Lecture 3: A* Search
10/7/2010

Luke Zettlemoyer
Based on slides from Dan Klein
Multiple slides from Stuart Russell or Andrew Moore
Announcements

- **Projects:**
  - Project 1 (Search) is out, due next Friday Oct 15th
  - You don’t need to submit answers the project’s discussion questions
  - Can talk to each other, but must write own solutions
Today

- A* Search
- Heuristic Design
- Graph search
Recap: Search

- **Search problem:**
  - States (configurations of the world)
  - Successor function: a function from states to lists of (state, action, cost) triples; drawn as a graph
  - Start state and goal test

- **Search tree:**
  - Nodes: represent plans for reaching states
  - Plans have costs (sum of action costs)

- **Search Algorithm:**
  - Systematically builds a search tree
  - Chooses an ordering of the fringe (unexplored nodes)
Example: Pancake Problem

Action: Flip over the top $n$ pancakes

Cost: Number of pancakes flipped
Example: Pancake Problem

BOUNDS FOR SORTING BY PREFIX REVERSAL

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Received 18 January 1978
Revised 28 August 1978

For a permutation $\sigma$ of the integers from 1 to $n$, let $f(\sigma)$ be the smallest number of prefix reversals that will transform $\sigma$ to the identity permutation, and let $f(n)$ be the largest such $f(\sigma)$ for all $\sigma$ in (the symmetric group) $S_n$. We show that $f(n) \leq (5n + 5)/3$, and that $f(n) \geq 17n/16$ for $n$ a multiple of 16. If, furthermore, each integer is required to participate in an even number of reversed prefixes, the corresponding function $g(n)$ is shown to obey $3n/2 - 1 \leq g(n) \leq 2n + 3$. 
Example: Pancake Problem

State space graph with costs as weights
function TREE-SEARCH(problem, strategy) returns a solution, or failure
initialize the search tree using the initial state of problem
loop do
  if there are no candidates for expansion then return failure
  choose a leaf node for expansion according to strategy
  if the node contains a goal state then return the corresponding solution
  else expand the node and add the resulting nodes to the search tree
end

Action: flip top two
Cost: 2

Path to reach goal:
Flip four, flip three
Total cost: 7
Uniform Cost Search

- Strategy: expand lowest path cost
- The good: UCS is complete and optimal!
- The bad:
  - Explores options in every “direction”
  - No information about goal location

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Example: Heuristic Function

Heuristic: the largest pancake that is still out of place

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Best First (Greedy)

- Strategy: expand a node that you think is closest to a goal state
  - Heuristic: estimate of distance to nearest goal for each state

- A common case:
  - Best-first takes you straight to the (wrong) goal

- Worst-case: like a badly-guided DFS
Example: Heuristic Function

$h(x)$
Combining UCS and Greedy

- **Uniform-cost** orders by path cost, or *backward cost* $g(n)$
- **Best-first** orders by goal proximity, or *forward cost* $h(n)$
- **A* Search** orders by the sum: $f(n) = g(n) + h(n)$

Example: Teg Grenager
When should A* terminate?

- Should we stop when we enqueue a goal?
- No: only stop when we dequeue a goal
Is A* Optimal?

What went wrong?
- Actual bad goal cost < estimated good goal cost
- We need estimates to be less than actual costs!
Admissible Heuristics

- A heuristic \( h \) is *admissible* (optimistic) if:

\[
h(n) \leq h^*(n)
\]

where \( h^*(n) \) is the true cost to a nearest goal

- Examples:

- Coming up with admissible heuristics is most of what’s involved in using A* in practice.
Optimality of A*: Blocking

Notation:
- $g(n) = \text{cost to node } n$
- $h(n) = \text{estimated cost from } n \text{ to the nearest goal (heuristic)}$
- $f(n) = g(n) + h(n) = \text{estimated total cost via } n$
- $G^*: \text{a lowest cost goal node}$
- $G: \text{another goal node}$
Optimality of A*: Blocking

Proof:

- What could go wrong?
- We’d have to have to pop a suboptimal goal $G$ off the fringe before $G^*$

- This can’t happen:
  - For all nodes $n$ on the best path to $G^*$
    - $f(n) < f(G)$
  - So, $G^*$ will be popped before $G$

\[
\begin{align*}
  f(n) &= g(n) + h(n) \\
  g(n) + h(n) &\leq g(G^*) \\
  g(G^*) &< g(G) \\
  g(G) &= f(G) \\
  f(n) &< f(G)
\end{align*}
\]
Properties of A*

Uniform-Cost

A*
UCS vs A* Contours

- Uniform-cost expanded in all directions

- A* expands mainly toward the goal, but does hedge its bets to ensure optimality
Which Algorithm?

- Uniform cost search (UCS):
Which Algorithm?

- A*, Manhattan Heuristic:
Which Algorithm?

- Best First / Greedy, Manhattan Heuristic:
Creating Heuristics

8-puzzle:

- What are the states?
- How many states?
- What are the actions?
- What states can I reach from the start state?
- What should the costs be?
8 Puzzle I

- **Heuristic**: Number of tiles misplaced

- **$h(\text{start}) = 8$**

- **Is it admissible?**

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<table>
<thead>
<tr>
<th>Average nodes expanded when optimal path has length…</th>
<th>…4 steps</th>
<th>…8 steps</th>
<th>…12 steps</th>
</tr>
</thead>
<tbody>
<tr>
<td>UCS</td>
<td>112</td>
<td>6,300</td>
<td>$3.6 \times 10^6$</td>
</tr>
<tr>
<td>TILES</td>
<td>13</td>
<td>39</td>
<td>227</td>
</tr>
</tbody>
</table>

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8 Puzzle II

- What if we had an easier 8-puzzle where any tile could slide any direction at any time, ignoring other tiles?
- Total Manhattan distance
- \( h(\text{start}) = 3 + 1 + 2 + \ldots \)  
  \( = 18 \)
- Admissible?

<table>
<thead>
<tr>
<th>TILES</th>
<th>13</th>
<th>39</th>
<th>227</th>
</tr>
</thead>
<tbody>
<tr>
<td>MANHATTAN</td>
<td>12</td>
<td>25</td>
<td>73</td>
</tr>
</tbody>
</table>

Average nodes expanded when optimal path has length…

- …4 steps
- …8 steps
- …12 steps
8 Puzzle III

- How about using the *actual cost* as a heuristic?
  - Would it be admissible?
  - Would we save on nodes expanded?
  - What’s wrong with it?

- With A*: a trade-off between quality of estimate and work per node!
Creating Admissible Heuristics

- Most of the work in solving hard search problems optimally is in coming up with admissible heuristics.

- Often, admissible heuristics are solutions to relaxed problems, where new actions are available.

- Inadmissible heuristics are often useful too (why?)

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Trivial Heuristics, Dominance

- Dominance: \( h_a \geq h_c \) if

\[
\forall n : h_a(n) \geq h_c(n)
\]

- Heuristics form a semi-lattice:
  - Max of admissible heuristics is admissible

\[
h(n) = \max(h_a(n), h_b(n))
\]

- Trivial heuristics
  - Bottom of lattice is the zero heuristic (what does this give us?)
  - Top of lattice is the exact heuristic

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A* Applications

- Pathing / routing problems
- Resource planning problems
- Robot motion planning
- Language analysis
- Machine translation
- Speech recognition
- ...

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Tree Search: Extra Work!

- Failure to detect repeated states can cause exponentially more work. Why?
Graph Search

- In BFS, for example, we shouldn’t bother expanding some nodes (which, and why?)
Graph Search

- Idea: never expand a state twice

- How to implement:
  - Tree search + list of expanded states (closed list)
  - Expand the search tree node-by-node, but…
  - Before expanding a node, check to make sure its state is new

- Python trick: store the closed list as a set, not a list

- Can graph search wreck completeness? Why/why not?

- How about optimality?
Optimality of A* Graph Search

Proof:

- New possible problem: nodes on path to $G^*$ that would have been in queue aren’t, because some worse $n'$ for the same state as some $n$ was dequeued and expanded first (disaster!)

- Let $p$ be the ancestor which was on the queue when $n'$ was expanded

- Assume $f(p) < f(n)$

- $f(n) < f(n')$ because $n'$ is suboptimal

- $p$ would have been expanded before $n'$

- Also, $n$ would have been expanded before $n'$
Consistency

- Wait, how do we know parents have better f-values than their successors?
- Could we pop some node $n$, and find its child $n'$ to have lower f value?

What can we require to prevent these inversions?
- Consistency: $c(n, a, n') \geq h(n) - h(n')$
- Real cost must always exceed reduction in heuristic
Optimality

- **Tree search:**
  - $A^*$ optimal if heuristic is admissible (and non-negative)
  - UCS is a special case ($h = 0$)

- **Graph search:**
  - $A^*$ optimal if heuristic is consistent
  - UCS optimal ($h = 0$ is consistent)

- Consistency implies admissibility

- In general, natural admissible heuristics tend to be consistent

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Summary: A*

- A* uses both backward costs and (estimates of) forward costs
- A* is optimal with admissible heuristics
- Heuristic design is key: often use relaxed problems
To Do:

- Keep up with the readings
- Get started on PS1
  - it is long; start soon
  - due in about a week