# CSE 573: Artificial Intelligence
Winter 2019

## Markov Models &
## Hidden Markov Models

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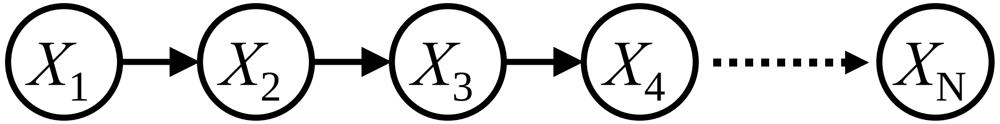
Many slides adapted from Pieter Abbeel, Dan Klein, Dan Weld,Stuart Russell, Andrew Moore & Luke Zettlemoyer

# Markov Models (Markov Chains)




- A Markov model defines
  - a joint probability distribution:

$$P(X_1, X_2, X_3, X_4) = P(X_1)P(X_2|X_1)P(X_3|X_2)P(X_4|X_3)$$

- More generally:

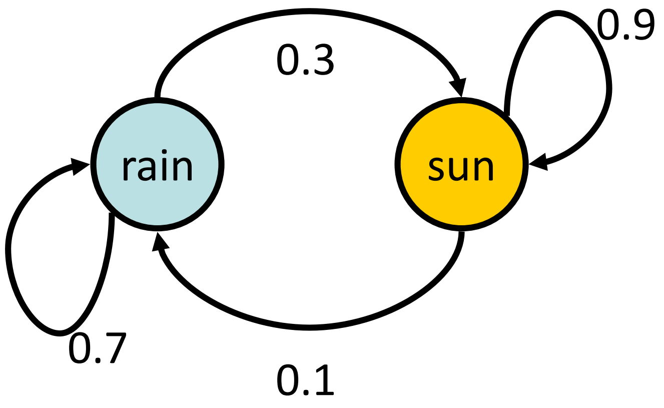
$$P(X_1, X_2, \ldots, X_T) = P(X_1)P(X_2|X_1)P(X_3|X_2)\ldots P(X_T|X_{T-1})$$

$$P(X_1, \ldots, X_n) = P(X_1)\prod_{t=2}^{N} P(X_t|X_{t-1})$$

- One common inference problem:
  - Compute marginals $P(X_t)$ for all time steps $t$

# Example Markov Chain: Weather

- Initial distribution: 1.0 sun



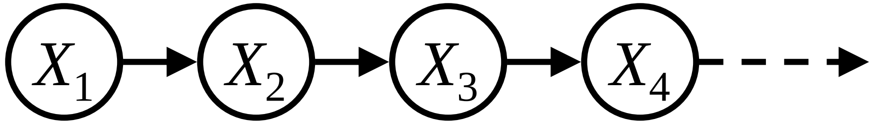

- What is the probability distribution after one step?

$$P(X_2 = \text{sun}) = \quad P(X_2 = \text{sun} | X_1 = \text{sun}) P(X_1 = \text{sun}) + P(X_2 = \text{sun} | X_1 = \text{rain}) P(X_1 = \text{rain})$$

$$0.9 \cdot 1.0 + 0.3 \cdot 0.0 = 0.9$$

# Mini-Forward Algorithm

- Question: What's P(X) on some day t?




$$P(x_1) = \text{known}$$

$$\begin{aligned} P(x_t) &= \sum_{x_{t-1}} P(x_{t-1}, x_t) \\ &= \sum_{x_{t-1}} P(x_t \mid x_{t-1}) P(x_{t-1}) \end{aligned}$$

*Forward simulation*

# Example Run of Mini-Forward Algorithm

- From initial observation of sun

$$\begin{array}{ccccc} \left\langle \begin{array}{c} 1.0 \\ 0.0 \end{array} \right\rangle & \left\langle \begin{array}{c} 0.9 \\ 0.1 \end{array} \right\rangle & \left\langle \begin{array}{c} 0.84 \\ 0.16 \end{array} \right\rangle & \left\langle \begin{array}{c} 0.804 \\ 0.196 \end{array} \right\rangle \Rightarrow & \left\langle \begin{array}{c} 0.75 \\ 0.25 \end{array} \right\rangle \\ \mathrm{P}(X_1) & \mathrm{P}(X_2) & \mathrm{P}(X_3) & \mathrm{P}(X_4) & \mathrm{P}(X_\infty) \end{array}$$

- From initial observation of rain

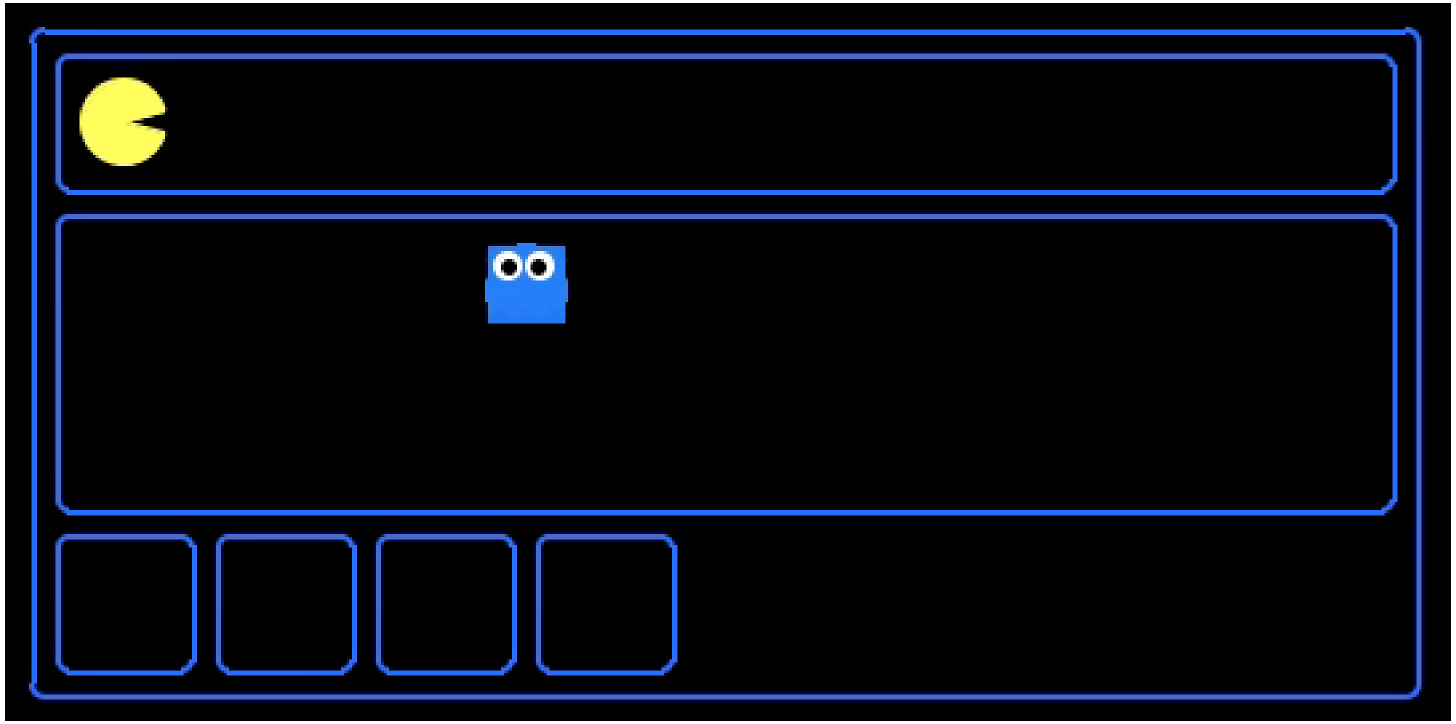
$$\begin{array}{ccccc} \left\langle \begin{array}{c} 0.0 \\ 1.0 \end{array} \right\rangle & \left\langle \begin{array}{c} 0.3 \\ 0.7 \end{array} \right\rangle & \left\langle \begin{array}{c} 0.48 \\ 0.52 \end{array} \right\rangle & \left\langle \begin{array}{c} 0.588 \\ 0.412 \end{array} \right\rangle \Rightarrow & \left\langle \begin{array}{c} 0.75 \\ 0.25 \end{array} \right\rangle \\ \mathrm{P}(X_1) & \mathrm{P}(X_2) & \mathrm{P}(X_3) & \mathrm{P}(X_4) & \mathrm{P}(X_\infty) \end{array}$$

- From yet another initial distribution $P(X_1)$:

$$\begin{array}{ccc} \left\langle \begin{array}{c} p \\ 1-p \end{array} \right\rangle & \cdots \Rightarrow & \left\langle \begin{array}{c} 0.75 \\ 0.25 \end{array} \right\rangle \\ \mathrm{P}(X_1) & & \mathrm{P}(X_\infty) \end{array}$$

# Pac-man Markov Chain

Pac-man knows the ghost's initial position, but gets no observations!



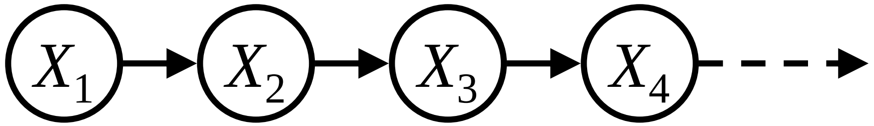
# Stationary Distributions

- For most chains:
  - Influence of the initial distribution gets less and less over time.
  - The distribution we end up in is independent of the initial distribution
- Stationary distribution:
  - The distribution we end up with is called the stationary distribution $P_\infty$ of the chain
  - It satisfies

$$P_\infty(X) = P_{\infty+1}(X) = \sum_x P(X|x)P_\infty(x)$$

# Example: Stationary Distributions

- Question: What’s P(X) at time t = infinity?




$$P_\infty(sun) = P(sun|sun)P_\infty(sun) + P(sun|rain)P_\infty(rain)$$
$$P_\infty(rain) = P(rain|sun)P_\infty(sun) + P(rain|rain)P_\infty(rain)$$

$$P_\infty(sun) = 0.9P_\infty(sun) + 0.3P_\infty(rain)$$
$$P_\infty(rain) = 0.1P_\infty(sun) + 0.7P_\infty(rain)$$

$$P_\infty(sun) = 3P_\infty(rain)$$
$$P_\infty(rain) = 1/3P_\infty(sun)$$

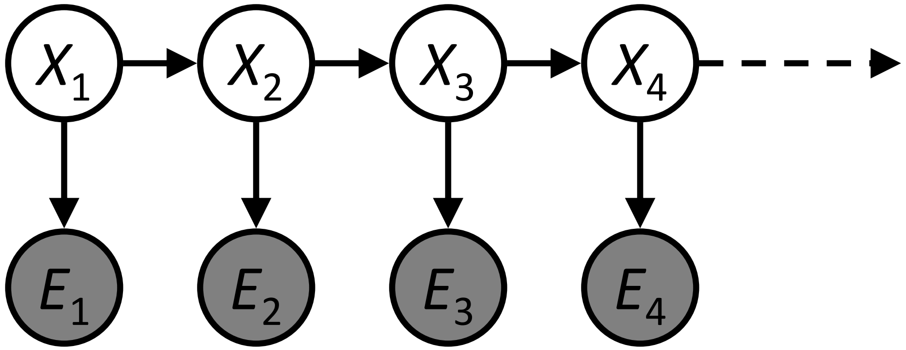
Also: $P_\infty(sun) + P_\infty(rain) = 1$

$$P_\infty(sun) = 3/4$$
$$P_\infty(rain) = 1/4$$

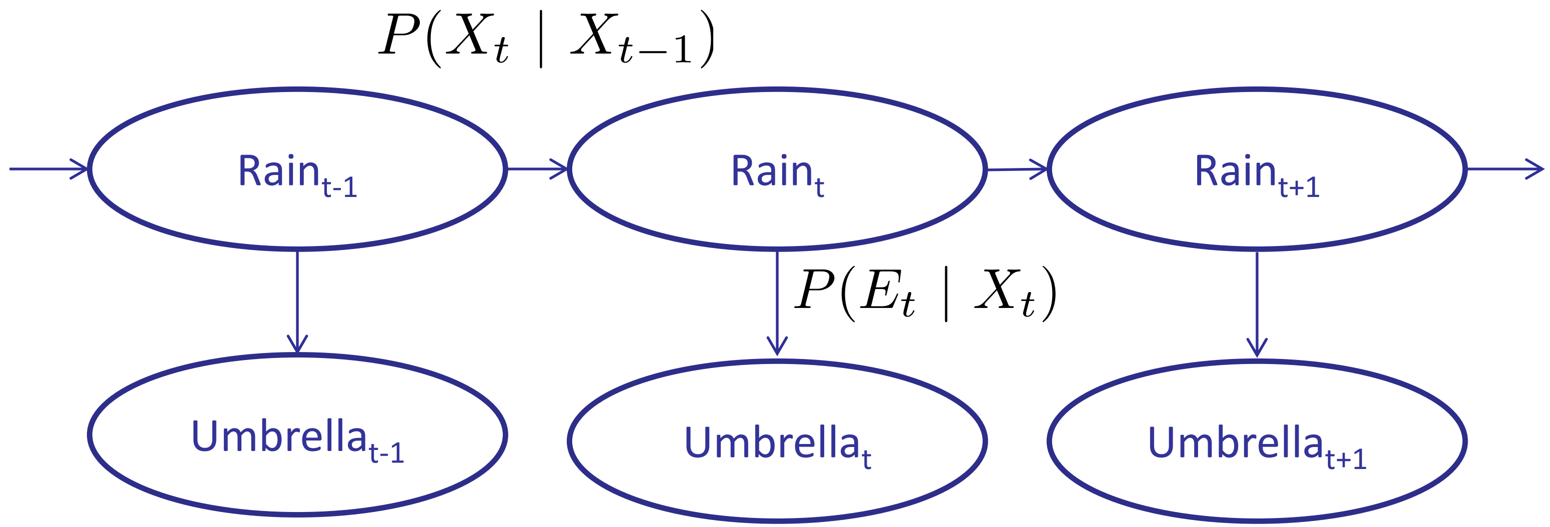
| $X_{t-1}$ | $X_t$ | $P(X_t \mid X_{t-1})$ |
|---|---|---|
| sun | sun | 0.9 |
| sun | rain | 0.1 |
| rain | sun | 0.3 |
| rain | rain | 0.7 |

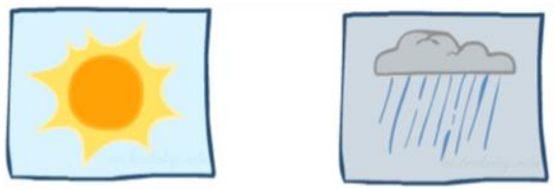
# Hidden Markov Models

- Markov chains not so useful for most agents
  - Need observations to update your beliefs
- Hidden Markov models (HMMs)
  - Underlying Markov chain over states X
  - You observe outputs (effects) at each time step



# Example: Weather HMM



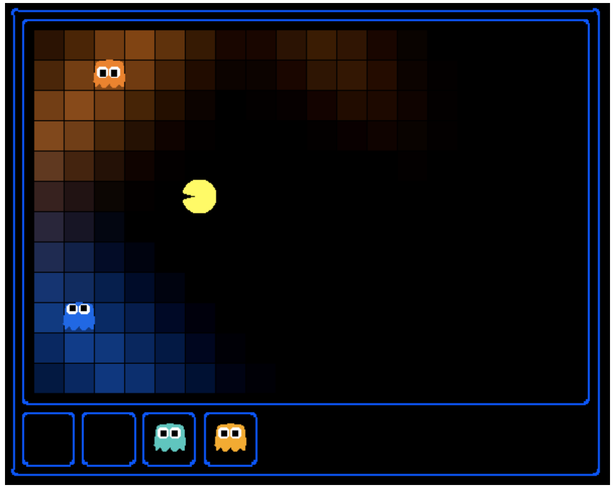
- An HMM is defined by:
  - Initial distribution: $P(X_1)$
  - Transitions: $P(X_t \mid X_{t-1})$
  - Emissions: $P(E_t \mid X_t)$

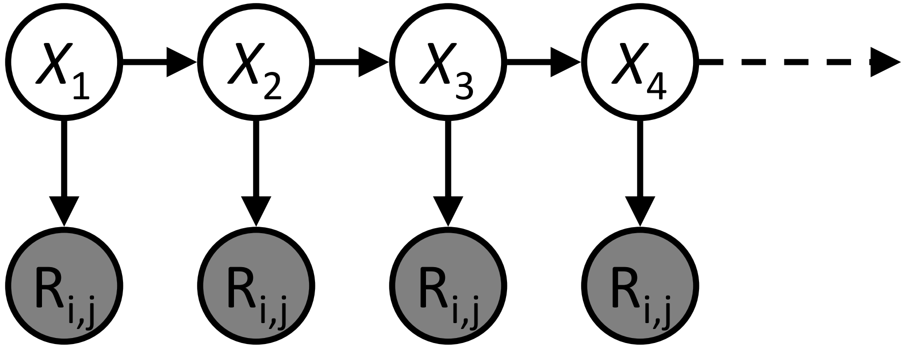
| $R_{t-1}$ | $R_t$ | $P(R_t \mid R_{t-1})$ |
|---|---|---|
| +r | +r | 0.7 |
| +r | -r | 0.3 |
| -r | +r | 0.3 |
| -r | -r | 0.7 |

| $R_t$ | $U_t$ | $P(U_t \mid R_t)$ |
|---|---|---|
| +r | +u | 0.9 |
| +r | -u | 0.1 |
| -r | +u | 0.2 |
| -r | -u | 0.8 |

# Example: Ghostbusters HMM

- $P(X_1)$ = uniform
- $P(X|X')$ = usually move clockwise, but sometimes move in a random direction or stay in place
- $P(R_{ij}|X)$ = same sensor model as before: red means close, green means far away.






| | | |
|---|---|---|
| 1/9 | 1/9 | 1/9 |
| 1/9 | 1/9 | 1/9 |
| 1/9 | 1/9 | 1/9 |

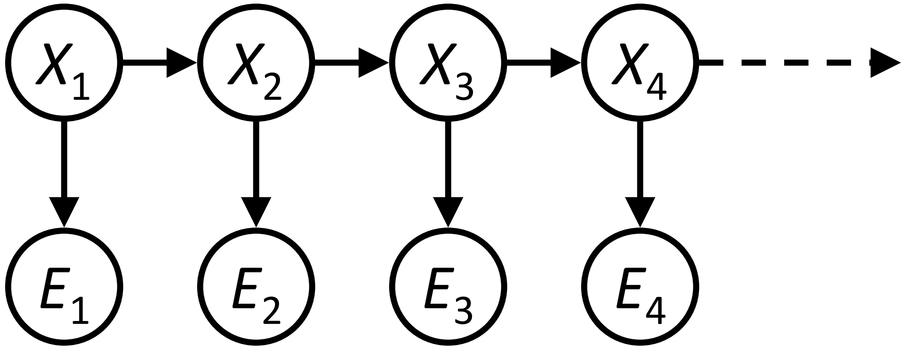
$P(X_1)$

| | | |
|---|---|---|
| 1/6 | 1/6 | 1/2 |
| 0 | 1/6 | 0 |
| 0 | 0 | 0 |

$P(X|X'=<1,2>)$

# Conditional Independence

- HMMs have two important independence properties:
  - Markov hidden process: future depends on past via the present
  - Current observation independent of all else given current state




- Quiz: does this mean that evidence variables are guaranteed to be independent?
  - [No, they tend to correlated by the hidden state]

# Real HMM Examples

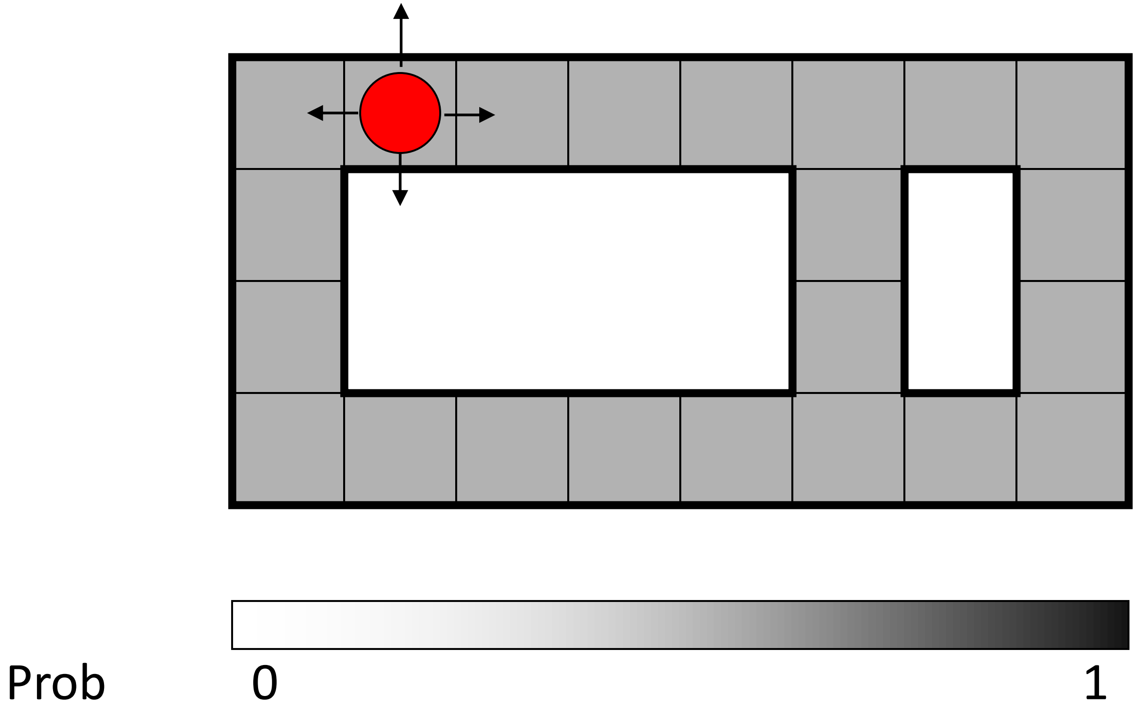
- Speech recognition HMMs:
  - Observations are acoustic signals (continuous valued)
  - States are specific positions in specific words (so, tens of thousands)
- Machine translation HMMs:
  - Observations are words (tens of thousands)
  - States are translation options
- Robot tracking:
  - Observations are range readings (continuous)
  - States are positions on a map (continuous)

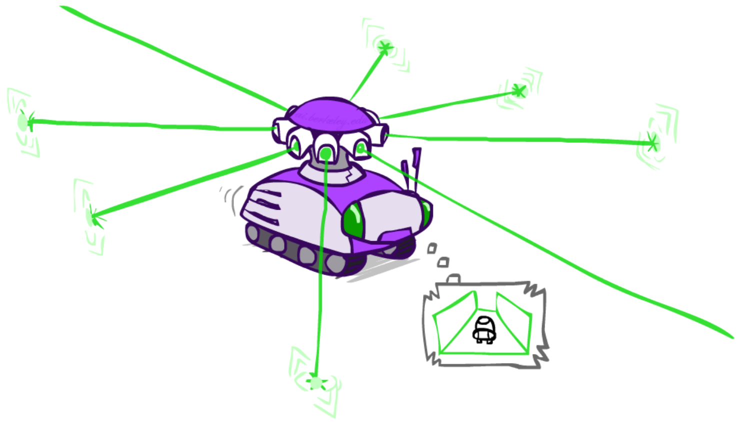
# Filtering / Monitoring

- Filtering, or monitoring, is the task of tracking the distribution $B_t(X) = P_t(X_t \mid e_1, \ldots, e_t)$ (the belief state) over time
- We start with $B_1(X)$ in an initial setting, usually uniform
- As time passes, or we get observations, we update $B(X)$
- The Kalman filter was invented in the 60's and first implemented as a method of trajectory estimation for the Apollo program

# Example: Robot Localization

*Example from Michael Pfeiffer*



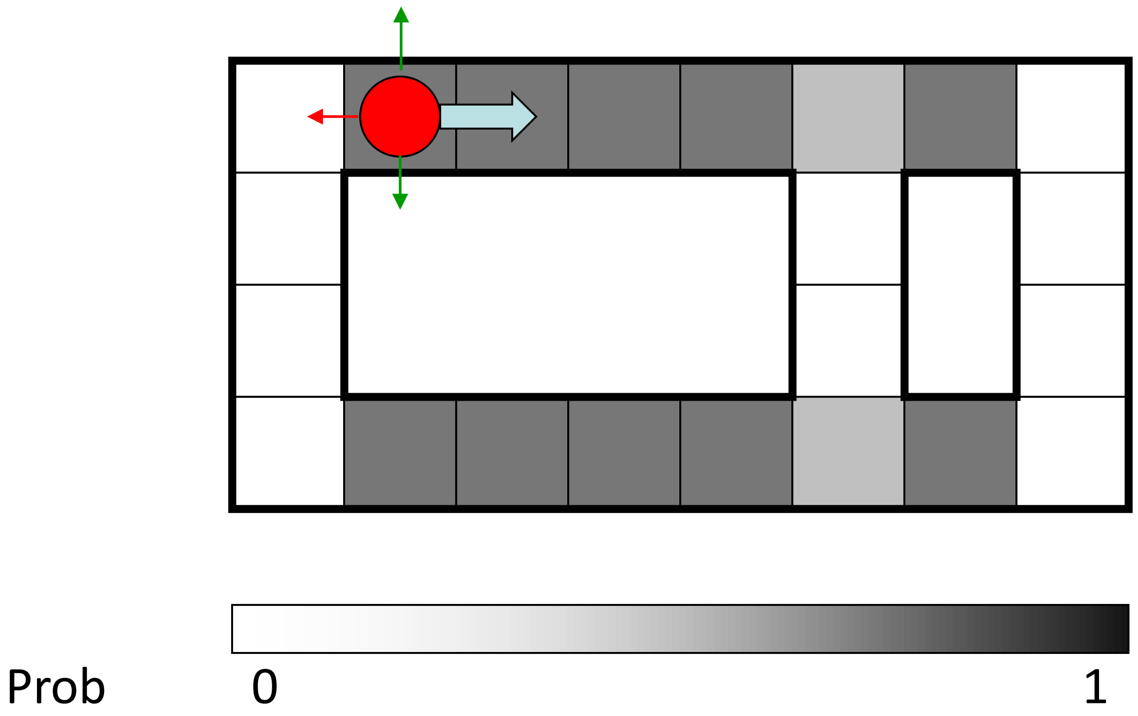



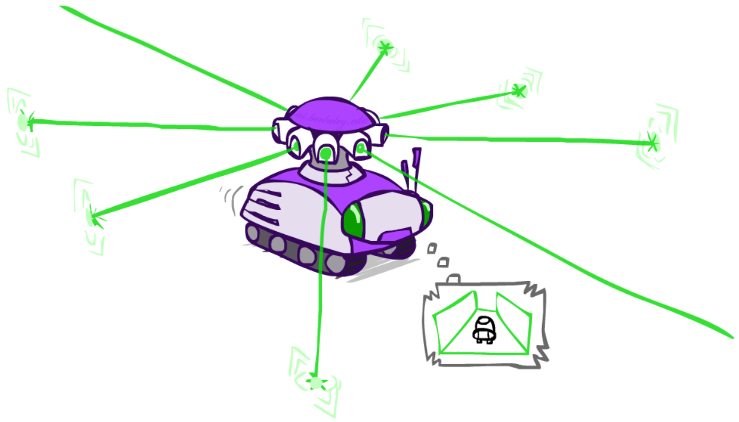
t=0

Sensor model: can read in which directions there is a wall, never more than 1 mistake

Motion model: may not execute action with small prob.

# Example: Robot Localization

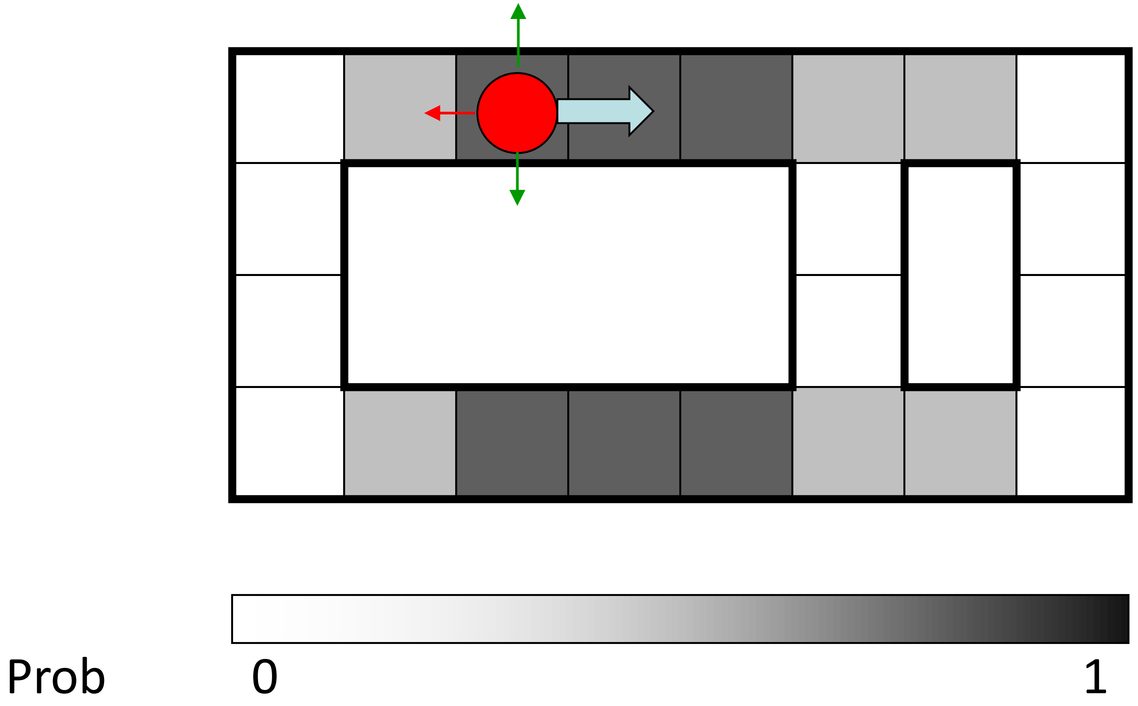


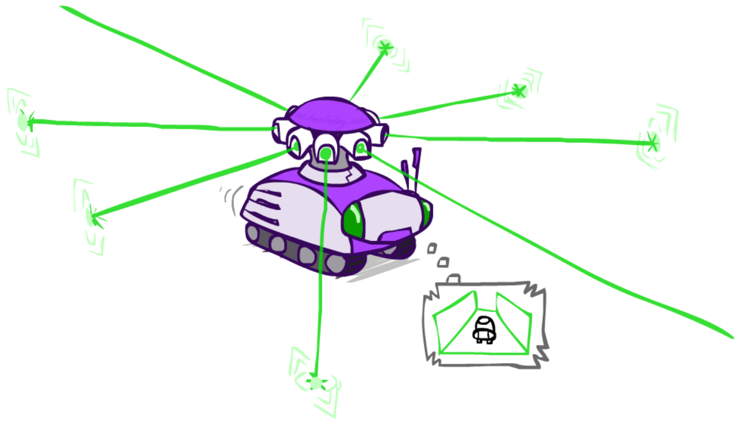



t=1

Lighter grey: was possible to get the reading, but less likely b/c required 1 mistake

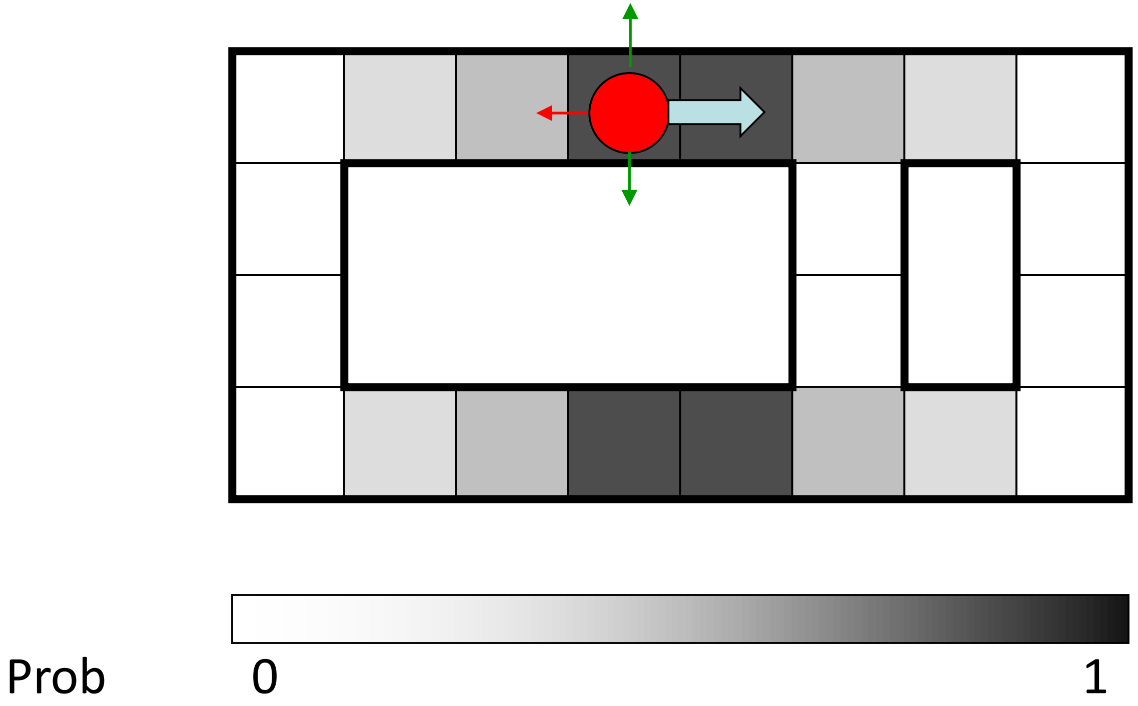
# Example: Robot Localization

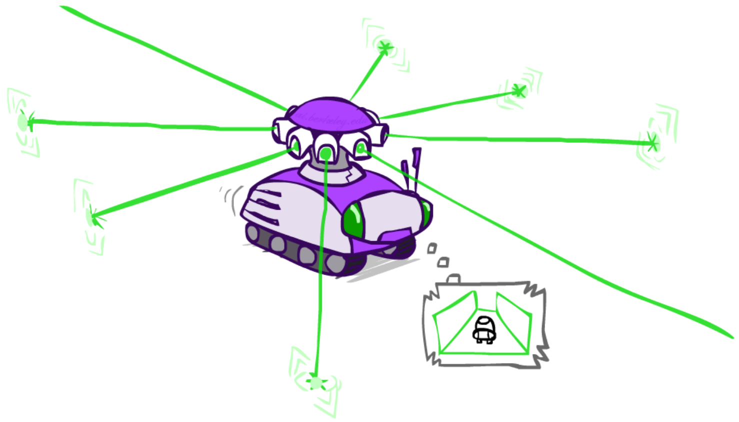


t=2

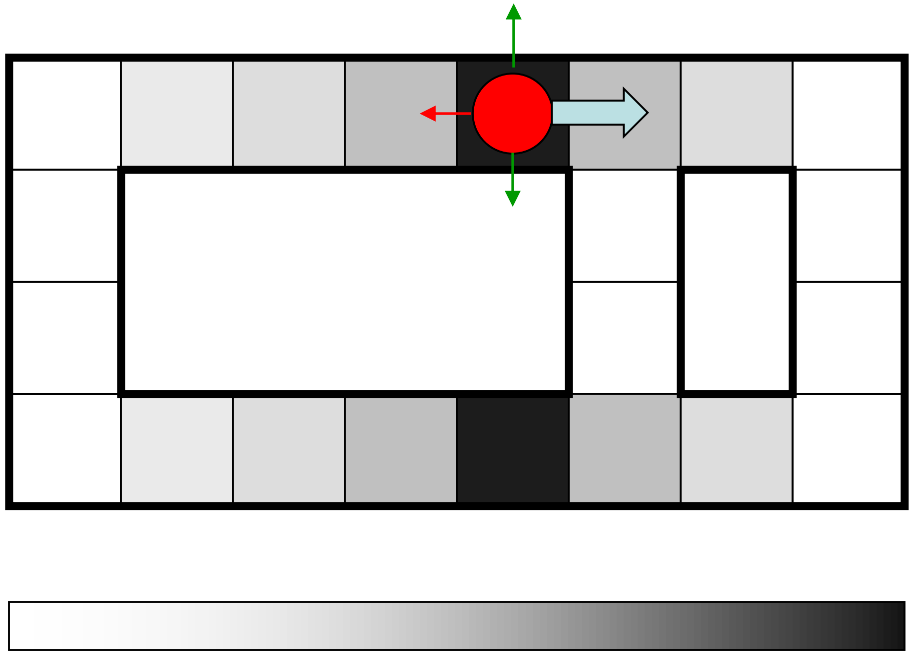
# Example: Robot Localization



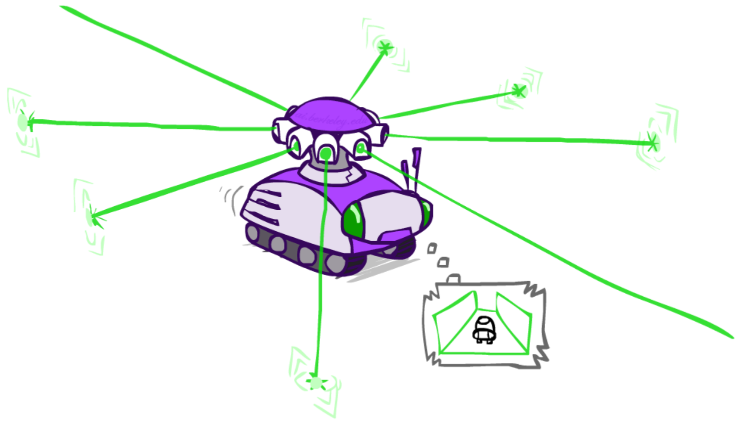
t=3

# Example: Robot Localization

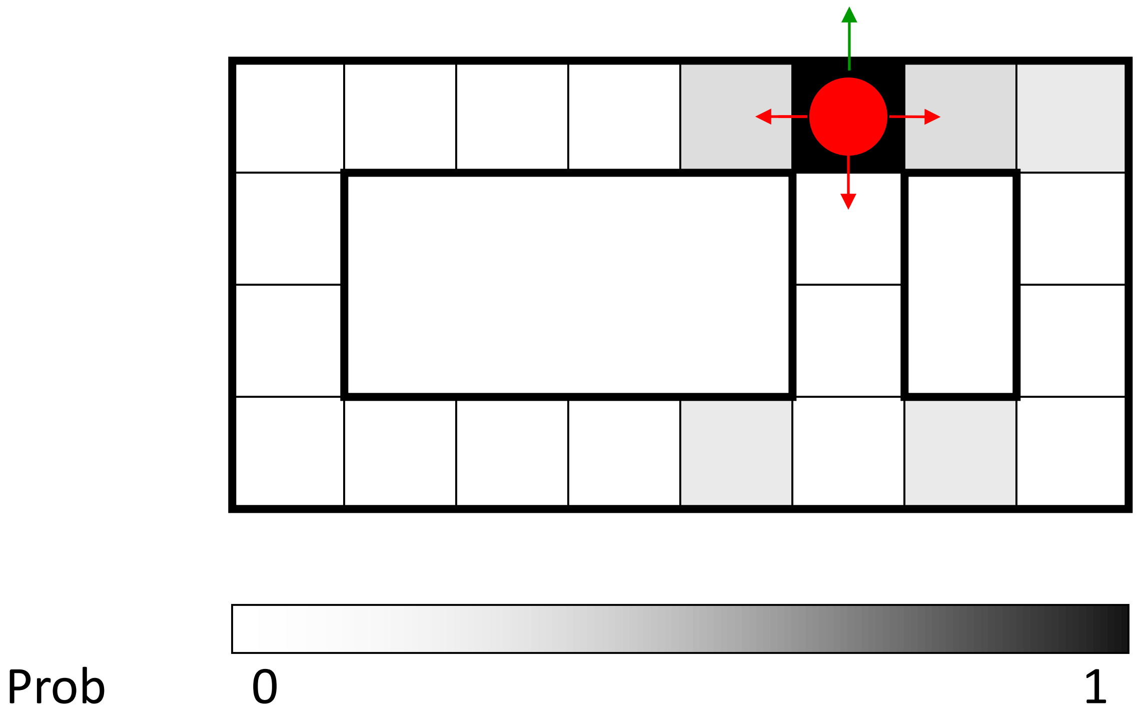


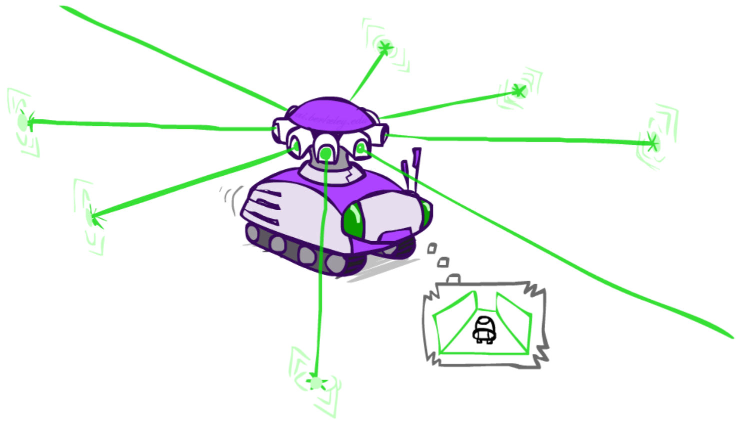
Prob 0 1

t=4



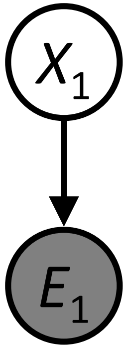
# Example: Robot Localization

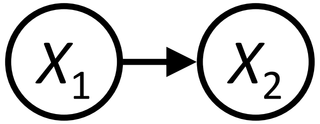


t=5

# Inference: Base Cases




$$P(X_1|e_1)$$

$$P(x_1|e_1) = P(x_1, e_1)/P(e_1)$$
$$\propto_{X_1} P(x_1, e_1)$$
$$= P(x_1)P(e_1|x_1)$$

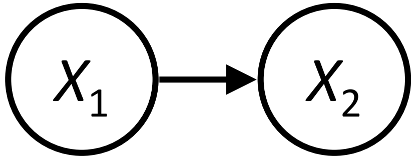
$$P(X_2)$$

$$P(x_2) = \sum_{x_1} P(x_1, x_2)$$
$$= \sum_{x_1} P(x_1)P(x_2|x_1)$$

# Passage of Time

- Assume we have current belief P(X | evidence to date)

$$B(X_t) = P(X_t|e_{1:t})$$




- Then, after one time step passes:

$$P(X_{t+1}|e_{1:t}) = \sum_{x_t} P(X_{t+1}, x_t|e_{1:t})$$
$$= \sum_{x_t} P(X_{t+1}|x_t, e_{1:t})P(x_t|e_{1:t})$$
$$= \sum_{x_t} P(X_{t+1}|x_t)P(x_t|e_{1:t})$$

- Or compactly:

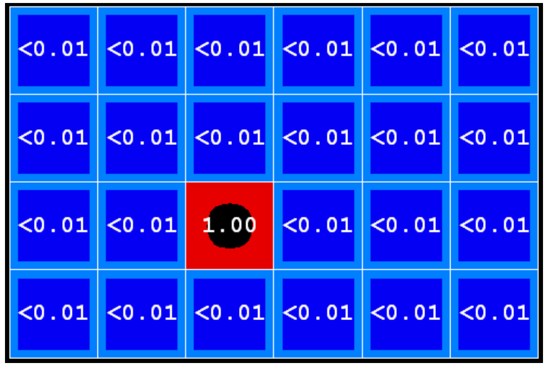
$$B'(X_{t+1}) = \sum_{x_t} P(X'|x_t)B(x_t)$$

- Basic idea: beliefs get "pushed" through the transitions
  - With the "B" notation, we have to be careful about what time step t the belief is about, and what evidence it includes

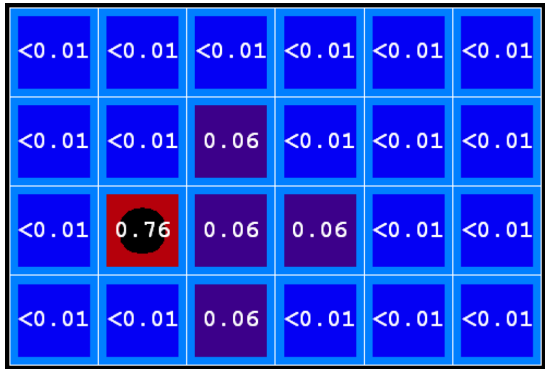
# Example: Passage of Time

- As time passes, uncertainty "accumulates"

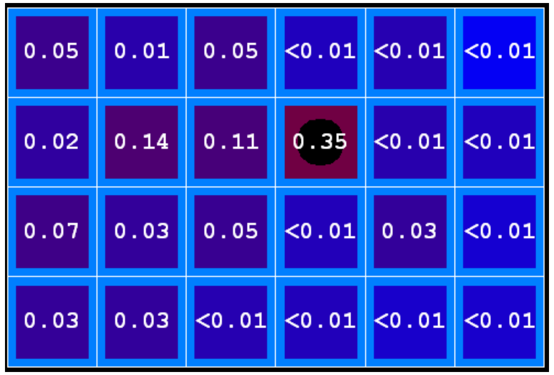
(Transition model: ghosts usually go clockwise)




T = 1




T = 2

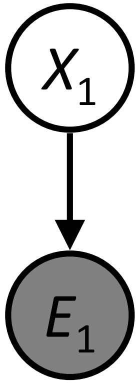



T = 5

$$B'(X') = \sum_x P(X'|x)B(x)$$

Transition model: ghosts usually go clockwise

# Observation




- Assume we have current belief P(X | previous evidence):

$$B'(X_{t+1}) = P(X_{t+1}|e_{1:t})$$

- Then, after evidence comes in:

$$\begin{aligned} P(X_{t+1}|e_{1:t+1}) &= P(X_{t+1}, e_{t+1}|e_{1:t})/P(e_{t+1}|e_{1:t}) \\ &\propto_{X_{t+1}} P(X_{t+1}, e_{t+1}|e_{1:t}) \\ &= P(e_{t+1}|e_{1:t}, X_{t+1})P(X_{t+1}|e_{1:t}) \\ &= P(e_{t+1}|X_{t+1})P(X_{t+1}|e_{1:t}) \end{aligned}$$

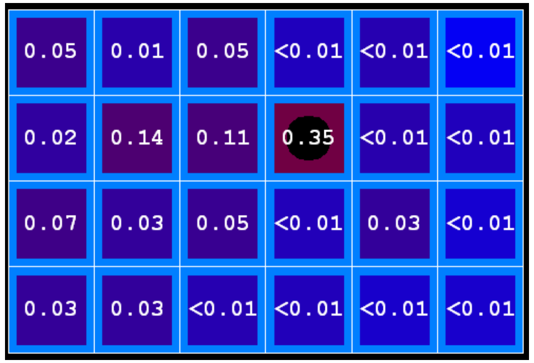
- Or, compactly:

$$B(X_{t+1}) \propto_{X_{t+1}} P(e_{t+1}|X_{t+1})B'(X_{t+1})$$

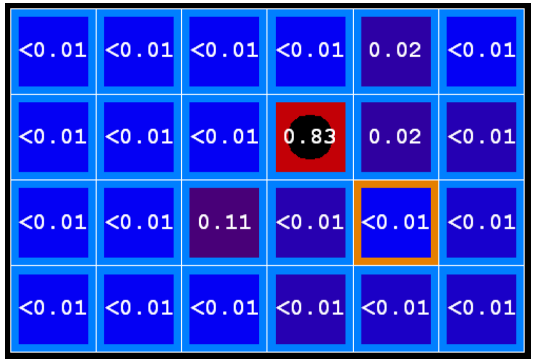
- Basic idea: beliefs “reweighted” by likelihood of evidence
- Unlike passage of time, we have to renormalize

# Example: Observation

- As we get observations, beliefs get reweighted, uncertainty "decreases"



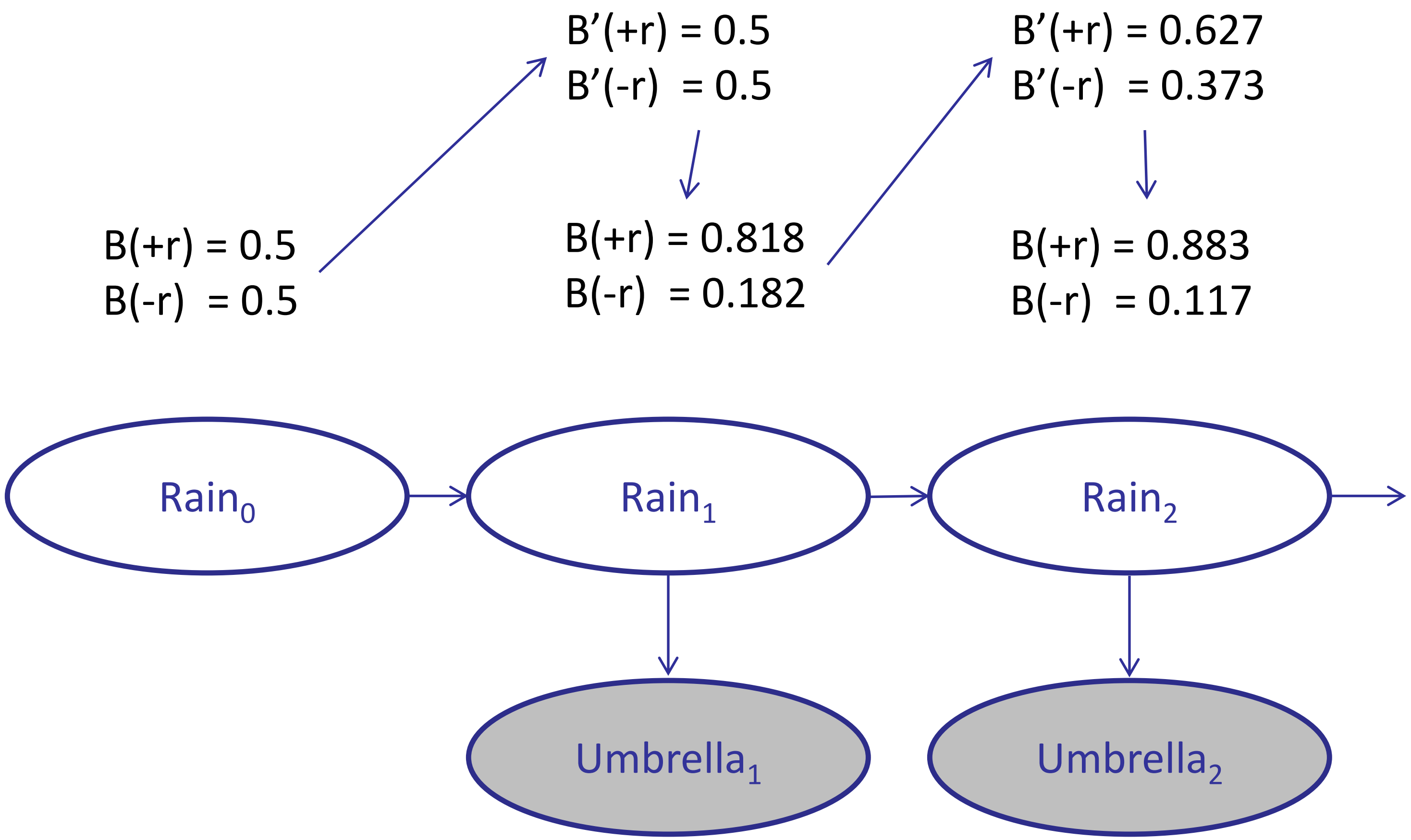

Before observation




After observation

$$B(X) \propto P(e|X)B'(X)$$

# Example: Weather HMM




| $R_t$ | $R_{t+1}$ | $P(R_{t+1} \mid R_t)$ |
|---|---|---|
| +r | +r | 0.7 |
| +r | -r | 0.3 |
| -r | +r | 0.3 |
| -r | -r | 0.7 |

| $R_t$ | $U_t$ | $P(U_t \mid R_t)$ |
|---|---|---|
| +r | +u | 0.9 |
| +r | -u | 0.1 |
| -r | +u | 0.2 |
| -r | -u | 0.8 |

# The Forward Algorithm

- We are given evidence at each time and want to know

$$B_t(X) = P(X_t|e_{1:t})$$

- We can derive the following updates

$$P(x_t|e_{1:t}) \propto_X P(x_t, e_{1:t})$$

$$= \sum_{x_{t-1}} P(x_{t-1}, x_t, e_{1:t})$$

$$= \sum_{x_{t-1}} P(x_{t-1}, e_{1:t-1}) P(x_t|x_{t-1}) P(e_t|x_t)$$

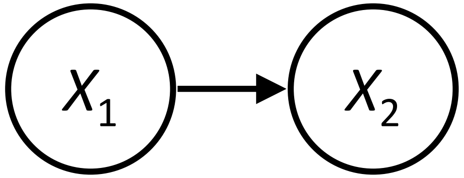
$$= P(e_t|x_t) \sum_{x_{t-1}} P(x_t|x_{t-1}) P(x_{t-1}, e_{1:t-1})$$

We can normalize as we go if we want to have P(x|e) at each time step, or just once at the end…

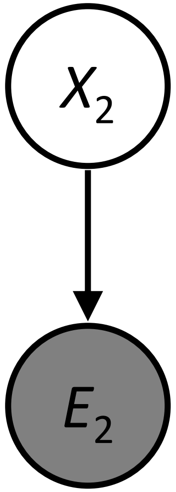
# Online Belief Updates

- Every time step, we start with current P(X | evidence)
- We update for time:

$$P(x_t|e_{1:t-1}) = \sum_{x_{t-1}} P(x_{t-1}|e_{1:t-1}) \cdot P(x_t|x_{t-1})$$



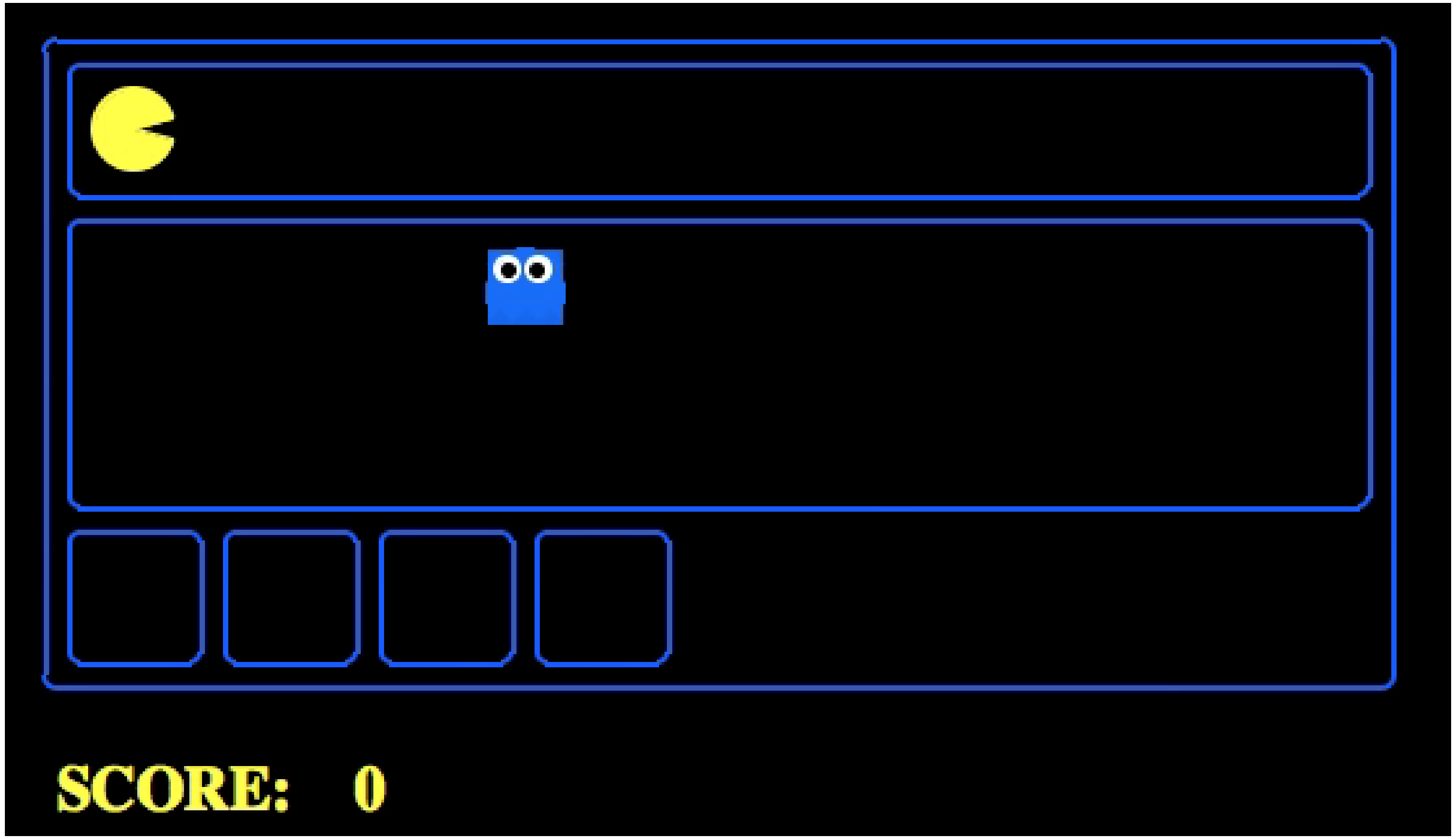

- We update for evidence:




$$P(x_t|e_{1:t}) \propto_X P(x_t|e_{1:t-1}) \cdot P(e_t|x_t)$$

- The forward algorithm does both at once (and doesn't normalize)

# Example Pac-man



# Summary: Filtering

- Filtering is the inference process of finding a distribution over $X_T$ given $e_1$ through $e_T$ : $P( X_T \mid e_{1:t} )$
- We first compute $P( X_1 \mid e_1 )$:

$$P(x_1|e_1) \propto P(x_1) \cdot P(e_1|x_1)$$

- For each t from 2 to T, we have $P( X_{t-1} \mid e_{1:t-1} )$
- **Elapse time:** compute $P( X_t \mid e_{1:t-1} )$

$$P(x_t|e_{1:t-1}) = \sum_{x_{t-1}} P(x_{t-1}|e_{1:t-1}) \cdot P(x_t|x_{t-1})$$

- **Observe:** compute $P(X_t \mid e_{1:t-1}, e_t) = P( X_t \mid e_{1:t} )$

$$P(x_t|e_{1:t}) \propto P(x_t|e_{1:t-1}) \cdot P(e_t|x_t)$$