CSE 473 Markov Decision Processes

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Many slides from Chris Bishop, Mausam, Dan Klein, Stuart Russell, Andrew Moore & Luke Zettlemoyer

Logistics

- PS 2 due ~~Tuesday~~ → Thursday 10/18
- PS 3 due Thursday 10/25

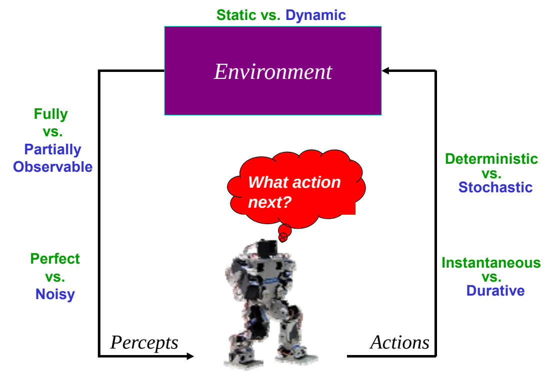
MDPs

Markov Decision Processes

- Planning Under Uncertainty
- Mathematical Framework
- Bellman Equations
- Value Iteration
- Real-Time Dynamic Programming
- Policy Iteration
- Reinforcement Learning

Andrey Markov
(1856-1922)

Planning Agent

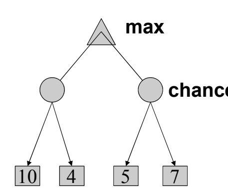


Objective of an MDP

- Find a policy π: V → D
- which optimizes
  - minimizes (discounted or undiscount) expected cost to reach a goal
  - maximizes (discounted or undiscount) expected reward
  - maximizes expected (reward-cost)
- given a ____ horizon
  - finite
  - infinite
  - indefinite

Review: Expectimax

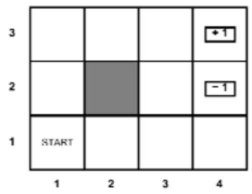
- What if we don't know what the result of an action will be? E.g.,
  - In solitaire, next card is unknown
  - In pacman, the ghosts act randomly
- Can do expectimax search
  - Max nodes as in minimax search
  - Chance nodes, like min nodes, except the outcome is uncertain - take average (expectation) of children
  - Calculate expected utilities

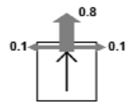


- Today, we formalize as an Markov Decision Process
  - Handle ***intermediate rewards*** & ***infinite plans***
  - More efficient processing

Grid World

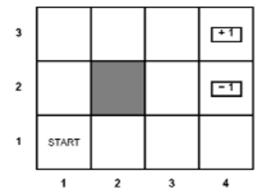
- Walls block the agent's path
- Agent's actions may go astray:
  - 80% of the time, North action takes the agent North (assuming no wall)
  - 10% - actually go West
  - 10% - actually go East
  - If there is a wall in the chosen direction, the agent stays put
- Small "living" reward each step
- Big rewards come at the end
- Goal: maximize sum of rewards





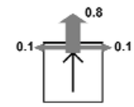
Markov Decision Processes

- An MDP is defined by:
  - A set of states s ∈ S
  - A set of actions a ∈ A
  - A transition function T(s,a,s')
    - Prob that a from s leads to s'
    - i.e., P(s' | s,a)
    - Also called "the model"
  - A reward function R(s, a, s')
    - Sometimes just R(s) or R(s')
  - A start state (or distribution)
  - Maybe a terminal state



- MDPs: non-deterministic search

Reinforcement learning: MDPs where we don't know the transition or reward functions



What is Markov about MDPs?

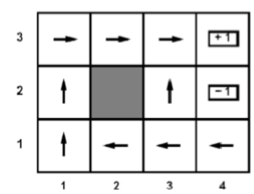
- Andrey Markov (1856-1922)
- "Markov" generally means that
  - conditioned on the present state,
  - the future is ***independent*** of the past
- For Markov decision processes, "Markov" means:

$$P(S_{t+1} = s' | S_t = s_t, A_t = a_t, S_{t-1} = s_{t-1}, A_{t-1}, \dots S_0 = s_0)$$
$$=$$
$$P(S_{t+1} = s' | S_t = s_t, A_t = a_t)$$

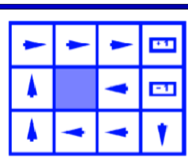
Solving MDPs

- In deterministic single-agent search problems, want an optimal plan, or sequence of actions, from start to a goal
- In an MDP, we want an optimal policy π*: S → A
  - A policy π gives an action for each state
  - An optimal policy maximizes expected utility if followed
  - Defines a reflex agent

Optimal policy when
R(s, a, s') = -0.03
for all non-terminals s

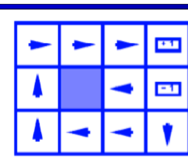


Example Optimal Policies

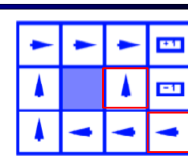


R(s) = -0.01

Example Optimal Policies



R(s) = -0.01



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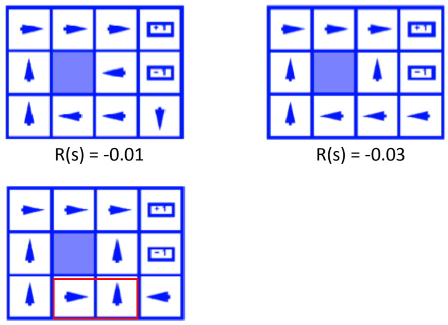
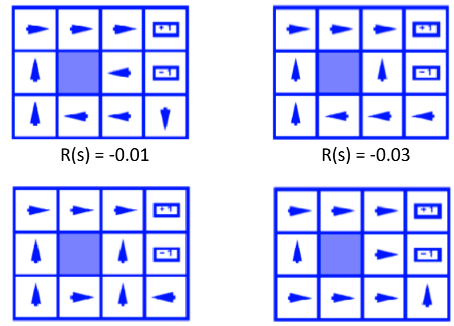
Example Optimal Policies



R(s) = -0.01 R(s) = -0.03 R(s) = -0.4

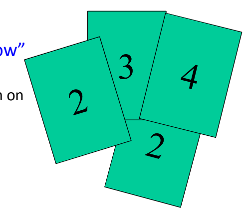
Example Optimal Policies



R(s) = -0.01 R(s) = -0.03 R(s) = -0.4 R(s) = -2.0

Example: High-Low

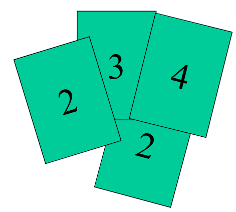
- Three card types: 2, 3, 4
  - Infinite deck, twice as many 2's
- Start with 3 showing
- After each card, you say "high" or "low"
- New card is flipped
  - If you're right, you win the points shown on the new card
  - Ties are no-ops (no reward)-0
  - If you're wrong, game ends



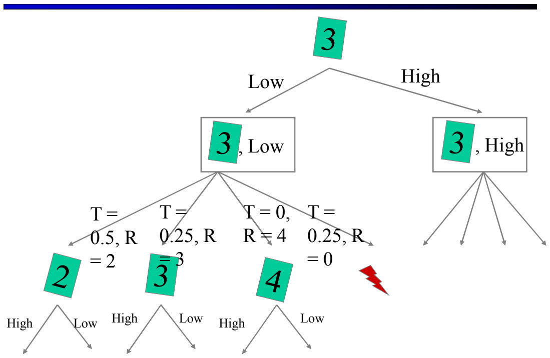
- Differences from expectimax problems:
  - #1: get rewards as you go
  - #2: you might play forever!

High-Low as an MDP

- States:
  - 2, 3, 4, done
- Actions:
  - High, Low
- Model: T(s, a, s'):
  - P(s'=4 | 4, Low) = 1/4
  - P(s'=3 | 4, Low) = 1/4
  - P(s'=2 | 4, Low) = 1/2
  - P(s'=done | 4, Low) = 0
  - P(s'=4 | 4, High) = 1/4
  - P(s'=3 | 4, High) = 0
  - P(s'=2 | 4, High) = 0
  - P(s'=done | 4, High) = 3/4
  - …
- Rewards: R(s, a, s'):
  - Number shown on s' if $s'<s \wedge$ a="high" …
  - 0 otherwise
- Start: 3

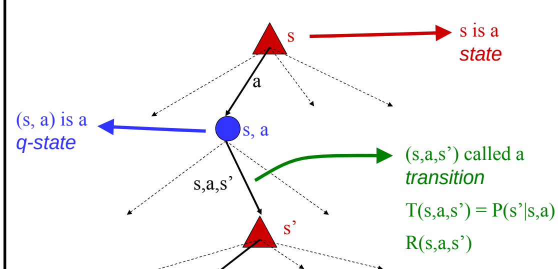


Search Tree: High-Low



MDP Search Trees

- Each MDP state gives an expectimax-like search tree



Utilities of Sequences

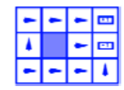
- In order to formalize optimality of a policy, need to understand utilities of sequences of rewards
- Typically consider stationary preferences:

$$[r, r_0, r_1, r_2, \ldots] \succ [r, r'_0, r'_1, r'_2, \ldots]$$
$$\Leftrightarrow$$
$$[r_0, r_1, r_2, \ldots] \succ [r'_0, r'_1, r'_2, \ldots]$$

- Theorem: only two ways to define stationary utilities
  - Additive utility:
    $$U([r_0, r_1, r_2, \ldots]) = r_0 + r_1 + r_2 + \cdots$$
  - Discounted utility:
    $$U([r_0, r_1, r_2, \ldots]) = r_0 + \gamma r_1 + \gamma^2 r_2 \cdots$$

Infinite Utilities?!

- Problem: infinite state sequences have infinite rewards
- Solutions:



  - Finite horizon:
    - Terminate episodes after a fixed T steps (e.g. life)
    - Gives nonstationary policies (π depends on time left)
  - Absorbing state: guarantee that for every policy, a terminal state will eventually be reached (like "done" for High-Low)
  - Discounting: for $0 < \gamma < 1$

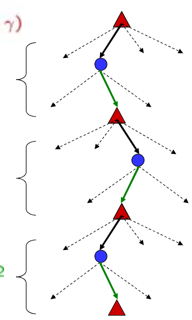
$$U([r_0, \ldots r_\infty]) = \sum_{t=0}^{\infty} \gamma^t r_t \leq R_{max}/(1-\gamma)$$

  - Smaller γ means smaller "horizon" – shorter term focus

Discounting

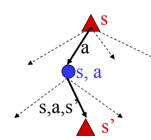
$$U([r_0, \ldots r_\infty]) = \sum_{t=0}^{\infty} \gamma^t r_t \leq R_{max}/(1-\gamma)$$

- Typically discount rewards by $\gamma < 1$ each time step
  - Sooner rewards have higher utility than later rewards
  - Also helps the algorithms converge



Recap: Defining MDPs

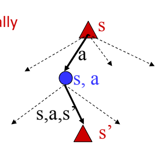
- Markov decision processes:
  - States S
  - Start state s_0
  - Actions A
  - Transitions P(s'|s, a)
    aka T(s,a,s')
  - Rewards R(s,a,s') (and discount γ)



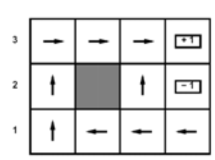
- MDP quantities so far:
  - Policy, π = Function that chooses an action for each state
  - Utility (aka "return") = sum of discounted rewards

Optimal Utilities

- Define the value of a state s:
  $V^*(s)$ = expected utility starting in s and acting optimally
- Define the value of a q-state (s,a):
  $Q^*(s,a)$ = expected utility starting in s, taking action a and thereafter acting optimally
- Define the optimal policy:
  $\pi^*(s)$ = optimal action from state s

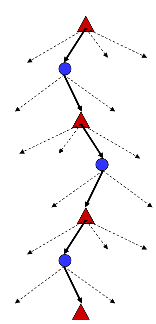


	1	2	3	4
3	0.812	0.868	0.912	+1
2	0.762		0.660	−1
1	0.705	0.655	0.611	0.388



Why Not Search Trees?

- Why not solve with expectimax?
- Problems:
  - This tree is usually infinite (why?)
  - Same states appear over and over (why?)
  - We would search once per state (why?)
- Idea: Value iteration
  - Compute optimal values for all states all at once using successive approximations
  - Will be a bottom-up dynamic program similar in cost to memoization
  - Do all planning offline, no replanning needed!



The Bellman Equations

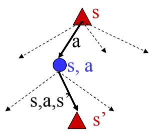
(1920-1984)

- Definition of "optimal utility" leads to a simple one-step look-ahead relationship between optimal utility values:

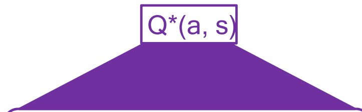
$$V^*(s) = \max_a Q^*(s,a)$$

$$Q^*(s,a) = \sum_{s'} T(s,a,s')\left[R(s,a,s') + \gamma V^*(s')\right]$$

$$V^*(s) = \max_a \sum_{s'} T(s,a,s')\left[R(s,a,s') + \gamma V^*(s')\right]$$



Bellman Equations for MDPs



$$V^*(s) = \max_{a \in Ap(s)} \sum_{s' \in \mathcal{S}} \mathcal{P}r(s'|s,a)\left[\mathcal{R}(s,a,s') + \gamma V^*(s')\right]$$

$$V^*(s) = \max_a Q^*(s,a)$$

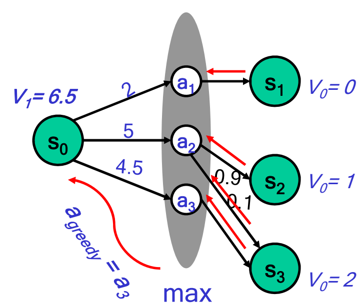
Bellman Backup (MDP)

- Given an estimate of V* function (say V_n)
- Backup V_n function at state s
  - calculate a new estimate (V_{n+1}) :

$$Q_{n+1}(s,a) = \sum_{s' \in \mathcal{S}} Pr(s'|s,a)\left[\mathbb{U}(s,a,s') + \gamma V_n(s')\right]$$
$$V_{n+1}(s) = \max_{a \in Ap(s)} \left[Q_{n+1}(s,a)\right]$$

- $Q_{n+1}(s,a)$: value/cost of the strategy:
  - execute action a in s, execute π_n subsequently
  - $\pi_n = \text{argmax}_{a \in Ap(s)} Q_n(s,a)$

Bellman Backup



$Q_1(s,a_1) = 2 + \gamma\, 0$
~ 2

$Q_1(s,a_2) = 5 + \gamma\, 0.9$~
$+ \gamma\, 0.1$~
~ 6.1

$Q_1(s,a_3) = 4.5 + \gamma\, 2$
~ 6.5

Value iteration [Bellman'57]

- assign an arbitrary assignment of V_0 to each state.
- repeat
  - for all states s
    - compute $V_{n+1}(s)$ by Bellman backup at s. ← Iteration n+1
- until $\max_s |V_{n+1}(s) - V_n(s)| < \varepsilon$

Residual(s)

ε-convergence

- Theorem: will converge to unique optimal values
  - Basic idea: approximations get refined towards optimal values
  - *Policy may converge* long before values do

Value Iteration

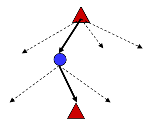
- Idea:
  - Start with $V_0^*(s) = 0$, which we know is right (why?)
  - Given V_i^*, calculate the values for all states for depth i+1:

$$V_{i+1}(s) \leftarrow \max_a \sum_{s'} T(s,a,s')\left[R(s,a,s') + \gamma V_i(s')\right]$$

  - This is called a value update or Bellman update
  - Repeat until convergence
- Theorem: will converge to unique optimal values
  - Basic idea: approximations get refined towards optimal values
  - *Policy may converge* long before values do

Value Estimates

- Calculate estimates $V_k^*(s)$
  - The optimal value considering only next k time steps (k rewards)
  - As $k \rightarrow \infty$, V_k approaches the optimal value
- Why:
  - If discounting, distant rewards become negligible
  - If terminal states reachable from everywhere, fraction of episodes not ending becomes negligible
  - Otherwise, can get infinite expected utility and then this approach actually won't work



Example: Bellman Updates

Example: γ=0.9, living reward=0, noise=0.2

V_0

	1	2	3	4
3	0	0	0 →	+1
2	0		0	−1
1	0	0	0	0

V_1

	1	2	3	4
3	?	?	?	+1
2	?		?	−1
1	?	?	?	?

$$V_{i+1}(s) = \max_a \sum_{s'} T(s,a,s')\left[R(s,a,s') + \gamma V_i(s')\right] = \max_a Q_{i+1}(s,a)$$

$$Q_1(\langle 3,3\rangle, \text{right}) = \sum_{s'} T(\langle 3,3\rangle, \text{right}, s')\left[R(\langle 3,3\rangle, \text{right}, s') + \gamma V_i(s')\right]$$

$$= 0.8 * [0.0 + 0.9 * 1.0] + 0.1 * [0.0 + 0.9 * 0.0] + 0.1 * [0.0 + 0.9 * 0.0]$$

Example: Value Iteration

V_1

	1	2	3	4
3	0	0	0.72	+1
2	0		0	−1
1	0	0	0	0

V_2

	1	2	3	4
3	0	0.52	0.78	+1
2	0		0.43	−1
1	0	0	0	0

- Information propagates outward from terminal states and eventually all states have correct value estimates

Example: Value Iteration

QuickTime™ and a GIF decompressor are needed to see this picture.

Practice: Computing Actions

- Which action should we chose from state s:
  - Given optimal values Q?

$$\arg\max_a Q^*(s,a)$$

  - Given optimal values V?

$$\arg\max_a \sum_{s'} T(s,a,s')[R(s,a,s') + \gamma V^*(s')]$$

  - Lesson: actions are easier to select from Q's!

Comments

- Decision-theoretic Algorithm
- Dynamic Programming
- Fixed Point Computation
- Probabilistic version of Bellman-Ford Algorithm
  - for shortest path computation
  - MDP_1 : Stochastic Shortest Path Problem
- Time Complexity
  - one iteration: $O(|V|^2|D|)$
  - number of iterations: poly(|V|, |D|, 1/(1-γ))
- Space Complexity: $O(|V|)$
- Factored MDPs = Planning under uncertainty
  - exponential space, exponential time

Convergence Properties

- $V_n \rightarrow V^*$ in the limit as $n \rightarrow 4$
- ε-convergence: V_n function is within ε of V^*
- Optimality: current policy is within $2\varepsilon\gamma/(1-\gamma)$ of optimal

- Monotonicity
  - $V_0 \leq_p V^* \Rightarrow V_n \leq_p V^*$ (V_n monotonic from below)
  - $V_0 \geq_p V^* \Rightarrow V_n \geq_p V^*$ (V_n monotonic from above)
  - otherwise V_n non-monotonic

Convergence

- Define the max-norm: $||U|| = \max_s |U(s)|$
- Theorem: For any two approximations U^t and V^t
  $$||U^{t+1} - V^{t+1}|| \leq \gamma \, ||U^t - V^t||$$
  - I.e. any distinct approximations must get closer to each other, so, in particular, any approximation must get closer to the true V* (aka U) and value iteration converges to a unique, stable, optimal solution
- Theorem:
  $$||U^{t+1} - U^t|| < \epsilon, \Rightarrow ||U^{t+1} - U|| < 2\epsilon\gamma/(1-\gamma)$$
  - I.e. once the change in our approximation is small, it must also be close to correct

Value Iteration Complexity

- Problem size:
  - |A| actions and |S| states
- Each Iteration
  - Computation: O(|A|·|S|^2)
  - Space: O(|S|)
- Num of iterations
  - Can be exponential in the discount factor γ

MDPs

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Asynchronous Value Iteration

- States may be backed up in any order
  - Instead of systematically, iteration by iteration
- Theorem:
  - As long as every state is backed up infinitely often…
  - Asynchronous value iteration converges to optimal

Asynchonous Value Iteration
Prioritized Sweeping

- Why backup a state if values of successors same?
- Prefer backing a state
  - whose successors had most change

- Priority Queue of (state, expected change in value)
- Backup in the order of priority
- After backing a state update priority queue
  - for all predecessors

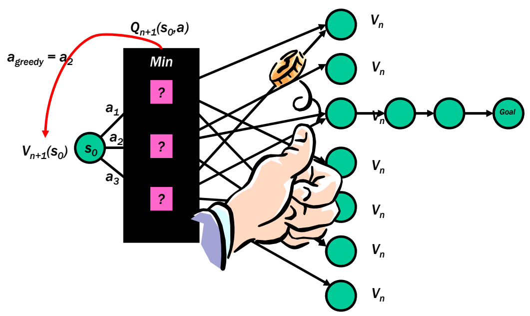
Asynchonous Value Iteration
Real Time Dynamic Programming
[Barto, Bradtke, Singh'95]

- Trial: simulate greedy policy starting from start state; perform Bellman backup on visited states
- RTDP:
  - Repeat Trials until value function converges

Why?

- Why is next slide saying min

RTDP Trial

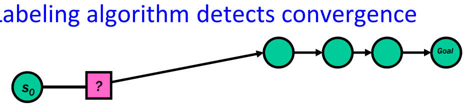


Comments

- Properties
  - if all states are visited infinitely often then $V_n \rightarrow V^*$
- Advantages
  - Anytime: more probable states explored quickly
- Disadvantages
  - complete convergence can be slow!

Labeled RTDP
[Bonet&Geffner ICAPS03]

- Stochastic Shortest Path Problems
  - Policy w/ min expected cost to reach goal
- Initialize $v^0(s)$ with admissible heuristic
  - Underestimates remaining cost
- Theorem:
  - if residual of $V^k(s) < \varepsilon$ and $V^k(s') < \varepsilon$ for all succ(s), s', in greedy graph
  - Then V^k is ε-consistent and will remain so
- Labeling algorithm detects convergence



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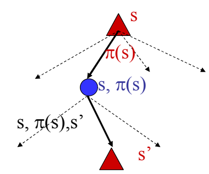
Changing the Search Space

- Value Iteration
  - Search in value space
  - Compute the resulting policy
- Policy Iteration
  - Search in policy space
  - Compute the resulting value

Utilities for Fixed Policies

- Another basic operation: compute the utility of a state s under a fix (general non-optimal) policy
- Define the utility of a state s, under a fixed policy π:
  - $V^{\pi}(s)$ = expected total discounted rewards (return) starting in s and following π
- Recursive relation (one-step look-ahead / Bellman equation):



$$V^{\pi}(s) = \sum_{s'} T(s, \pi(s), s')[R(s, \pi(s), s') + \gamma V^{\pi}(s')]$$

Policy Evaluation

- How do we calculate the V's for a fixed policy?
- Idea one: modify Bellman updates

$$V_0^{\pi}(s) = 0$$

$$V_{i+1}^{\pi}(s) \leftarrow \sum_{s'} T(s, \pi(s), s')[R(s, \pi(s), s') + \gamma V_i^{\pi}(s')]$$

- Idea two: it's just a linear system, solve with Matlab (or whatever)

Policy Iteration

- Problem with value iteration:
  - Considering all actions each iteration is slow: takes |A| times longer than policy evaluation
  - But policy doesn't change each iteration, time wasted
- Alternative to value iteration:
  - Step 1: Policy evaluation: calculate utilities for a fixed policy (not optimal utilities!) until convergence (fast)
  - Step 2: Policy improvement: update policy using one-step lookahead with resulting converged (but not optimal!) utilities (slow but infrequent)
  - Repeat steps until policy converges

Policy Iteration

- Policy evaluation: with fixed current policy π, find values with simplified Bellman updates:
  - Iterate until values converge

$$V_{i+1}^{\pi_k}(s) \leftarrow \sum_{s'} T(s, \pi_k(s), s') \left[R(s, \pi_k(s), s') + \gamma V_i^{\pi_k}(s')\right]$$

- Policy improvement: with fixed utilities, find the best action according to one-step look-ahead

$$\pi_{k+1}(s) = \arg\max_a \sum_{s'} T(s, a, s') \left[R(s, a, s') + \gamma V^{\pi_k}(s')\right]$$

Policy iteration [Howard'60]

- assign an arbitrary assignment of π_0 to each state.
- repeat
  - Policy Evaluation: compute V_{n+1}: the evaluation of π_n — costly: $O(n^3)$
  - Policy Improvement: for all states s
    - compute $\pi_{n+1}(s)$: $\text{argmax}_{a \in Ap(s)} Q_{n+1}(s,a)$
- until $\pi_{n+1} = \pi_n$

Modified Policy Iteration → approximate by value iteration using fixed policy

Advantage

- searching in a finite (policy) space as opposed to uncountably infinite (value) space ⇒ convergence faster.
- all other properties follow!

Modified Policy iteration

- assign an arbitrary assignment of π_0 to each state.
- repeat
  - Policy Evaluation: compute V_{n+1} the *approx.* evaluation of π_n
  - Policy Improvement: for all states s
    - compute $\pi_{n+1}(s)$: $\text{argmax}_{a \in Ap(s)} Q_{n+1}(s,a)$
- until $\pi_{n+1} = \pi_n$

Advantage

- probably the most competitive synchronous dynamic programming algorithm.

Policy Iteration Complexity

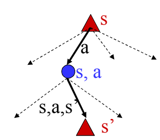
- Problem size:
  - |A| actions and |S| states
- Each Iteration
  - Computation: $O(|S|^3 + |A| \cdot |S|^2)$
  - Space: $O(|S|)$
- Num of iterations
  - Unknown, but can be faster in practice
  - Convergence is guaranteed

Comparison

- In value iteration:
  - Every pass (or "backup") updates both utilities (explicitly, based on current utilities) and policy (possibly implicitly, based on current policy)
- In policy iteration:
  - Several passes to update utilities with frozen policy
  - Occasional passes to update policies
- Hybrid approaches (asynchronous policy iteration):
  - Any sequences of partial updates to either policy entries or utilities will converge if every state is visited infinitely often

Recap: MDPs

- Markov decision processes:
  - States S
  - Actions A
  - Transitions P(s'|s,a) (or T(s,a,s'))
  - Rewards R(s,a,s') (and discount ©)
  - Start state s_0



- Quantities:
  - Returns = sum of discounted rewards
  - Values = expected future returns from a state (optimal, or for a fixed policy)
  - Q-Values = expected future returns from a q-state (optimal, or for a fixed policy)