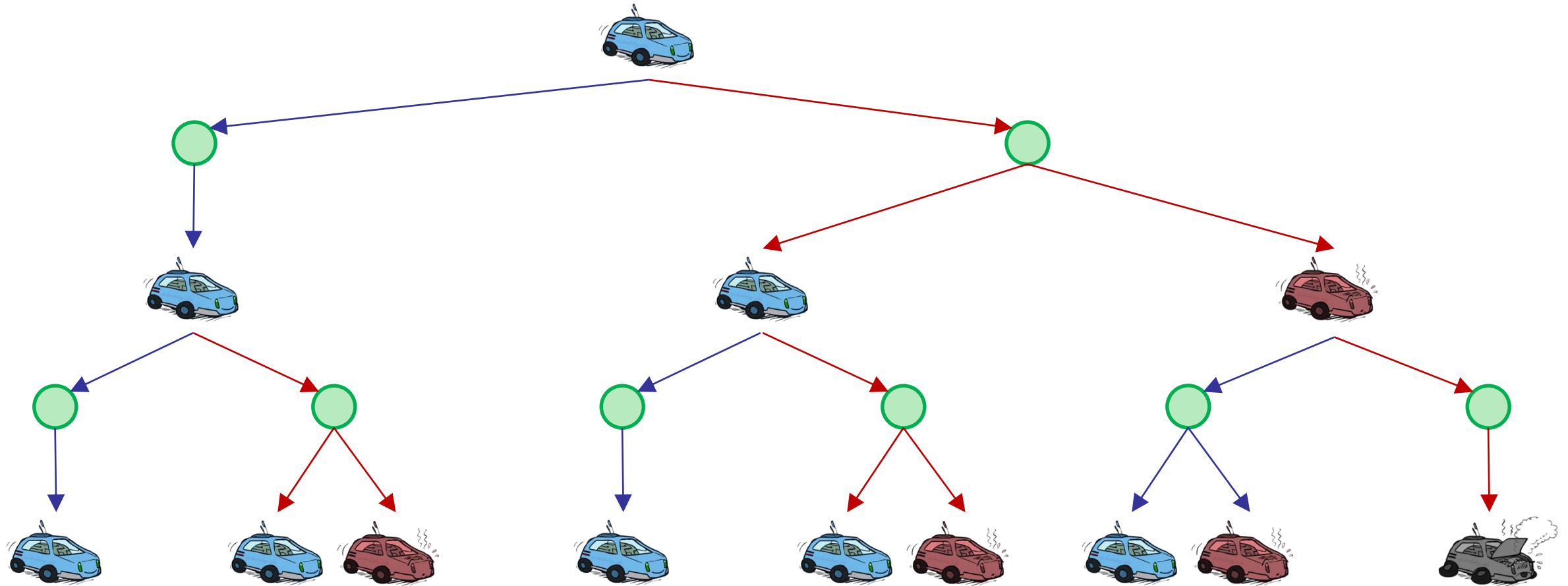
$$V^*(s) = \max_a Q^*(s, a)$$

$$Q^*(s, a) = \sum_{s'} T(s, a, s') [R(s, a, s') + \gamma V^*(s')]$$

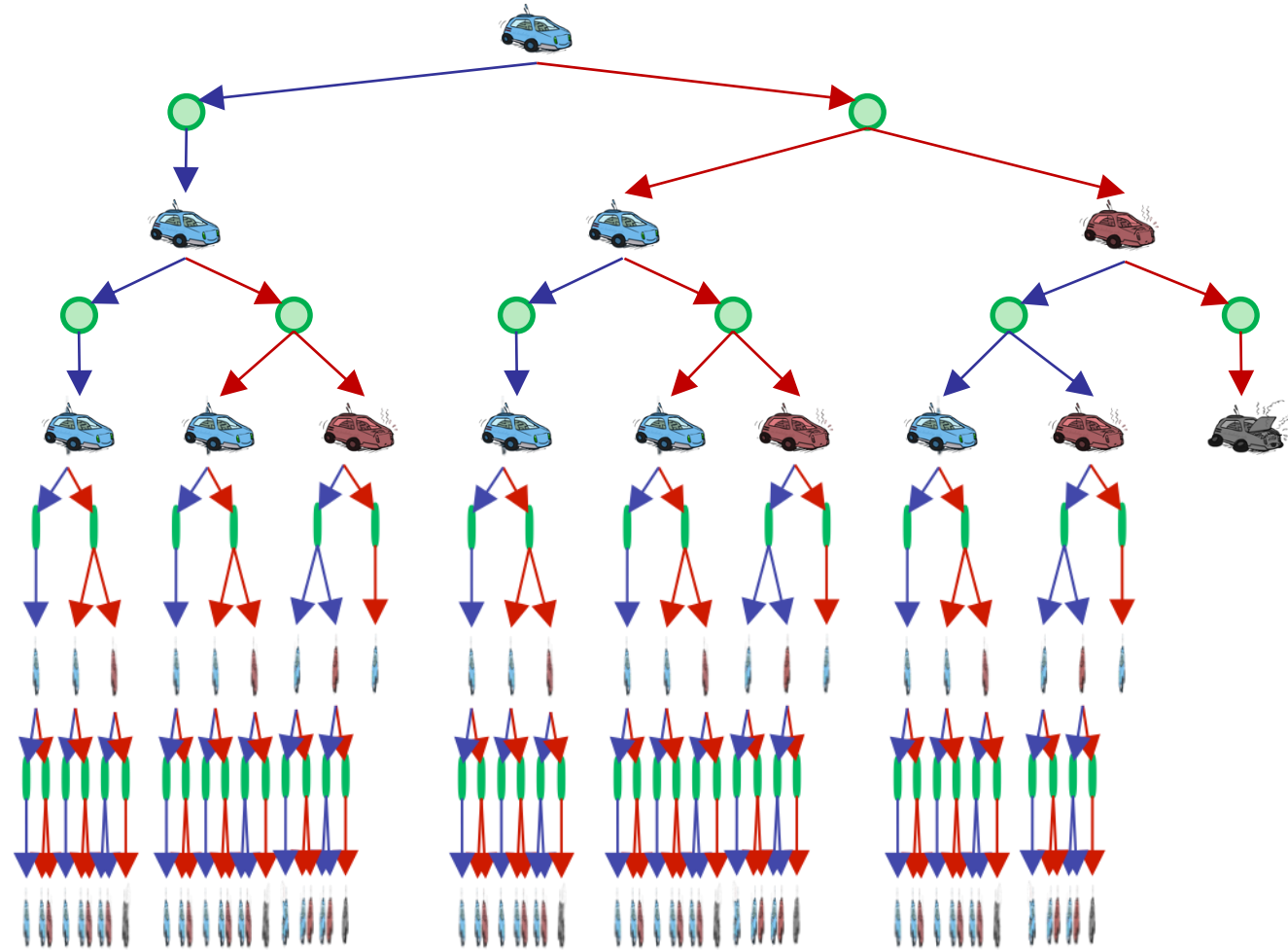
$$V^*(s) = \max_a \sum_{s'} T(s, a, s') [R(s, a, s') + \gamma V^*(s')]$$



Racing Search Tree

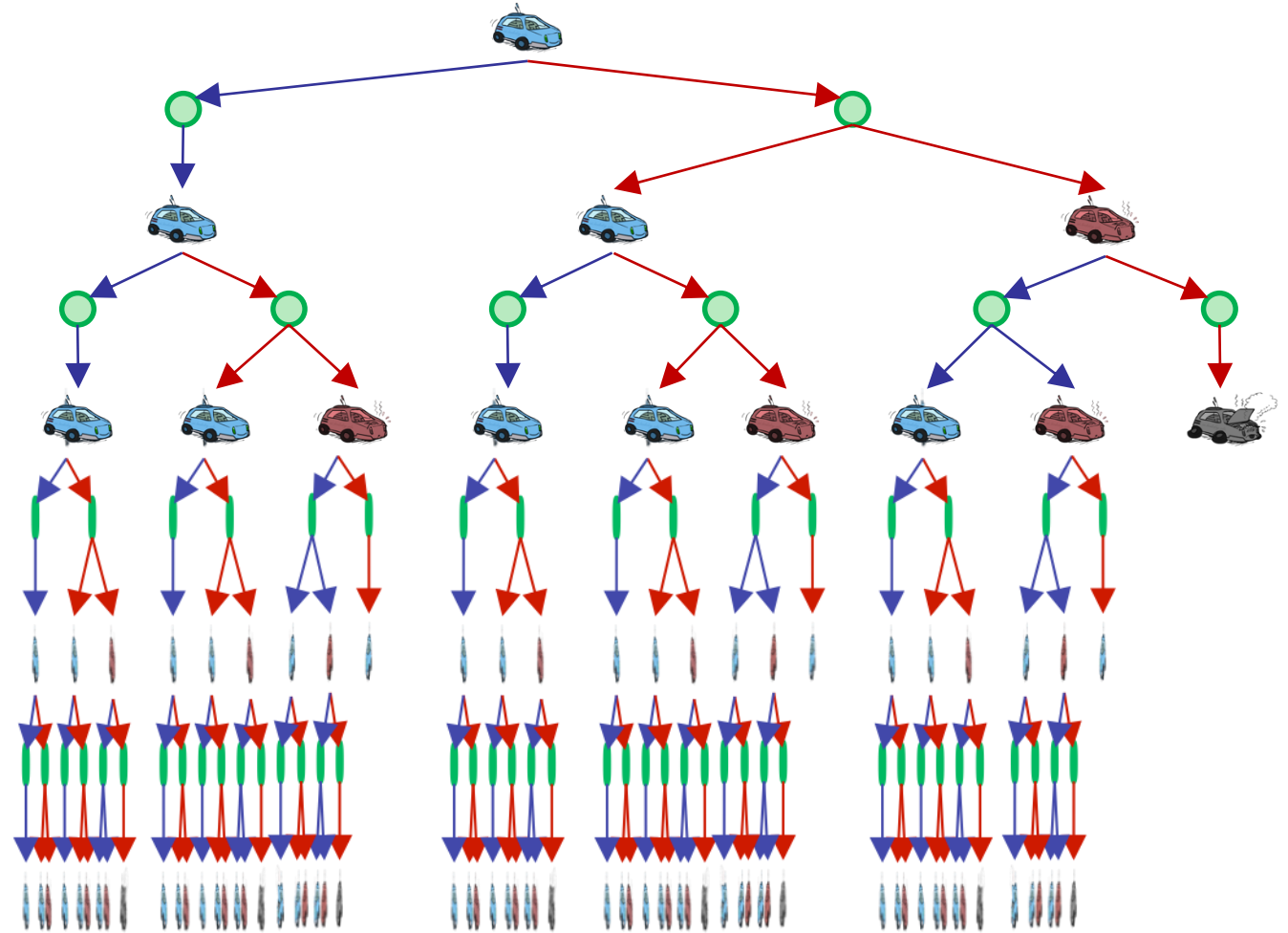


Racing Search Tree



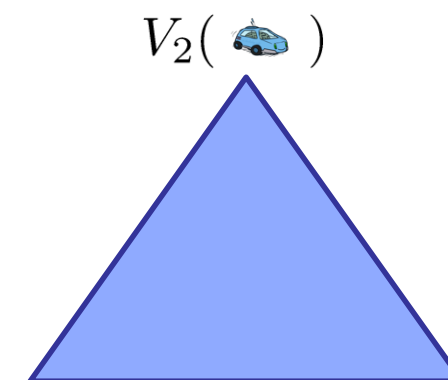
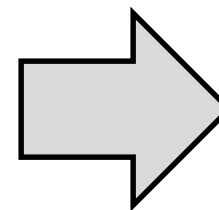
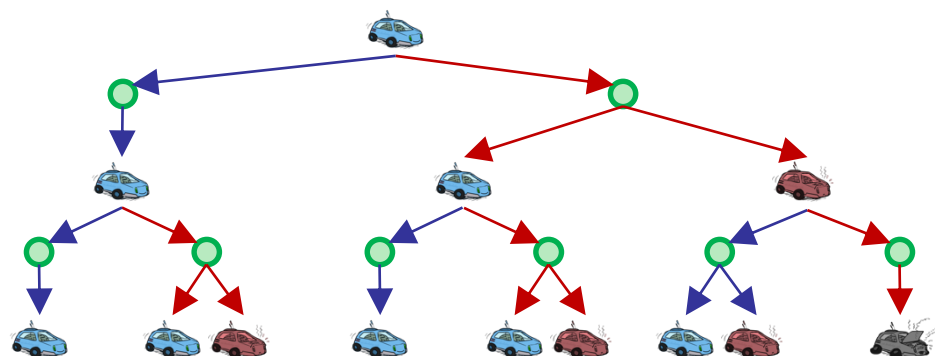
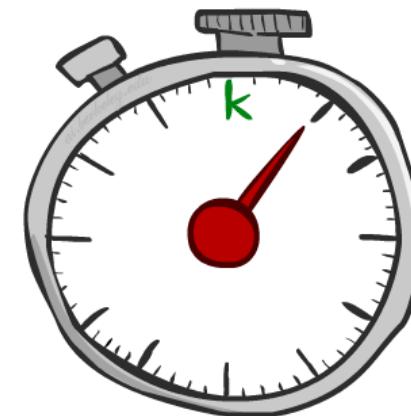
Racing Search Tree

- We're doing way too much work with expectimax!
- Problem: States are repeated
 - Idea quantities: Only compute needed once
- Problem: Tree goes on forever
 - Idea: Do a depth-limited computation, but with increasing depths until change is small
 - Note: deep parts of the tree

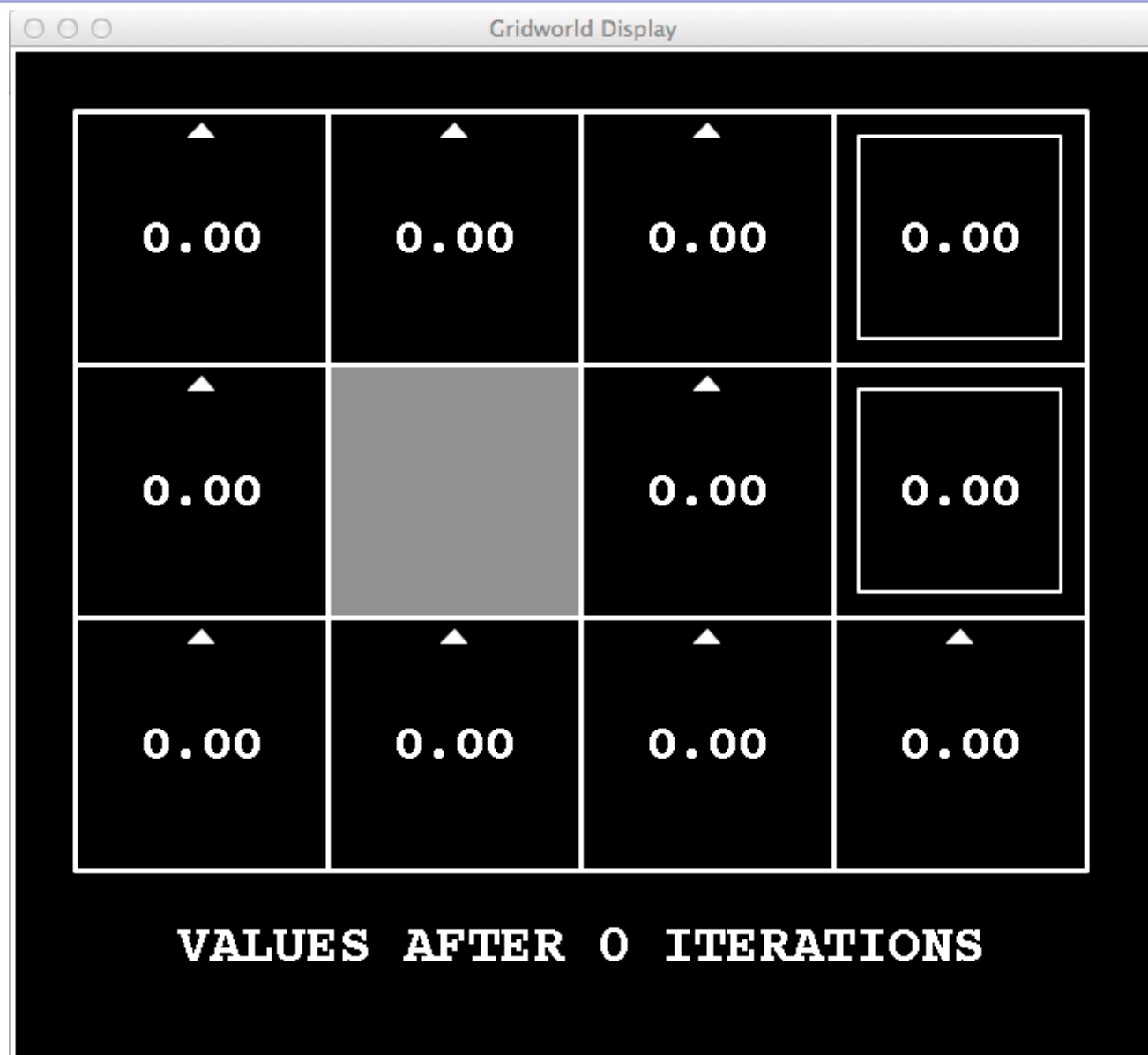


Time-Limited Values

- Key idea: time-limited values
- Define $V_k(s)$ to be the optimal value of s if the game ends in k more time steps
 - Equivalently, it's what a depth- k expectimax would give from s

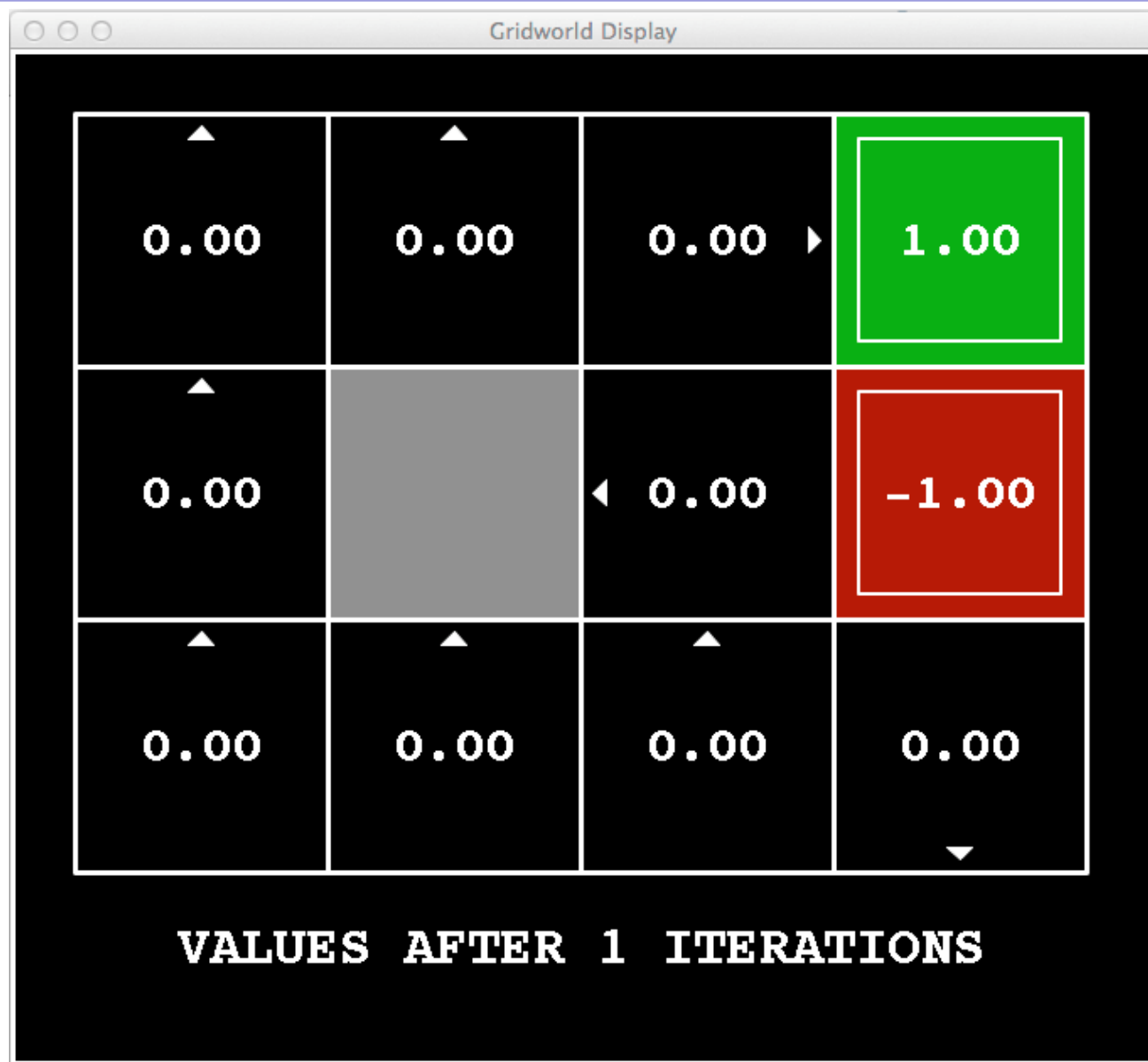


$k=0$



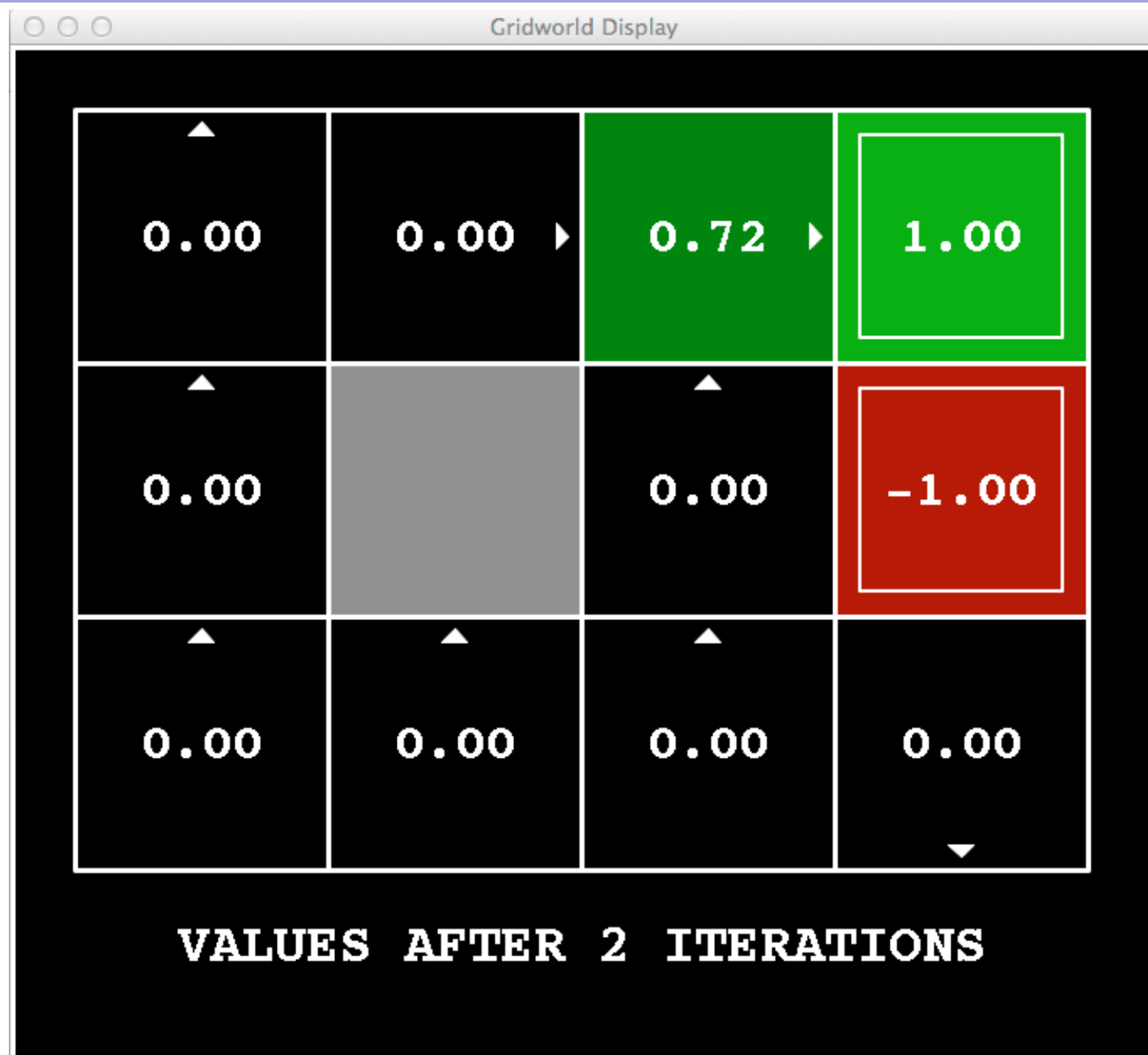
Noise = 0.2
Discount = 0.9
Living reward = 0

k=1



Noise = 0.2
Discount = 0.9
Living reward = 0

k=2



Noise = 0.2
Discount = 0.9
Living reward = 0

k=3



Noise = 0.2
Discount = 0.9
Living reward = 0

k=4



Noise = 0.2
Discount = 0.9
Living reward = 0

k=5



Noise = 0.2
Discount = 0.9
Living reward = 0

k=6



Noise = 0.2
Discount = 0.9
Living reward = 0

k=7



Noise = 0.2
Discount = 0.9
Living reward = 0

k=8



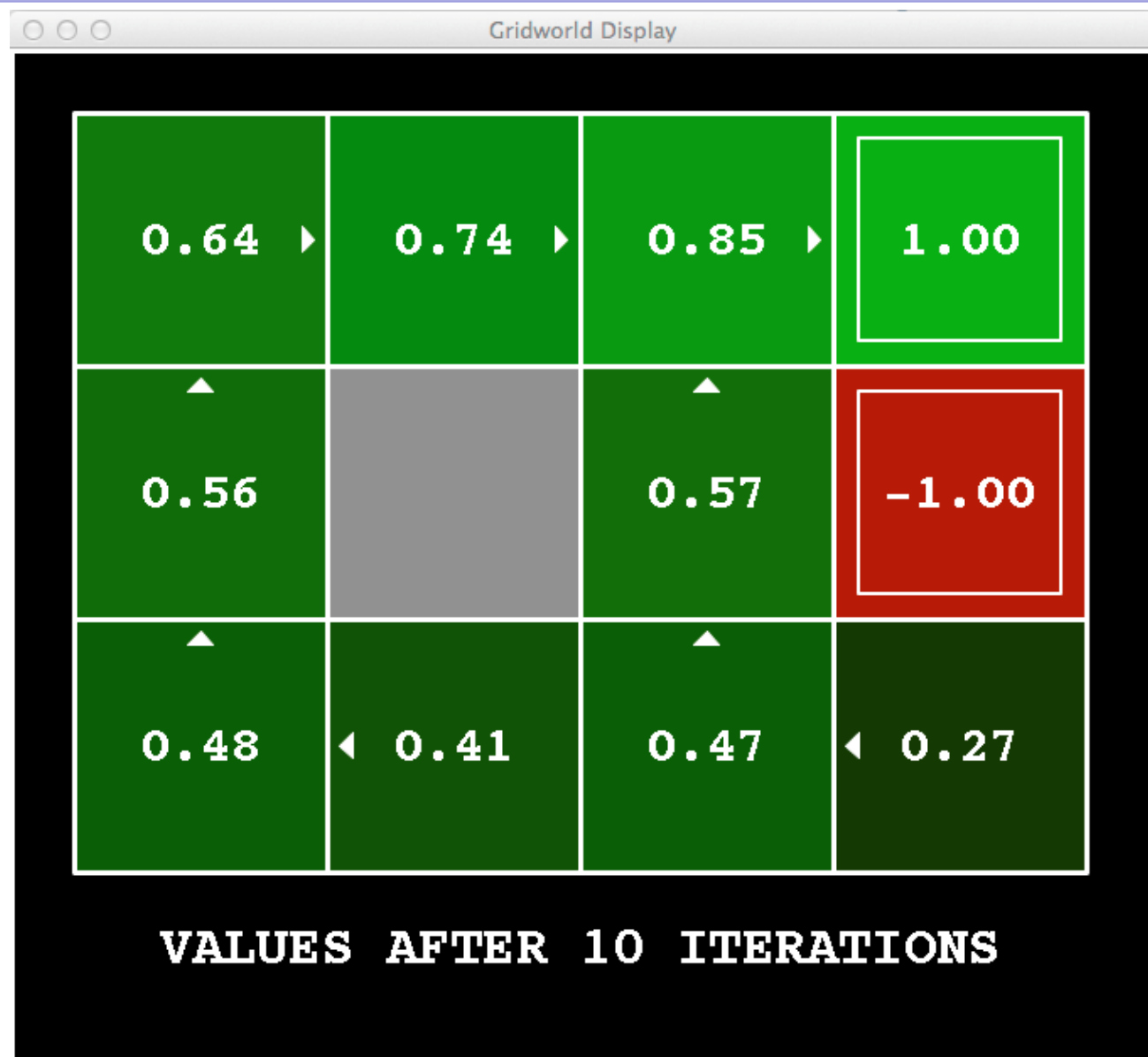
Noise = 0.2
Discount = 0.9
Living reward = 0

k=9



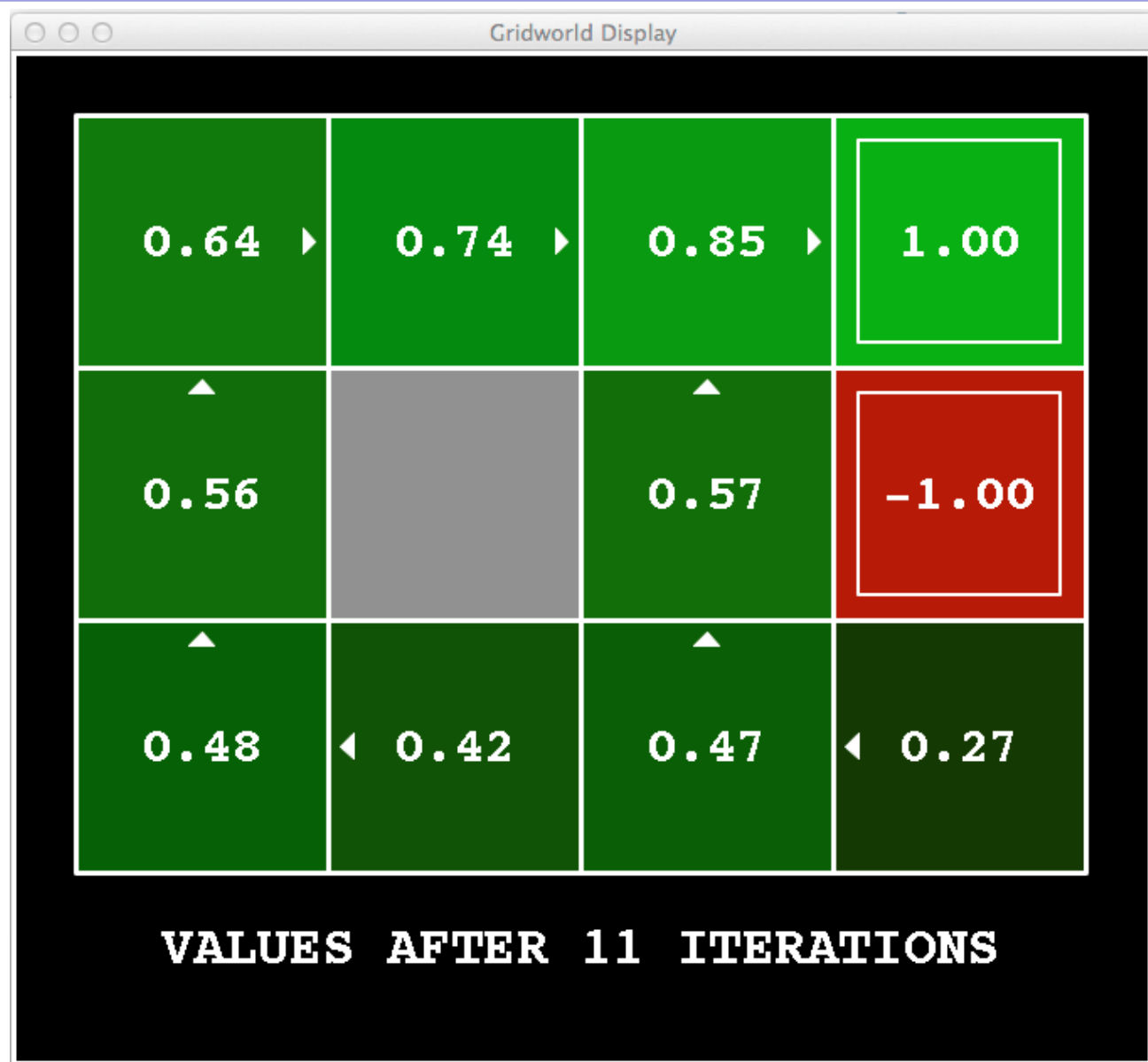
Noise = 0.2
Discount = 0.9
Living reward = 0

k=10



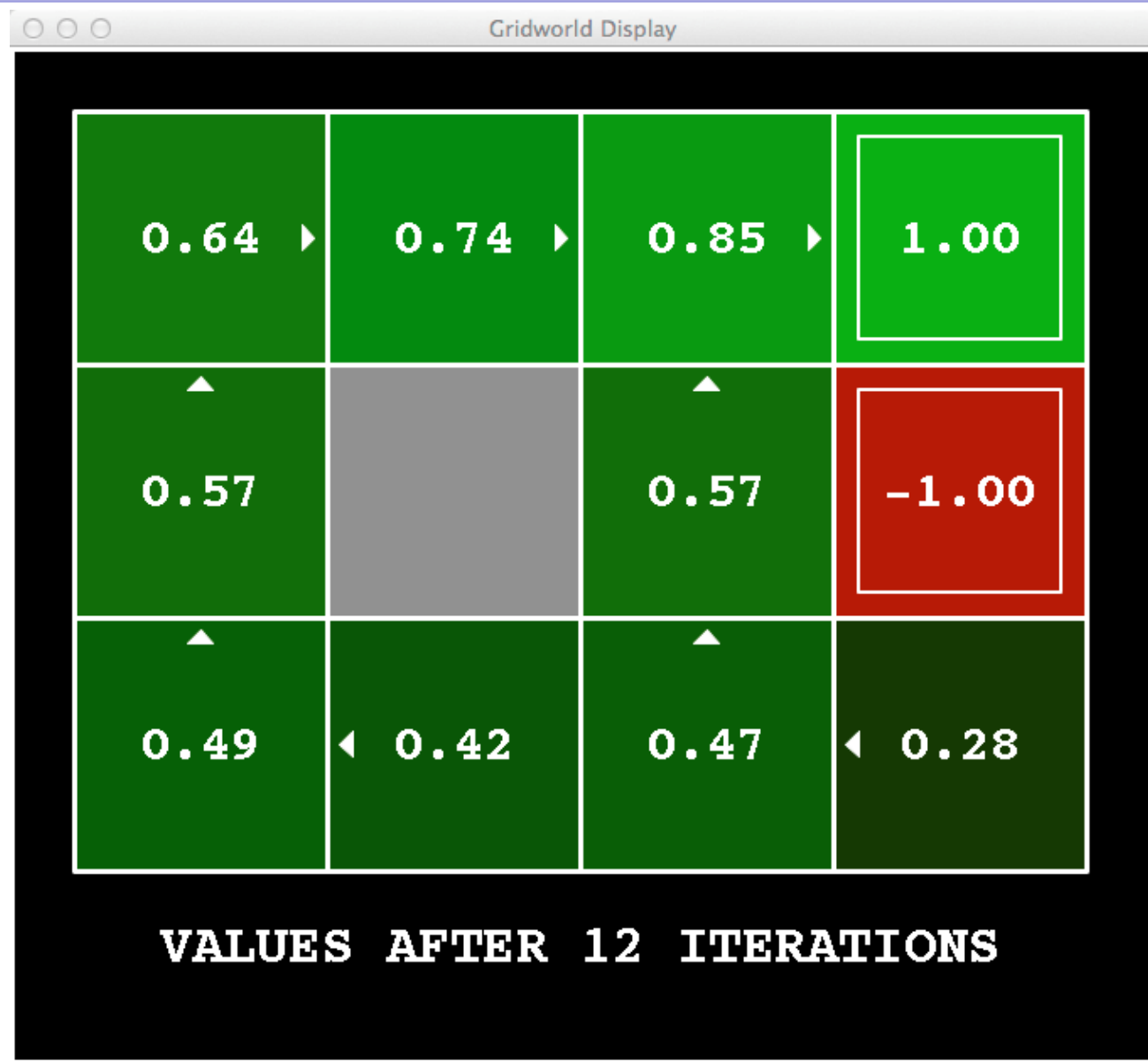
Noise = 0.2
Discount = 0.9
Living reward = 0

k=11



Noise = 0.2
Discount = 0.9
Living reward = 0

k=12



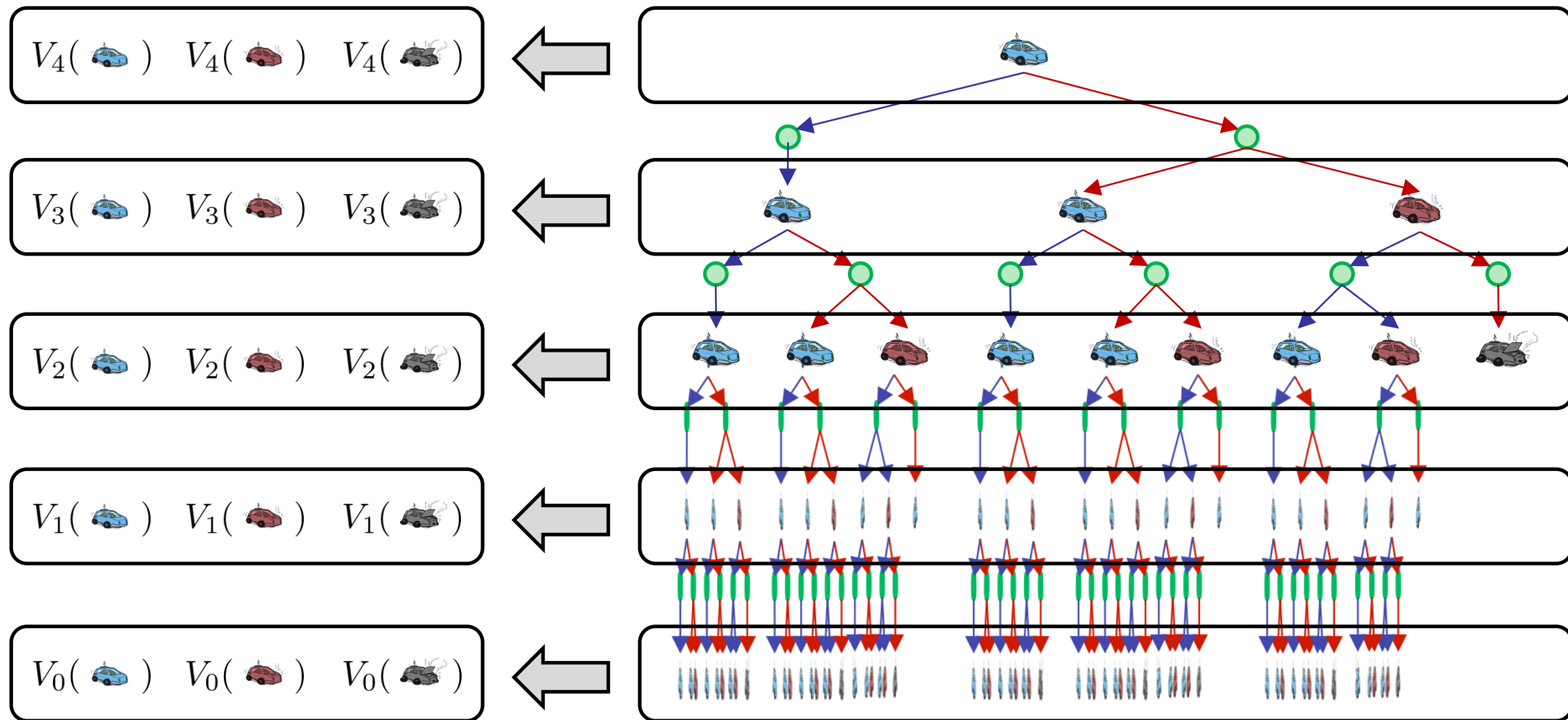
Noise = 0.2
Discount = 0.9
Living reward = 0

k=100

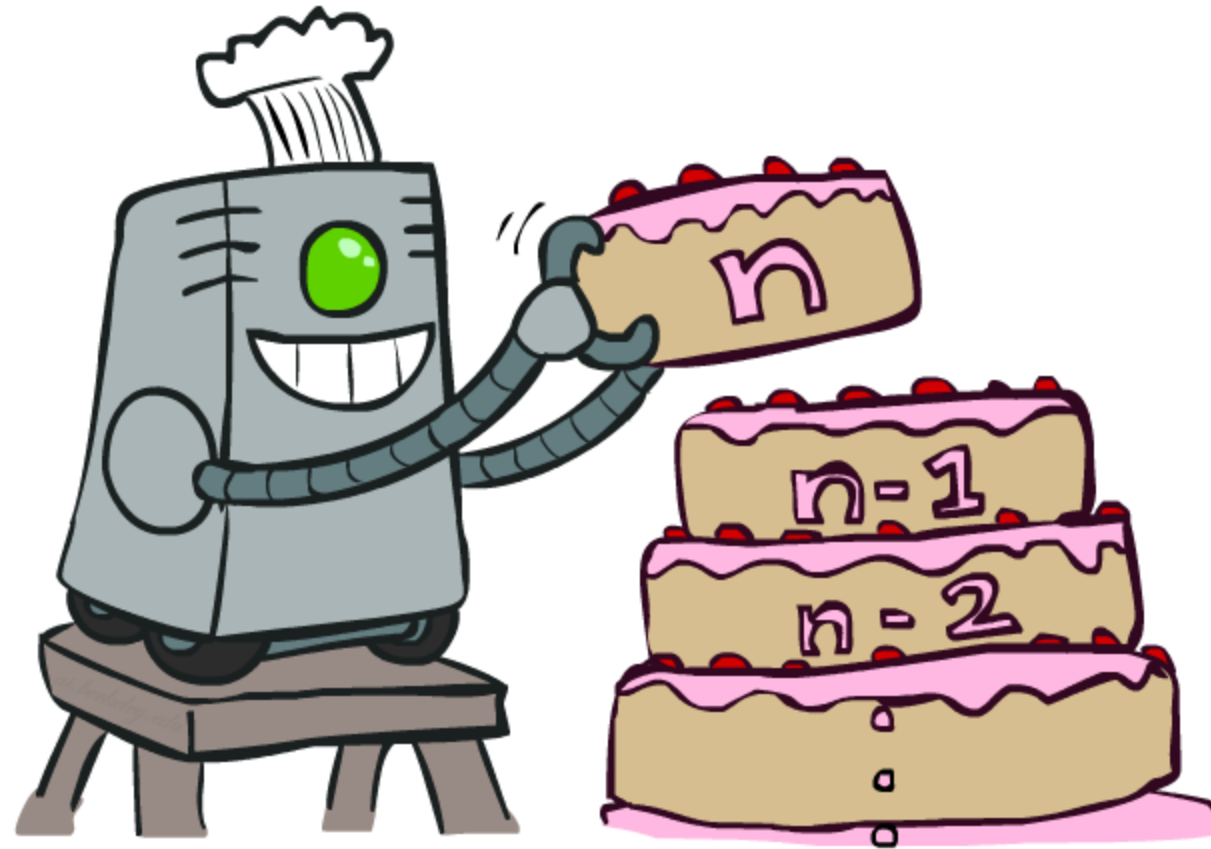


Noise = 0.2
Discount = 0.9
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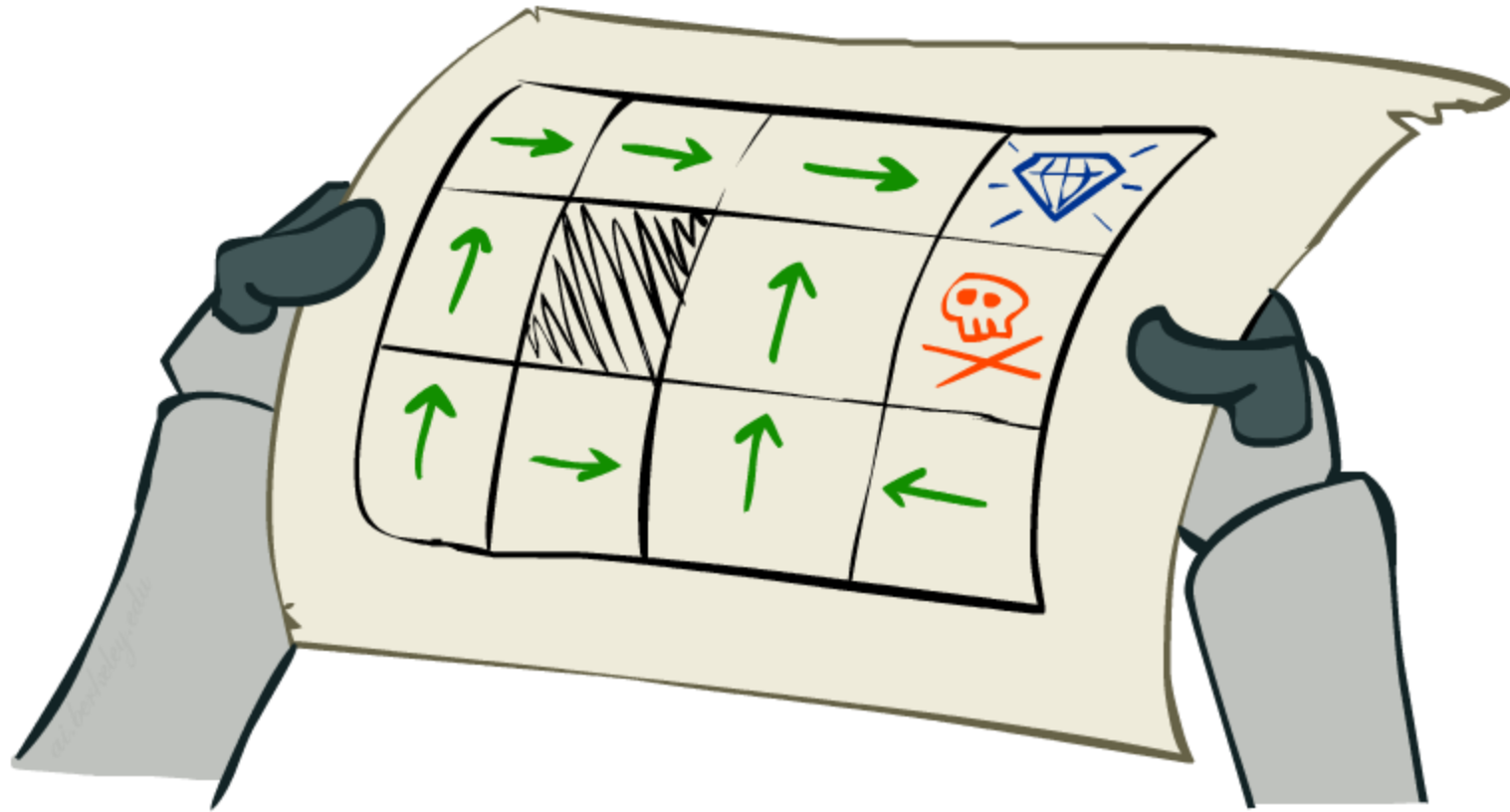
Computing Time-Limited Values



Value Iteration



Solving MDPs

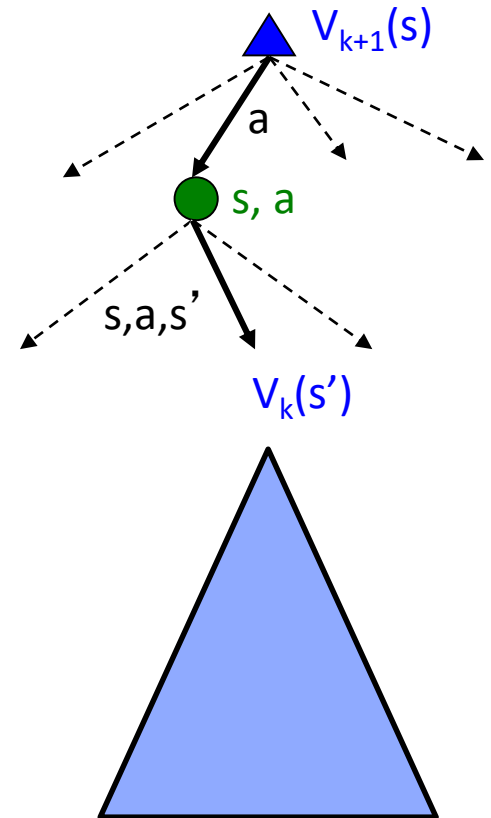


Value Iteration

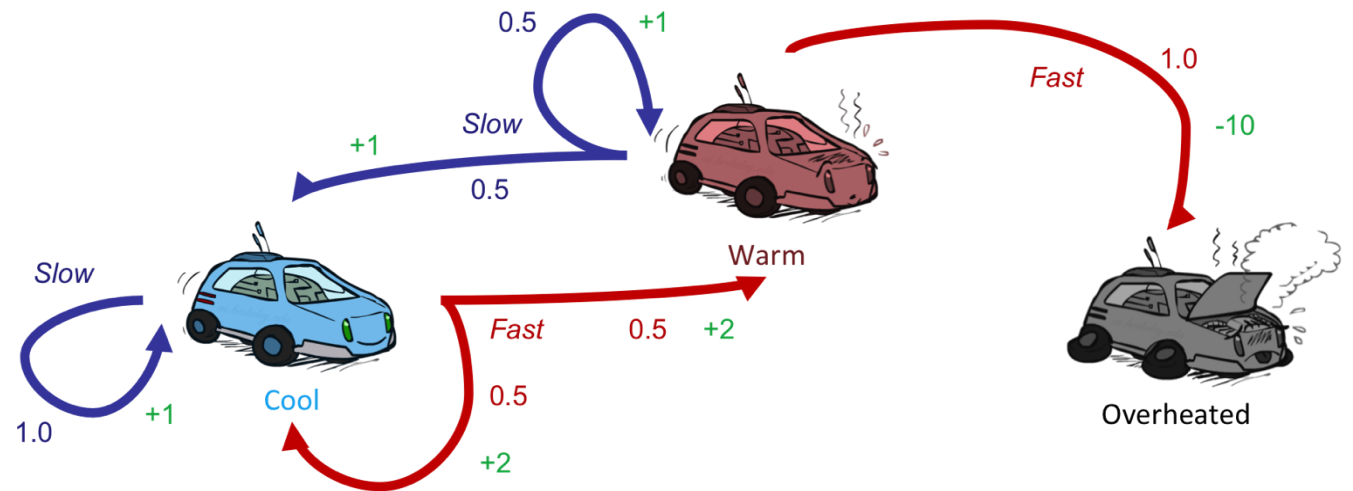
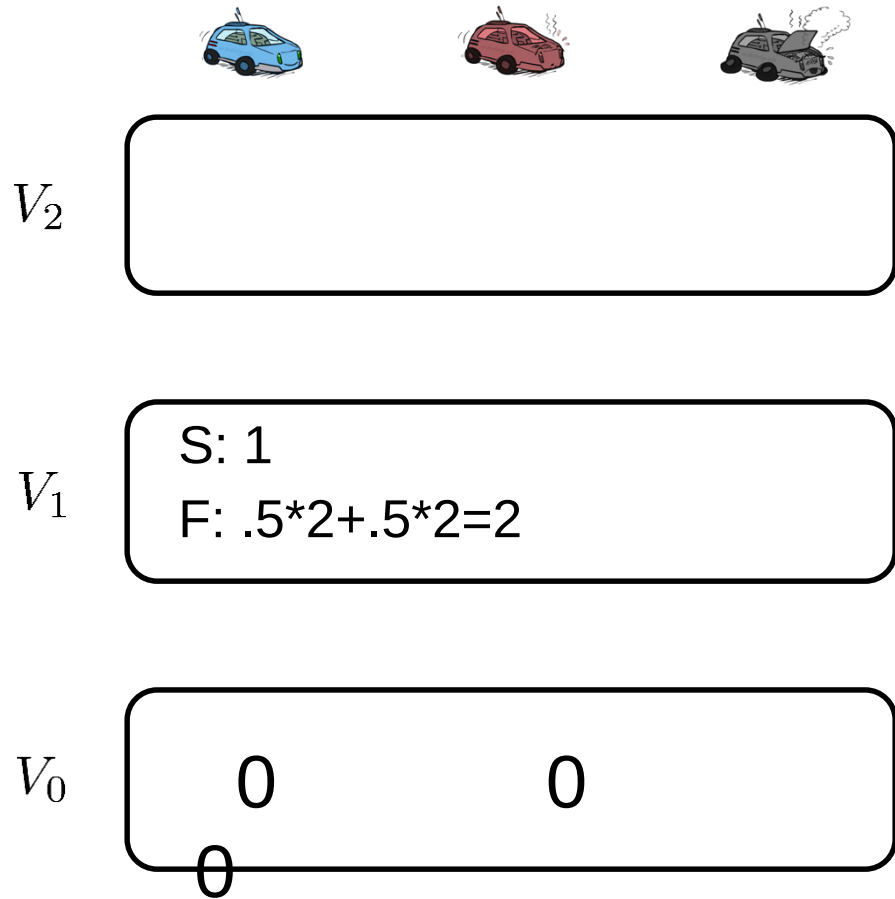
- Start with $V_0(s) = 0$: no time steps left means an expected reward sum of zero
- Given vector of $V_k(s)$ values, do one ply of expectimax from each state:

$$V_{k+1}(s) \leftarrow \max_a \sum_{s'} T(s, a, s') [R(s, a, s') + \gamma V_k(s')]$$

- Repeat until convergence
- Complexity of each iteration: $O(S^2A)$
- Theorem: will converge to unique optimal values
 - Basic idea: approximations get refined towards optimal values
 - Policy may converge long before values do



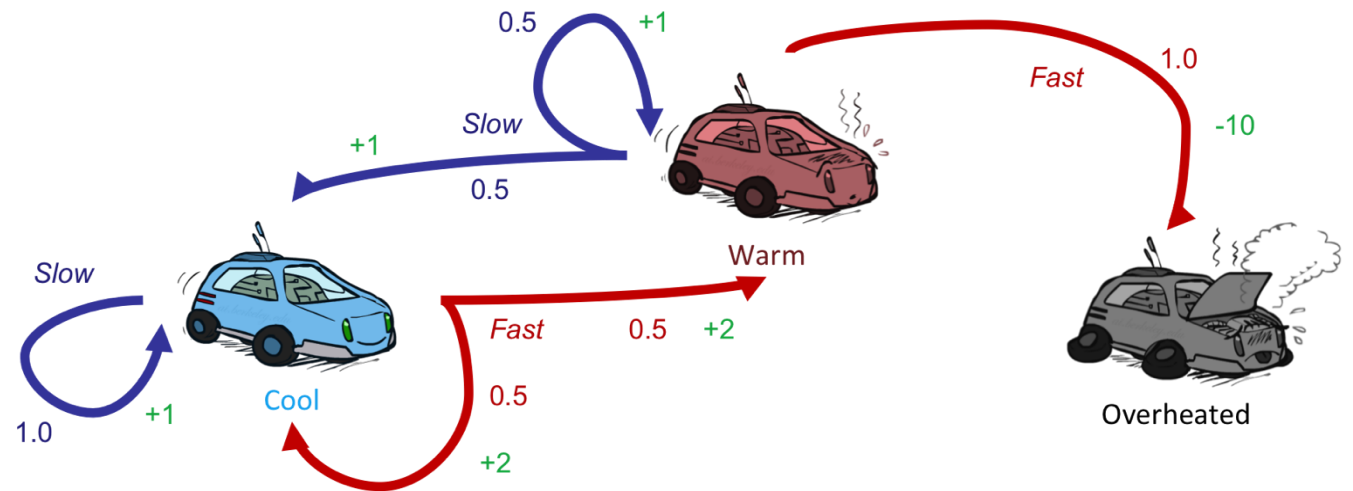
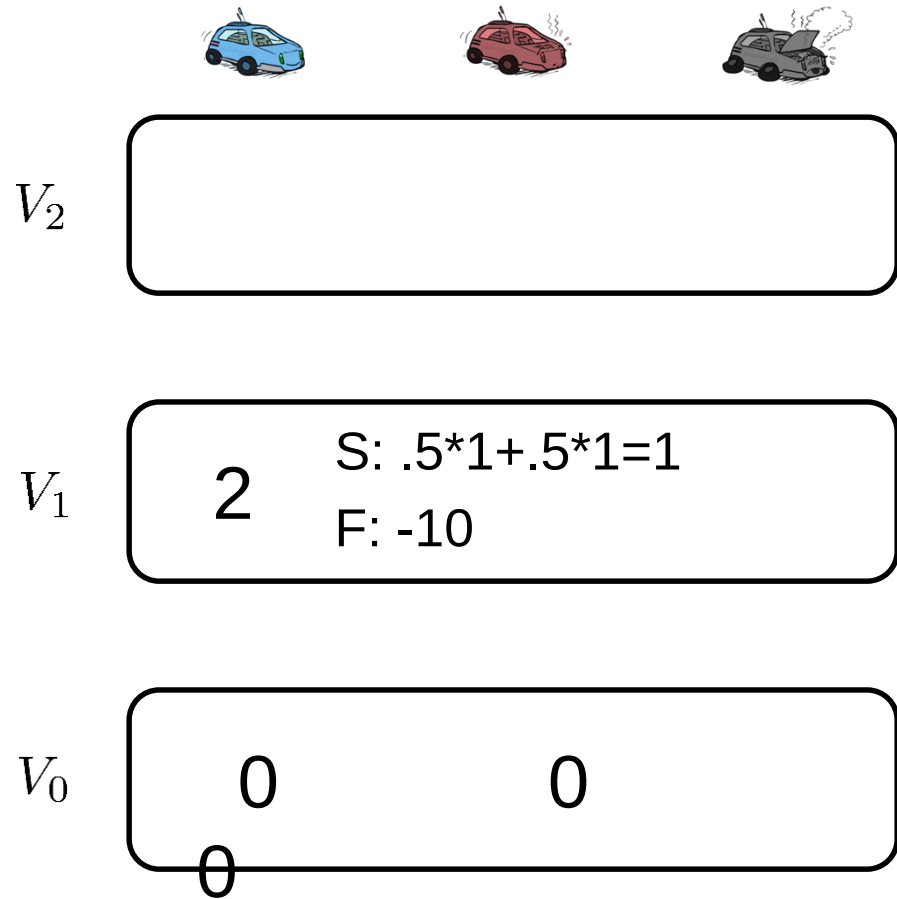
Example: Value Iteration



Assume no discount!

$$V_{k+1}(s) \leftarrow \max_a \sum_{s'} T(s, a, s') [R(s, a, s') + \gamma V_k(s')]$$

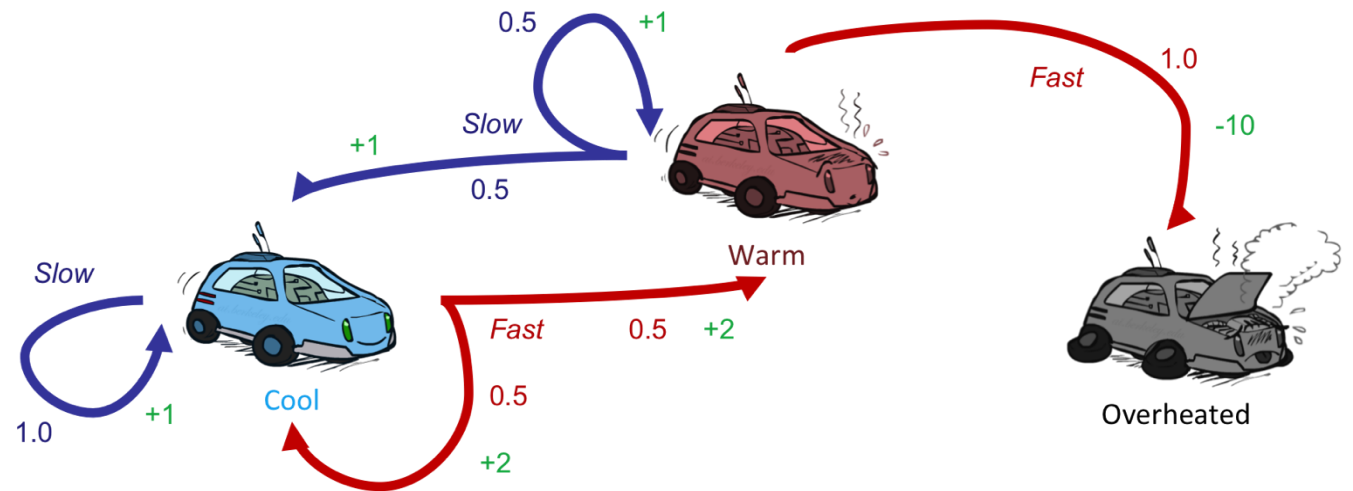
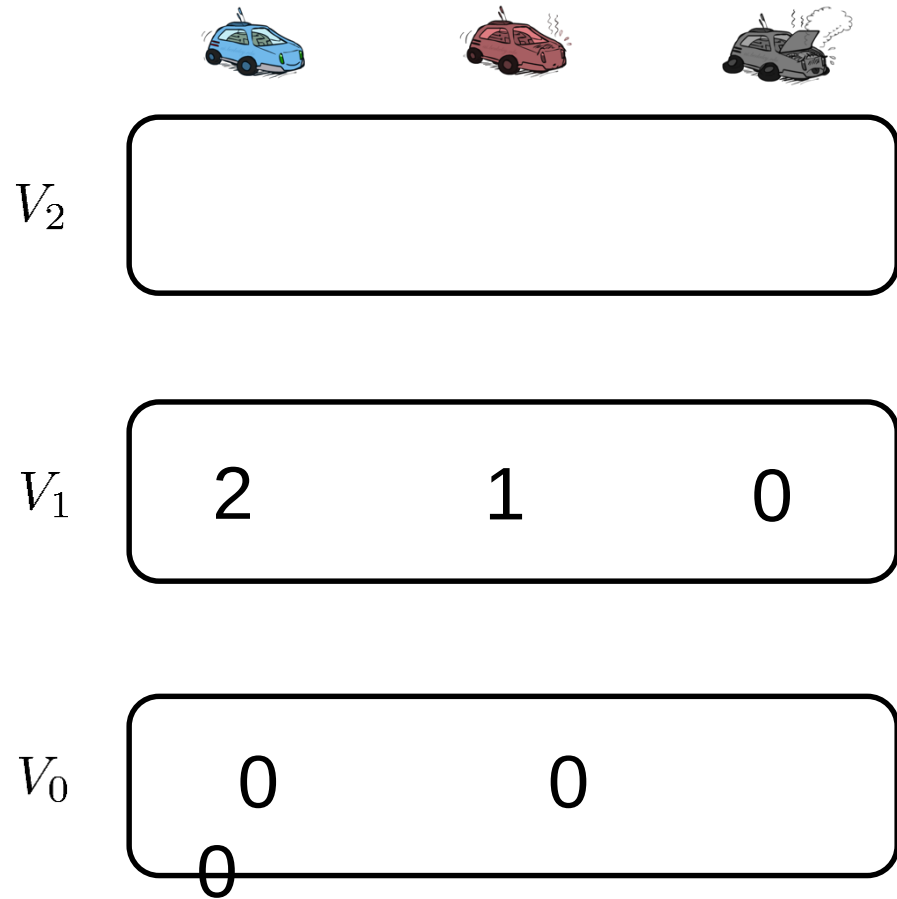
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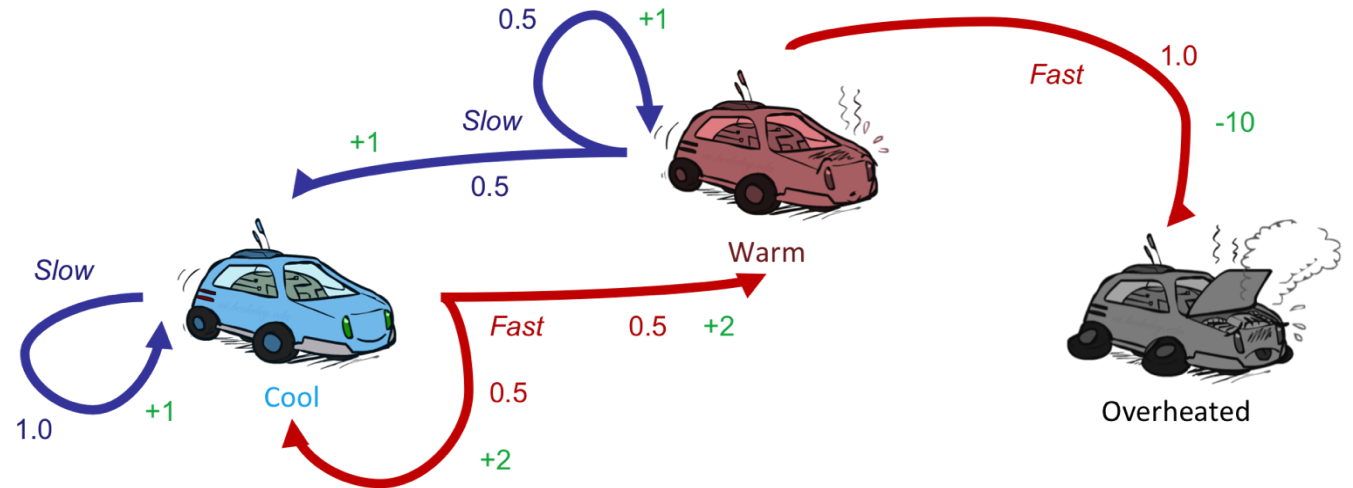
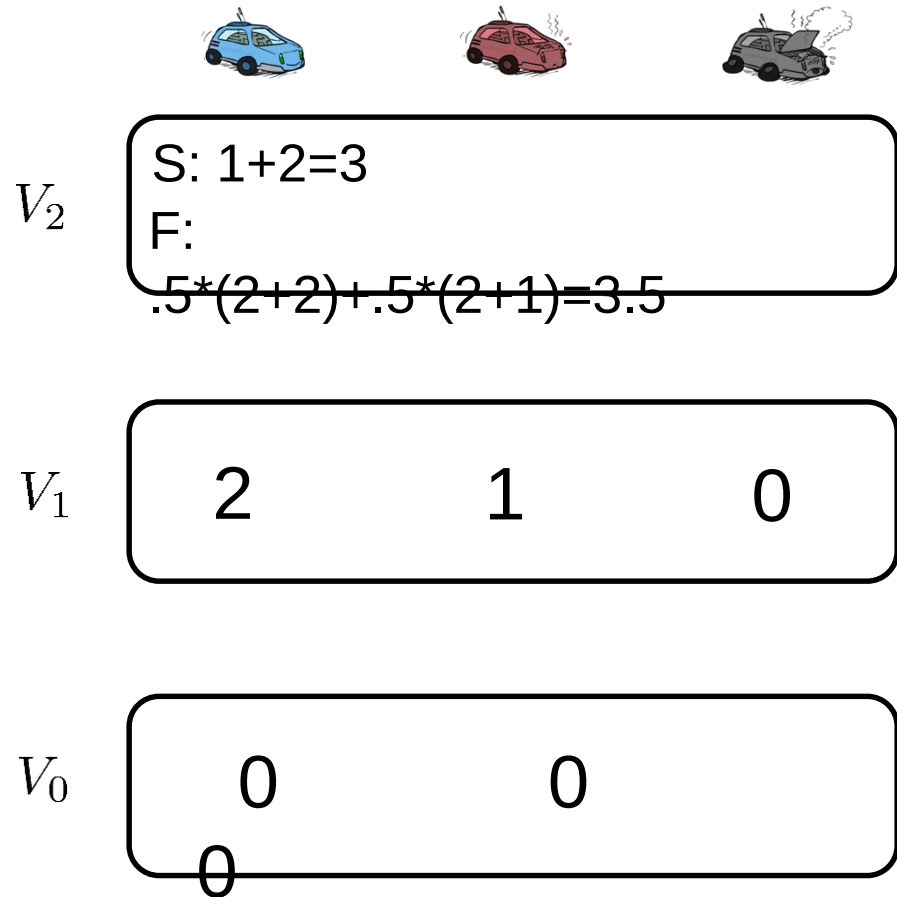
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Example: Value Iteration

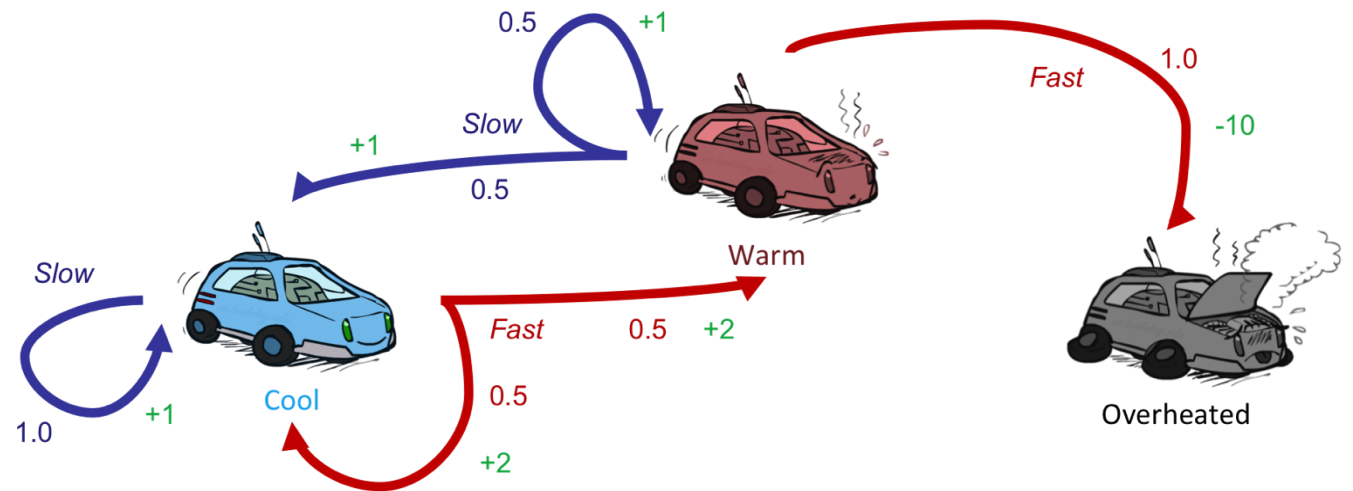


Assume no discount!

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Example: Value Iteration

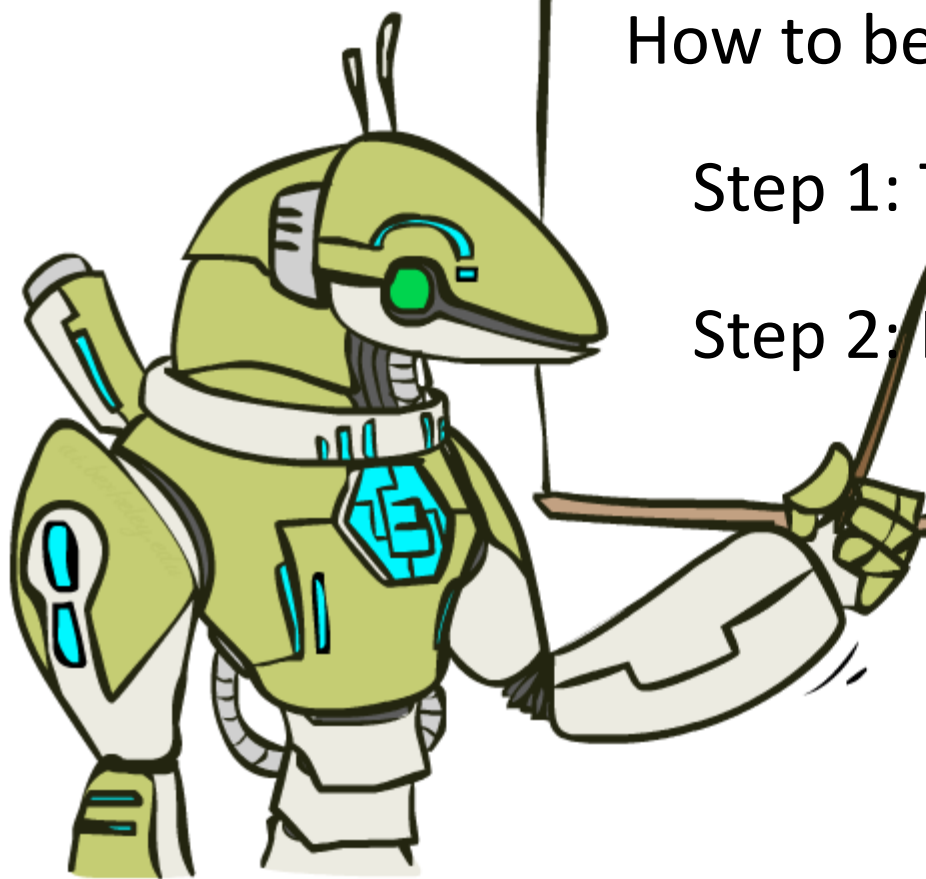
			
V_2	3.5	2.5	0
V_1	2	1	0
V_0	0	0	0



Assume no discount!

$$V_{k+1}(s) \leftarrow \max_a \sum_{s'} T(s, a, s') [R(s, a, s') + \gamma V_k(s')]$$

The Bellman Equations



How to be optimal:

Step 1: Take correct first action

Step 2: Keep being optimal

The Bellman Equations

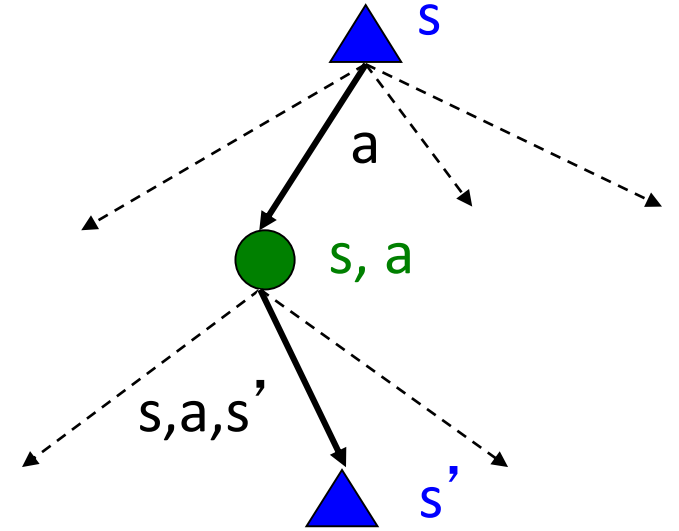
- Definition of “optimal utility” via expectimax recurrence gives a simple one-step lookahead relationship amongst optimal utility values

$$V^*(s) = \max_a Q^*(s, a)$$

$$Q^*(s, a) = \sum_{s'} T(s, a, s') [R(s, a, s') + \gamma V^*(s')]$$

$$V^*(s) = \max_a \sum_{s'} T(s, a, s') [R(s, a, s') + \gamma V^*(s')]$$

- These are the Bellman equations, and they characterize optimal values in a way we'll use over and over



Value Iteration

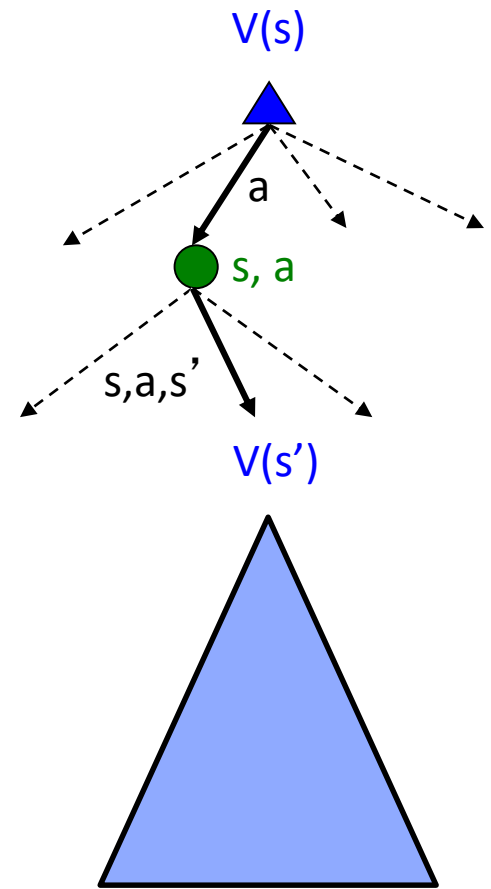
- Bellman equations **characterize** the optimal values:

$$V^*(s) = \max_a \sum_{s'} T(s, a, s') [R(s, a, s') + \gamma V^*(s')]$$

- Value iteration **computes** them:

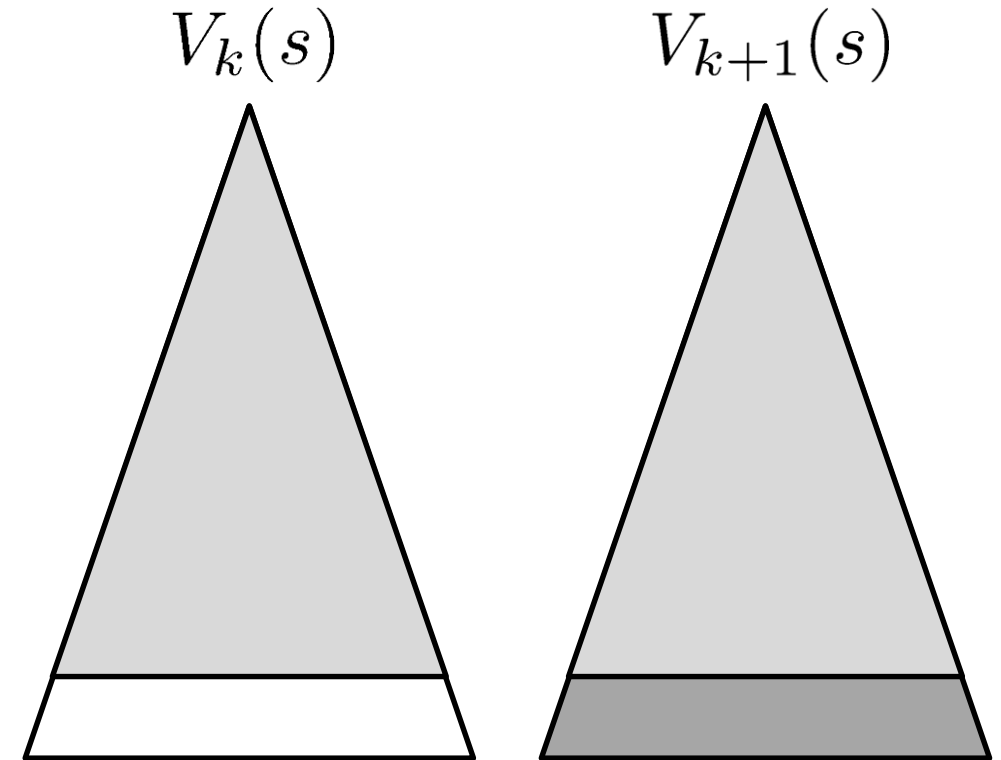
$$V_{k+1}(s) \leftarrow \max_a \sum_{s'} T(s, a, s') [R(s, a, s') + \gamma V_k(s')]$$

- Value iteration is just a fixed point solution method
 - ... though the V_k vectors are also interpretable as time-limited values



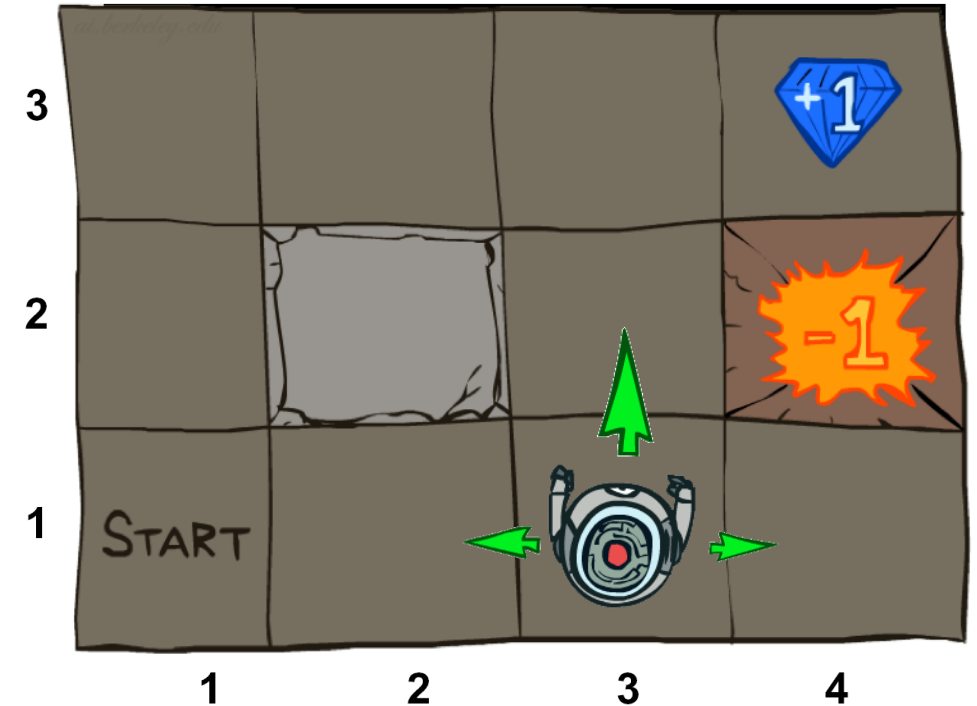
Convergence*

- How do we know the V_k vectors are going to converge?
- Case 1: If the tree has maximum depth M , then V_M holds the actual untruncated values
- Case 2: If the discount is less than 1
 - Sketch: For any state V_k and V_{k+1} can be viewed as depth $k+1$ expectimax results in nearly identical search trees
 - The difference is that on the bottom layer, V_{k+1} has actual rewards while V_k has zeros
 - That last layer is at best all R_{MAX}
 - It is at worst R_{MIN}
 - But everything is discounted by γ^k that far out
 - So V_k and V_{k+1} are at most $\gamma^k \max |R|$ different
 - So as k increases, the values converge



Recap: Markov Decision Processes

- An MDP is defined by:
 - A **set of states** $s \in S$
 - A **set of actions** $a \in A$
 - A **transition function** $T(s, a, s')$
 - Probability that a from s leads to s' , i.e., $P(s' | s, a)$
 - Also called the model or the dynamics
 - A **reward function** $R(s, a, s')$
 - Sometimes just $R(s)$ or $R(s')$
 - A **start state**
 - Maybe a **terminal state**
- MDPs are non-deterministic search problems
 - One way to solve them is with expectimax search
 - We'll have a new tool soon



Recap: MDPs

- Search problems in uncertain environments
 - Model uncertainty with transition function
 - Assign utility to states. How? Using reward functions
- Decision making and search in MDPs <-- Find a sequence of actions that maximize expected sum of rewards
 - Value of a state
 - Q-Value of a state
 - Policy for a state

The Bellman Equations

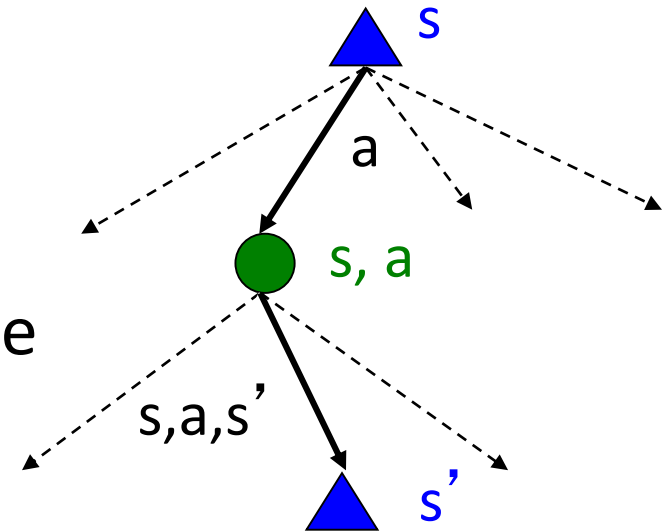
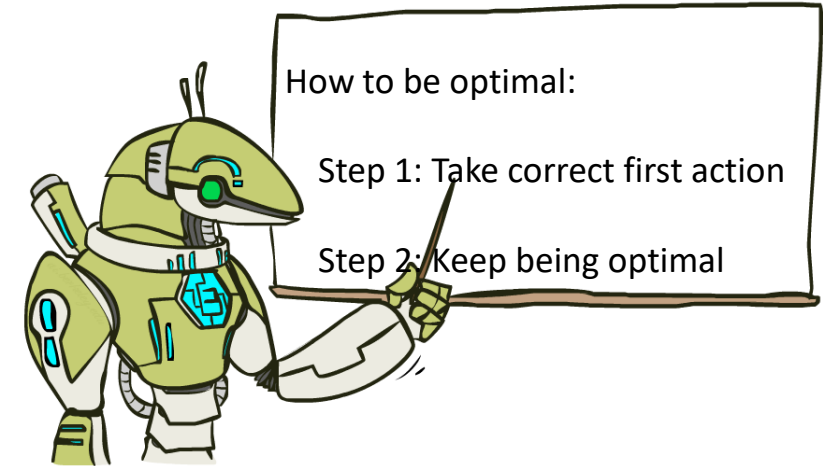
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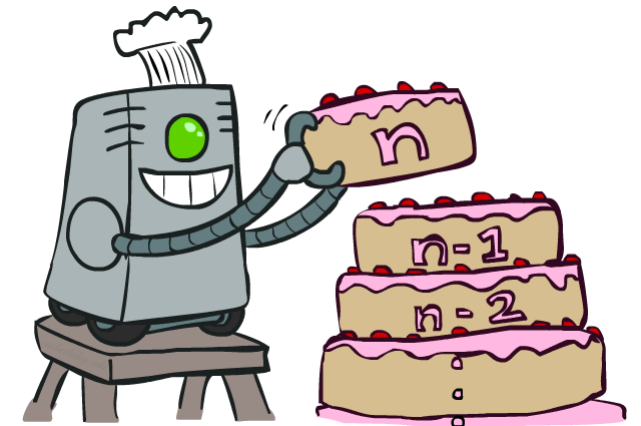
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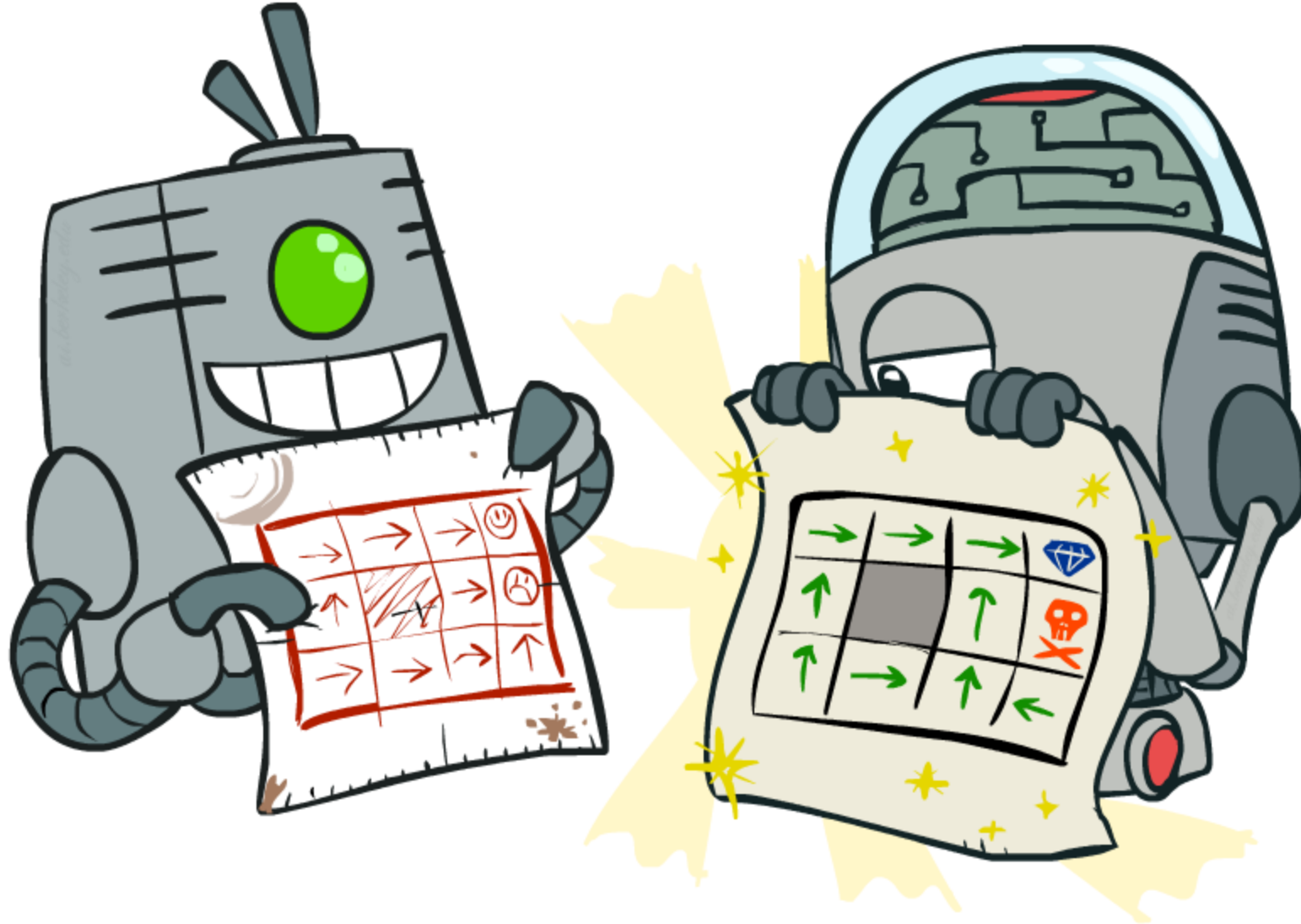


Solving MDPs

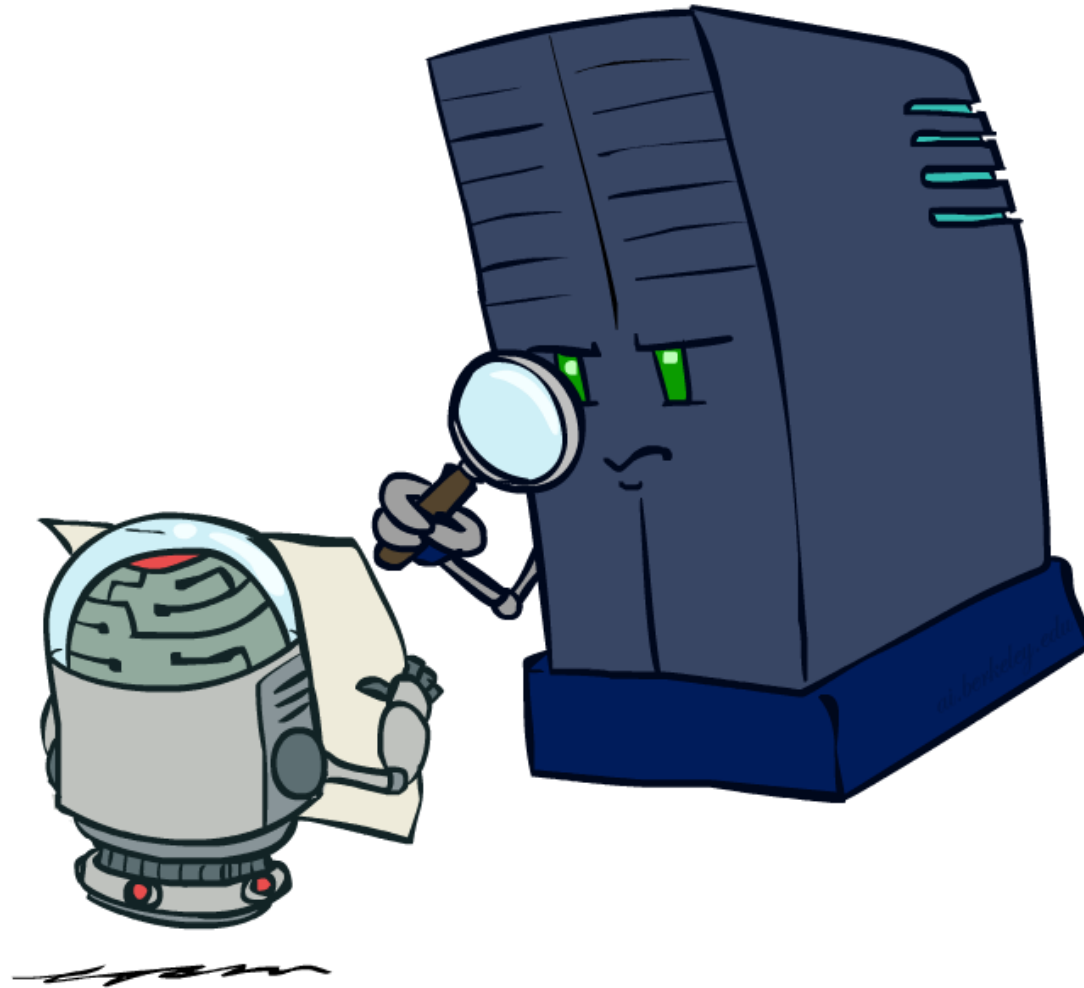
- Finding the best policy $\rightarrow$ mapping of actions to states
- So far, we have talked about one method
 - Value iteration: computes the **optimal** values of states



Policy Methods

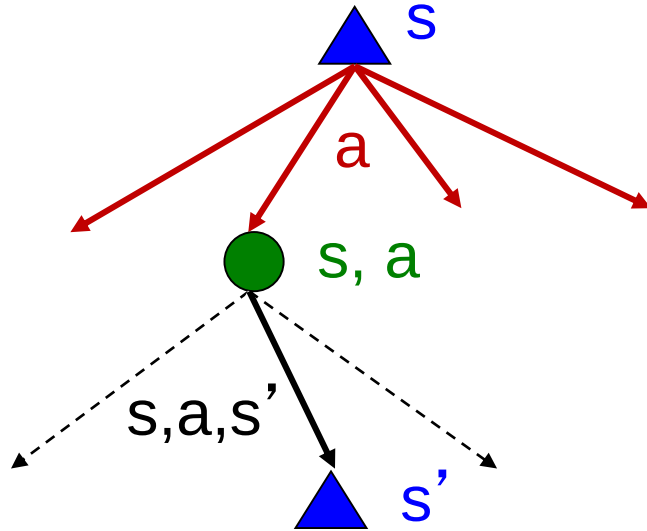


Policy Evaluation

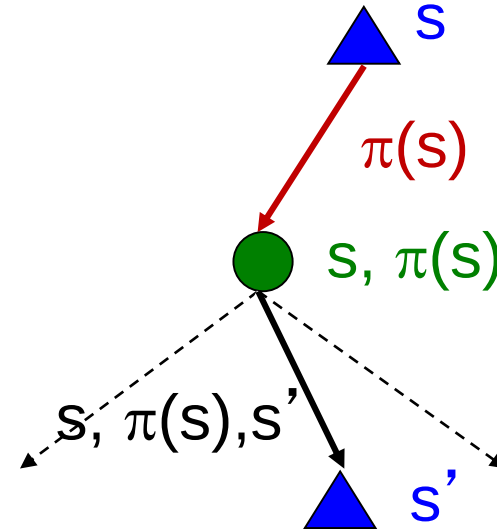


Fixed Policies

Do the optimal action



Do what π says to do

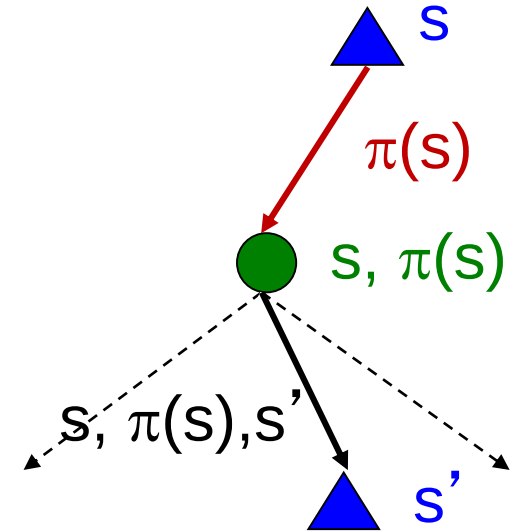


- Expectimax trees max over all actions to compute the optimal values
- If we fixed some policy $\pi(s)$, then the tree would be simpler – only one action per state
 - ... though the tree's value would depend on which policy we fixed

Utilities for a Fixed Policy

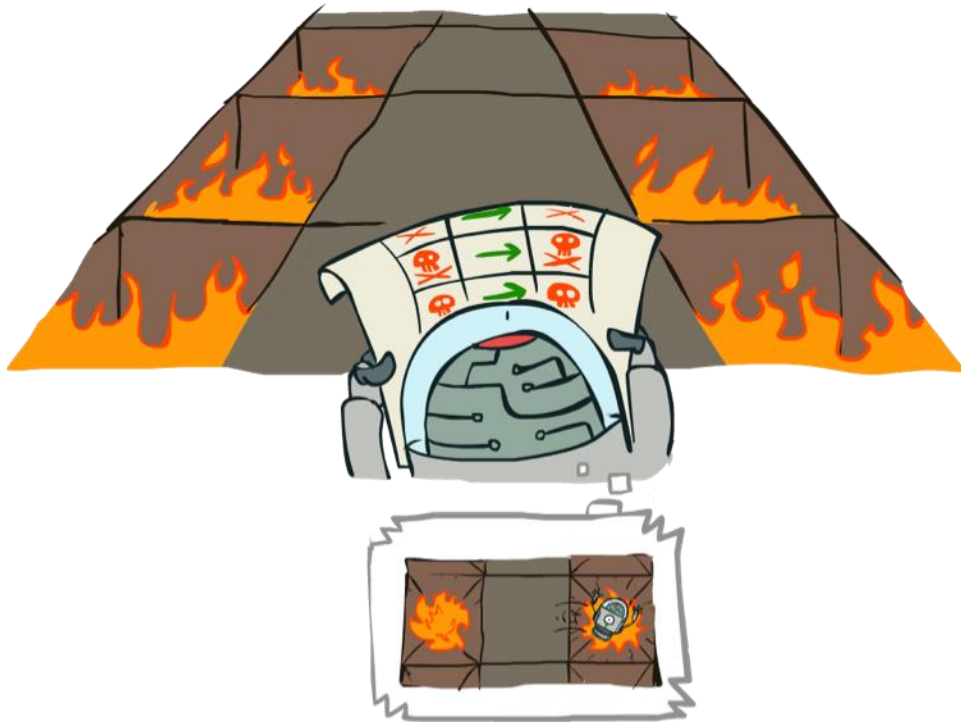
- Another basic operation: compute the utility of a state s under a fixed (generally non-optimal) policy
- Define the utility of a state s , under a fixed policy π :
 $V^\pi(s)$ = expected total discounted rewards starting in s and following π
- Recursive relation (one-step look-ahead / Bellman equation):

$$V^\pi(s) = \sum_{s'} T(s, \pi(s), s') [R(s, \pi(s), s') + \gamma V^\pi(s')]$$

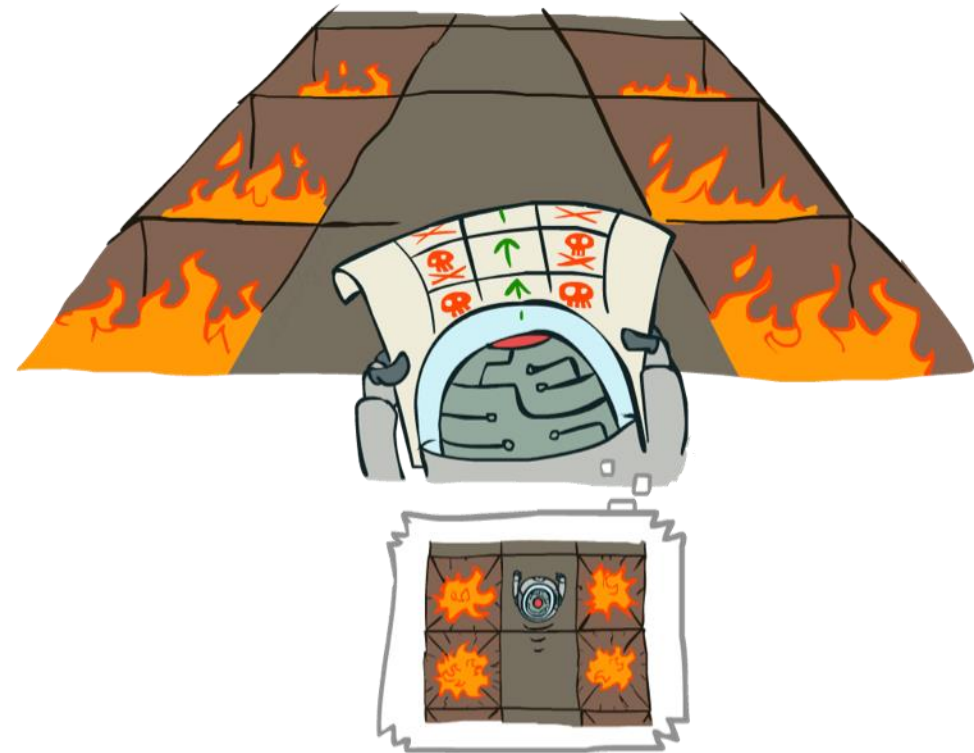


Example: Policy Evaluation

Always Go Right



Always Go Forward

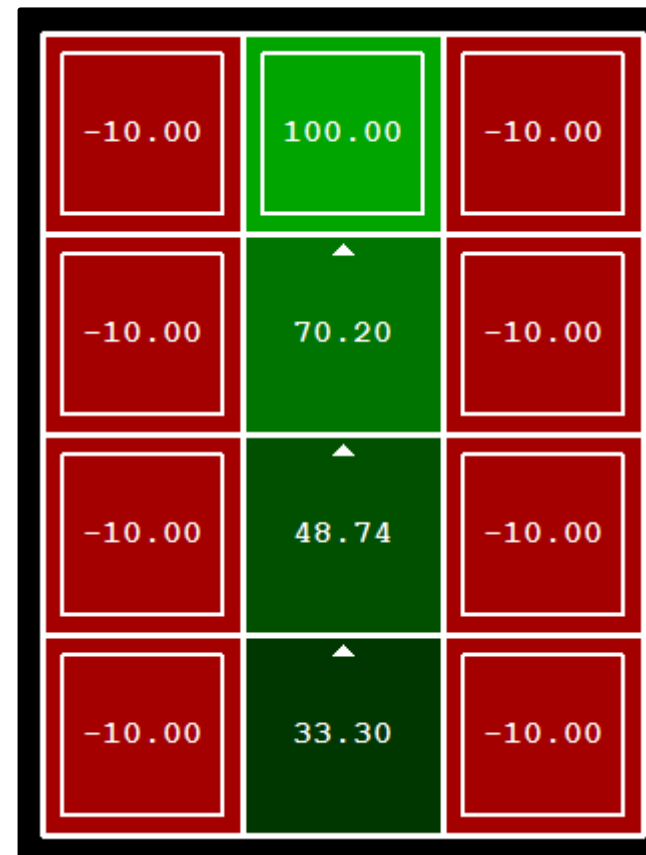


Example: Policy Evaluation

Always Go Right



Always Go Forward

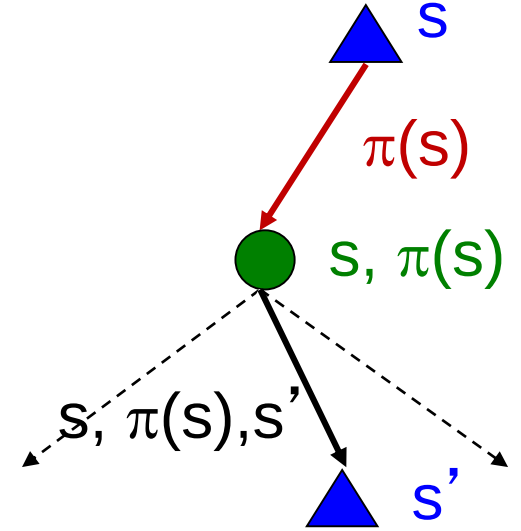


Policy Evaluation

- How do we calculate the V 's for a fixed policy π ?
- Idea 1: Turn recursive Bellman equations into updates (like value iteration)

$$V_0^\pi(s) = 0$$

$$V_{k+1}^\pi(s) \leftarrow \sum_{s'} T(s, \pi(s), s') [R(s, \pi(s), s') + \gamma V_k^\pi(s')]$$

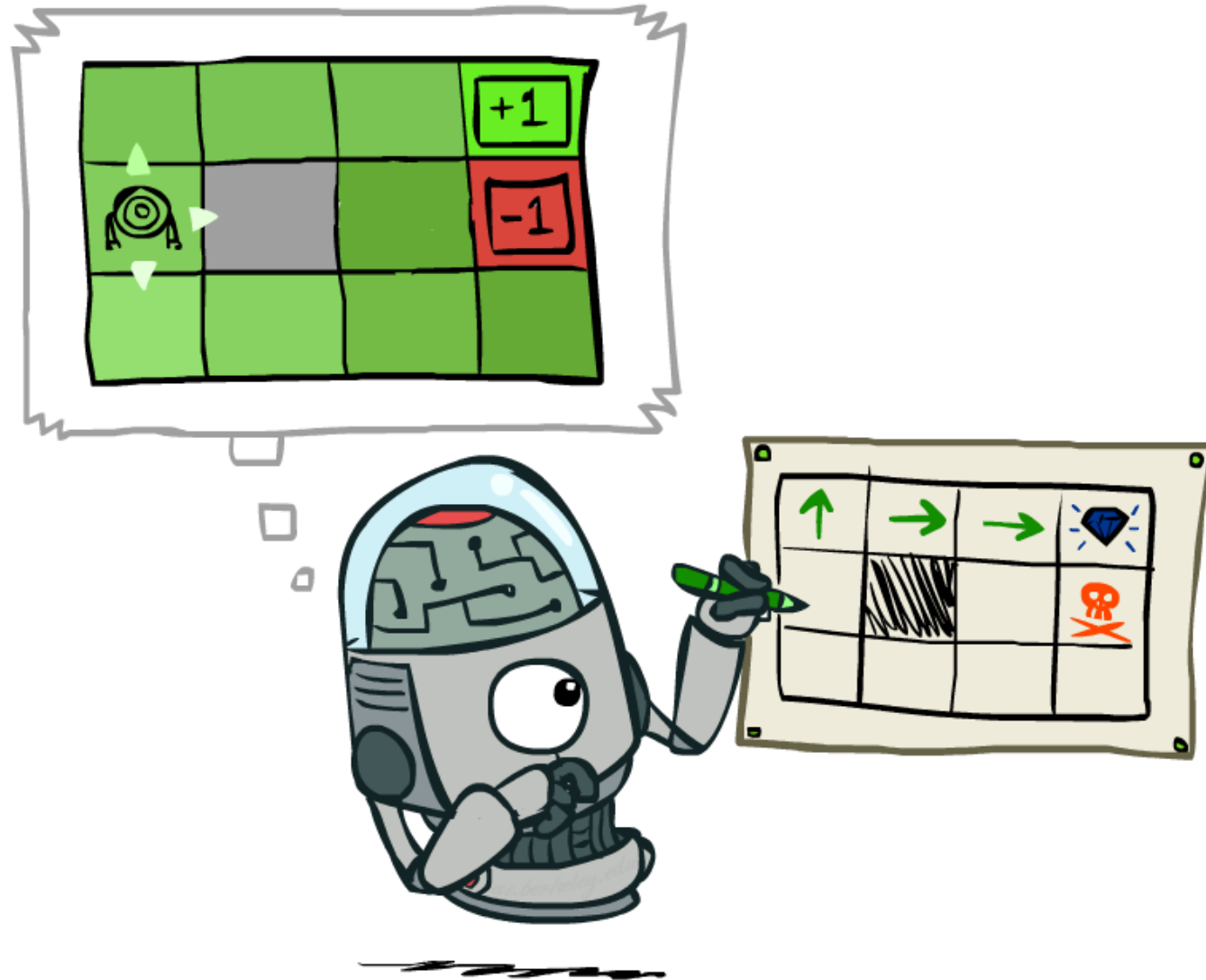


- Efficiency: $O(S^2)$ per iteration
- Idea 2: Without the maxes, the Bellman equations are just a linear system
 - Solve with Matlab (or your favorite linear system solver)

Let's think...

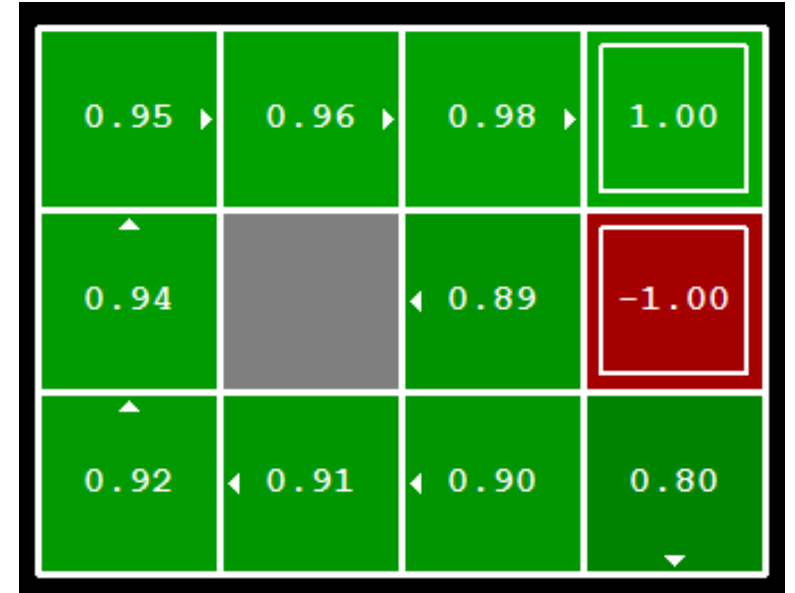
- Take a minute, think about value iteration and policy evaluation
 - Write down the biggest questions you have about them.

Policy Extraction



Computing Actions from Values

- Let's imagine we have the optimal values $V^*(s)$
- How should we act?
 - It's not obvious!
- We need to do a mini-expectimax (one step)



$$\pi^*(s) = \arg \max_a \sum_{s'} T(s, a, s') [R(s, a, s') + \gamma V^*(s')]$$

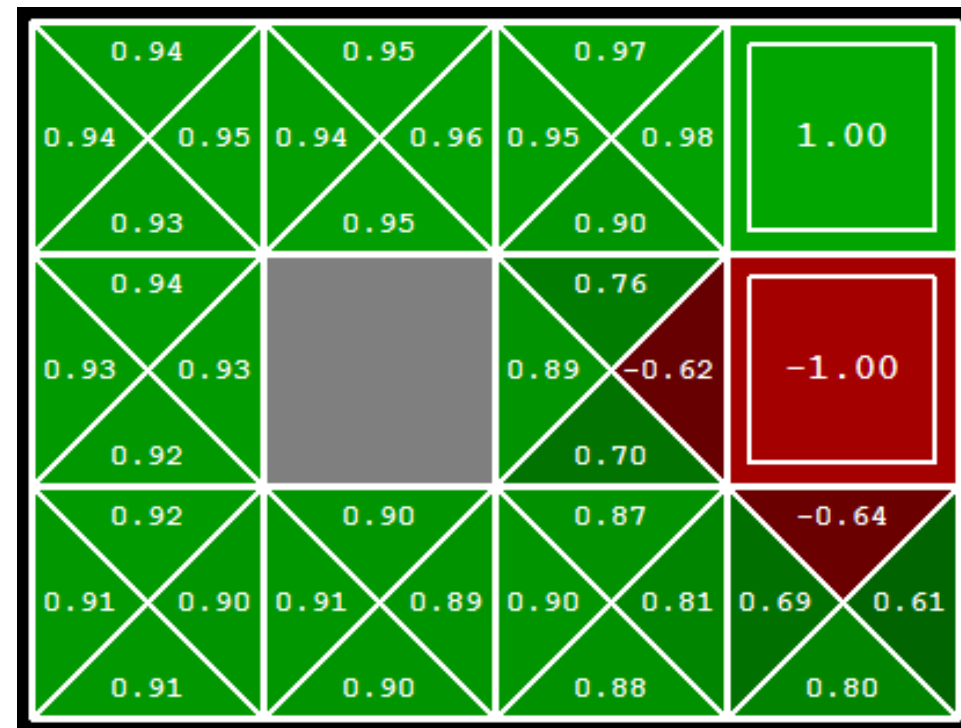
- This is called **policy extraction**, since it gets the policy implied by the values

Computing Actions from Q-Values

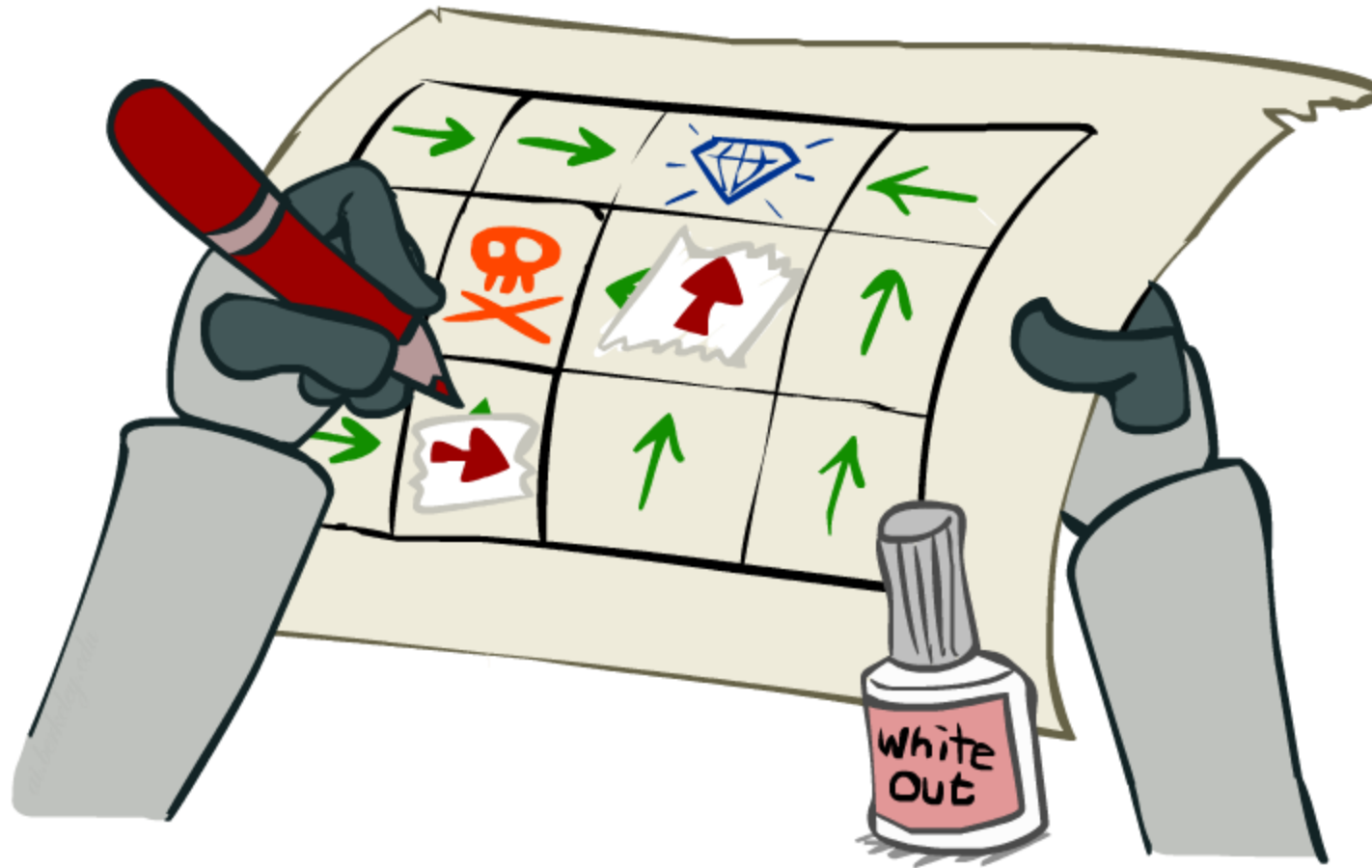
- Let's imagine we have the optimal q-values:
- How should we act?
 - Completely trivial to decide!

$$\pi^*(s) = \arg \max_a Q^*(s, a)$$

- Important lesson: actions are easier to select from q-values than values!



Policy Iteration

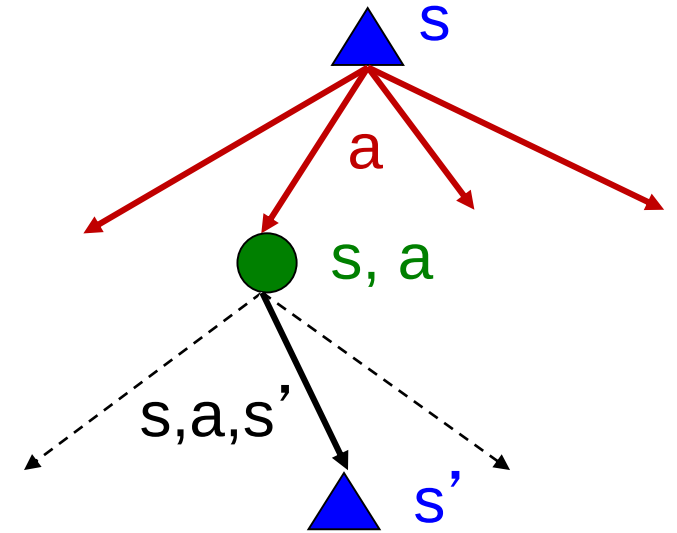


Problems with Value Iteration

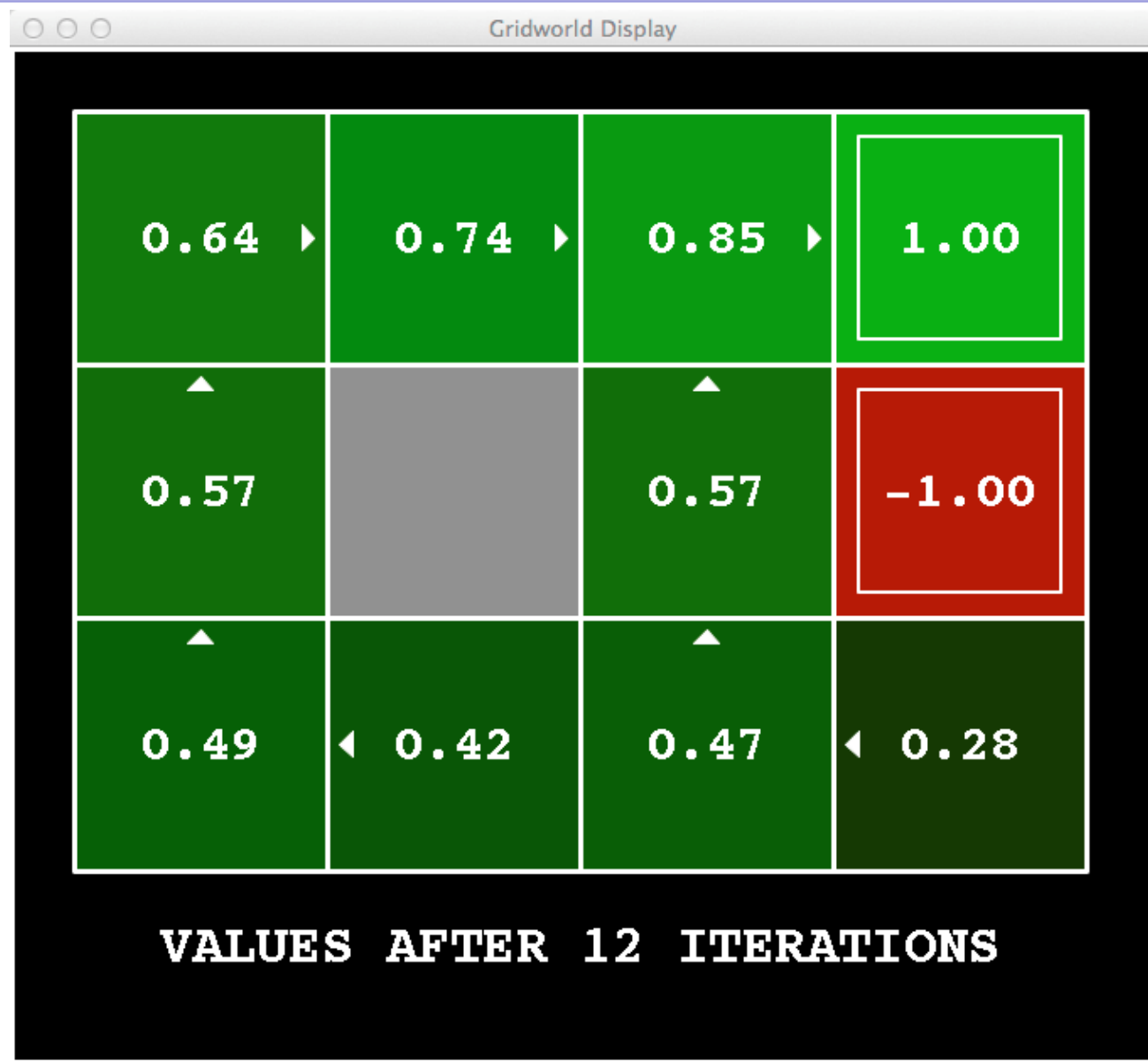
- Value iteration repeats the Bellman updates:

$$V_{k+1}(s) \leftarrow \max_a \sum_{s'} T(s, a, s') [R(s, a, s') + \gamma V_k(s')]$$

- Problem 1: It's slow – $O(S^2A)$ per iteration
- Problem 2: The “max” at each state rarely changes
- Problem 3: The policy often converges long before the values



k=12



Noise = 0.2
Discount = 0.9
Living reward = 0

k=100



Noise = 0.2
Discount = 0.9
Living reward = 0

Policy Iteration

- Alternative approach for optimal values:
 - **Step 1: Policy evaluation:** calculate utilities for some fixed policy (not optimal utilities!) until convergence
 - **Step 2: Policy improvement:** update policy using one-step look-ahead with resulting converged (but not optimal!) utilities as future values
 - Repeat steps until policy converges
- This is **policy iteration**
 - It's still optimal!
 - Can converge (much) faster under some conditions

Policy Iteration

- Evaluation: For fixed current policy π , find values with policy evaluation:
 - Iterate until values converge:

$$V_{k+1}^{\pi_i}(s) \leftarrow \sum_{s'} T(s, \pi_i(s), s') [R(s, \pi_i(s), s') + \gamma V_k^{\pi_i}(s')]$$

- Improvement: For fixed values, get a better policy using policy extraction
 - One-step look-ahead:

$$\pi_{i+1}(s) = \arg \max_a \sum_{s'} T(s, a, s') [R(s, a, s') + \gamma V^{\pi_i}(s')]$$

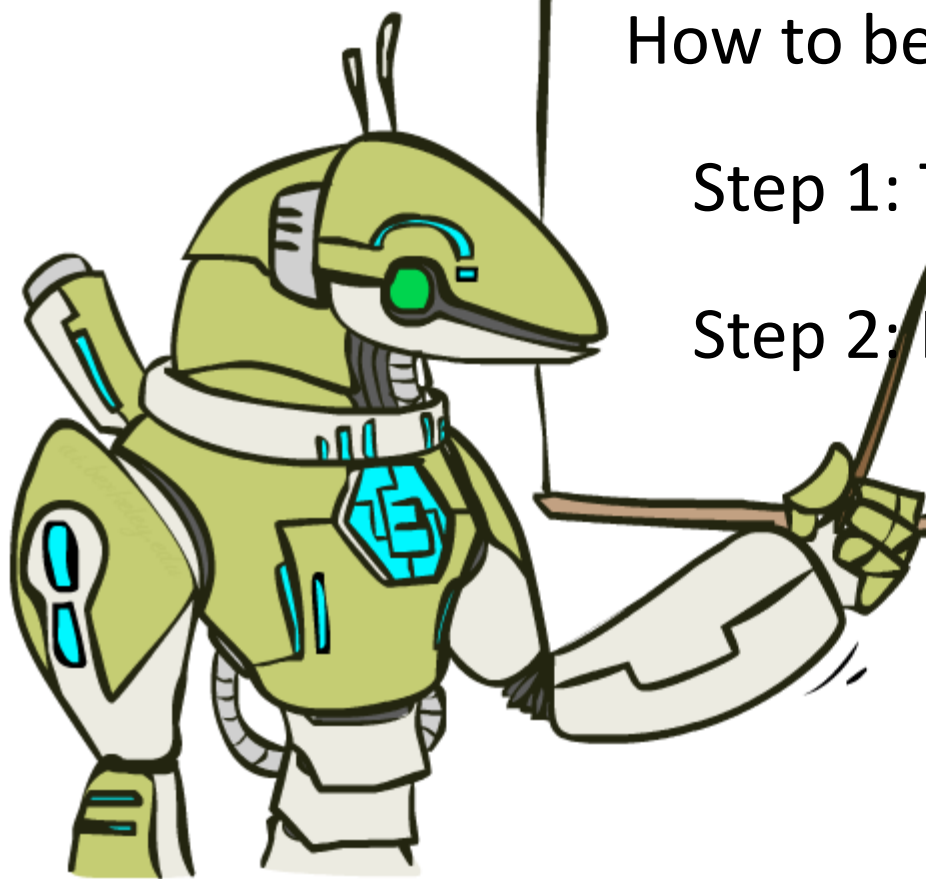
Comparison

- Both value iteration and policy iteration compute the same thing (all optimal values)
- In value iteration:
 - Every iteration updates both the values and (implicitly) the policy
 - We don't track the policy, but taking the max over actions implicitly recomputes it
- In policy iteration:
 - We do several passes that update utilities with fixed policy (each pass is fast because we consider only one action, not all of them)
 - After the policy is evaluated, a new policy is chosen (slow like a value iteration pass)
 - The new policy will be better (or we're done)
- Both are dynamic programs for solving MDPs

Summary: MDP Algorithms

- So you want to....
 - Compute optimal values: use value iteration or policy iteration
 - Compute values for a particular policy: use policy evaluation
 - Turn your values into a policy: use policy extraction (one-step lookahead)
- These all look the same!
 - They basically are – they are all variations of Bellman updates
 - They all use one-step lookahead expectimax fragments
 - They differ only in whether we plug in a fixed policy or max over actions

The Bellman Equations



How to be optimal:

Step 1: Take correct first action

Step 2: Keep being optimal

Next Topic: Reinforcement Learning!
