UNIVERSITY of WASHINGTON

Spatial Algebra & Kinematics

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CSE 571

Things to cover today:

- Monogram notion
- Spatial algebra
- Rigid-body motions and rotation matrices
- Time Derivatives of Rotations
- Homogeneous transformation matrices
- Forward kinematics

Time Derivatives of Rotations

Rotation matrix

$$\mathbf{R}(\mathbf{t})$$

Orthogonality

$$R^T(t)R(t) = I$$

$$R(t)R^T(t) = I$$

$\xrightarrow{\frac{d}{dt}(\cdot)}$

$$\dot{R}^T R + R^T \dot{R} = 0$$

$$R\dot{R}^T + \dot{R}R^T = 0$$

Use product rules

So $\dot{R}R^T$ = -$\dot{R}R^T$ -> For special R = I.

$$\dot{R} = -\dot{R}^T$$

Is skew symmetric.

Skew symmetric Matrix

Suppose we have a 3D vector of angular velocities:

$$\omega = [\,\omega_x \;\; \omega_y \;\; \omega_z\,]^T \in \mathbb{R}^3$$

Then the skew-symmetric matrix from this vector is:

$$S(\omega) = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix} \in \mathbb{R}^{3\times 3}$$

$$\mathrm{S}(\boldsymbol{\omega})^{\mathrm{T}} = -\mathrm{S}(\boldsymbol{\omega})$$
$$\mathrm{S}(\boldsymbol{\omega}) = -\mathrm{S}(\boldsymbol{\omega})^{\mathrm{T}}$$

$$S(\omega)^T = \begin{bmatrix} 0 & \omega_z & -\omega_y \\ -\omega_z & 0 & \omega_x \\ \omega_y & -\omega_x & 0 \end{bmatrix}$$

$$-S(\omega) = \begin{bmatrix} 0 & \omega_z & -\omega_y \\ -\omega_z & 0 & \omega_x \\ \omega_y & -\omega_x & 0 \end{bmatrix}$$

Skew symmetric Matrix and Cross Products

Recall the calculation of instantaneous linear velocity of a rotating body:

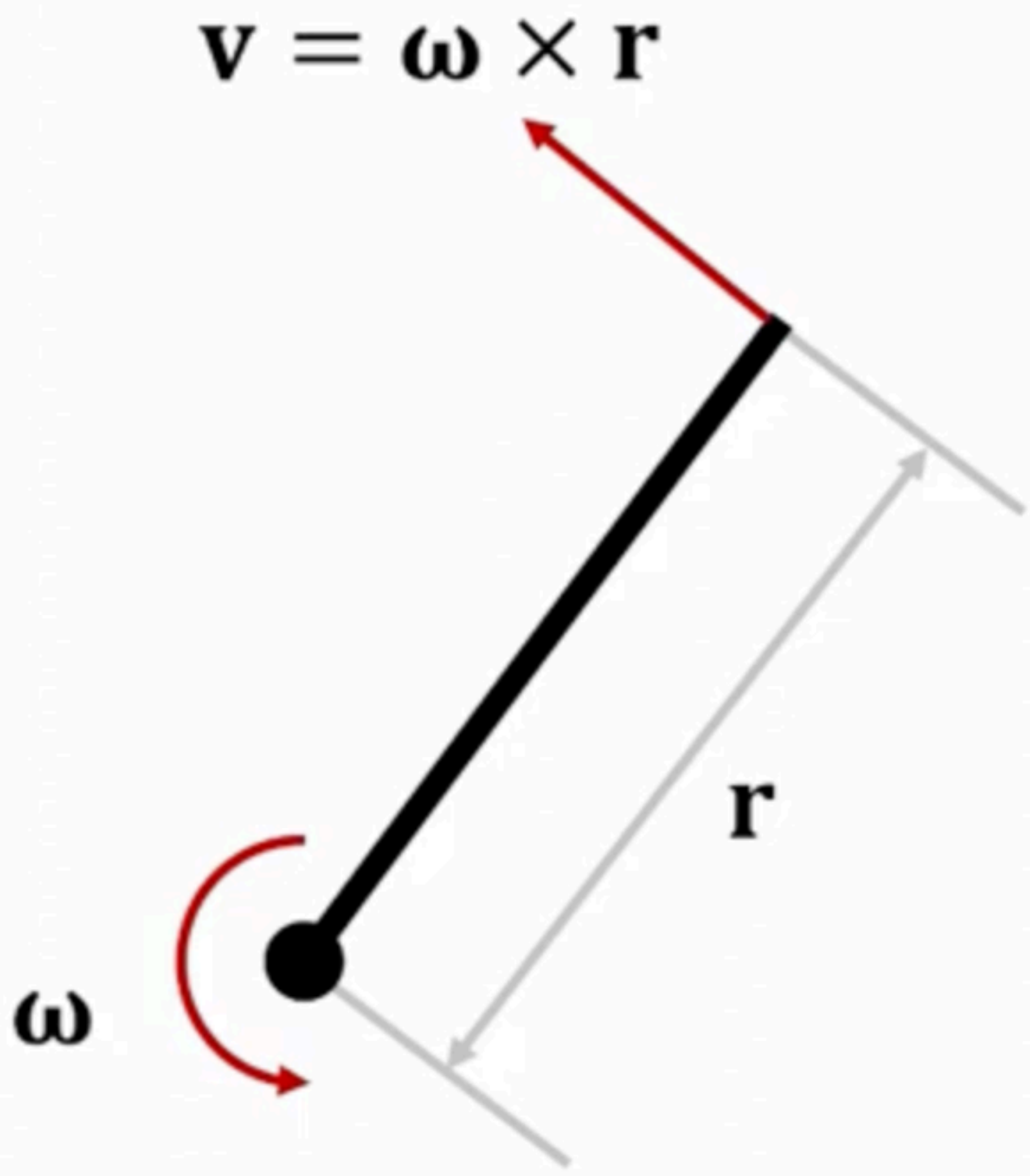
$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r} \text{ (cross product)}$$

This can be re–written as:

$$\mathbf{v} = S(\boldsymbol{\omega})\mathbf{r}$$

And by exploiting properties of the skew–symmetric matrix:

$$\mathbf{v} = -S(\mathbf{r})\boldsymbol{\omega}$$



Time Derivatives of Rotations

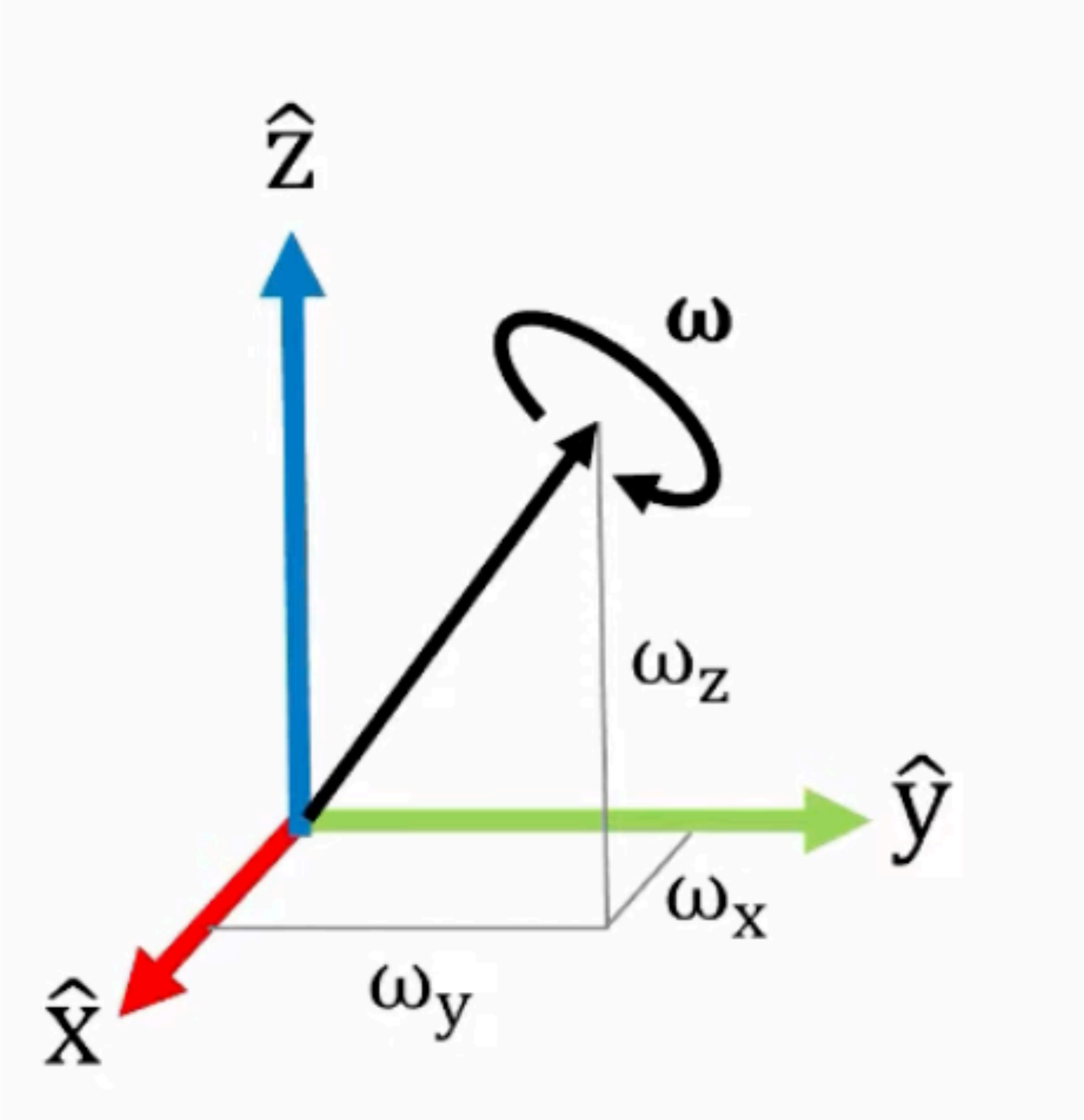
The time derivative of the rotation matrix is a skew-symmetric:

$$\dot{\mathbf{R}}\mathbf{R}^{\mathrm{T}} = -\mathbf{R}\dot{\mathbf{R}}^{\mathrm{T}}$$
$$\dot{\mathbf{R}} = -\dot{\mathbf{R}}^{\mathrm{T}}$$

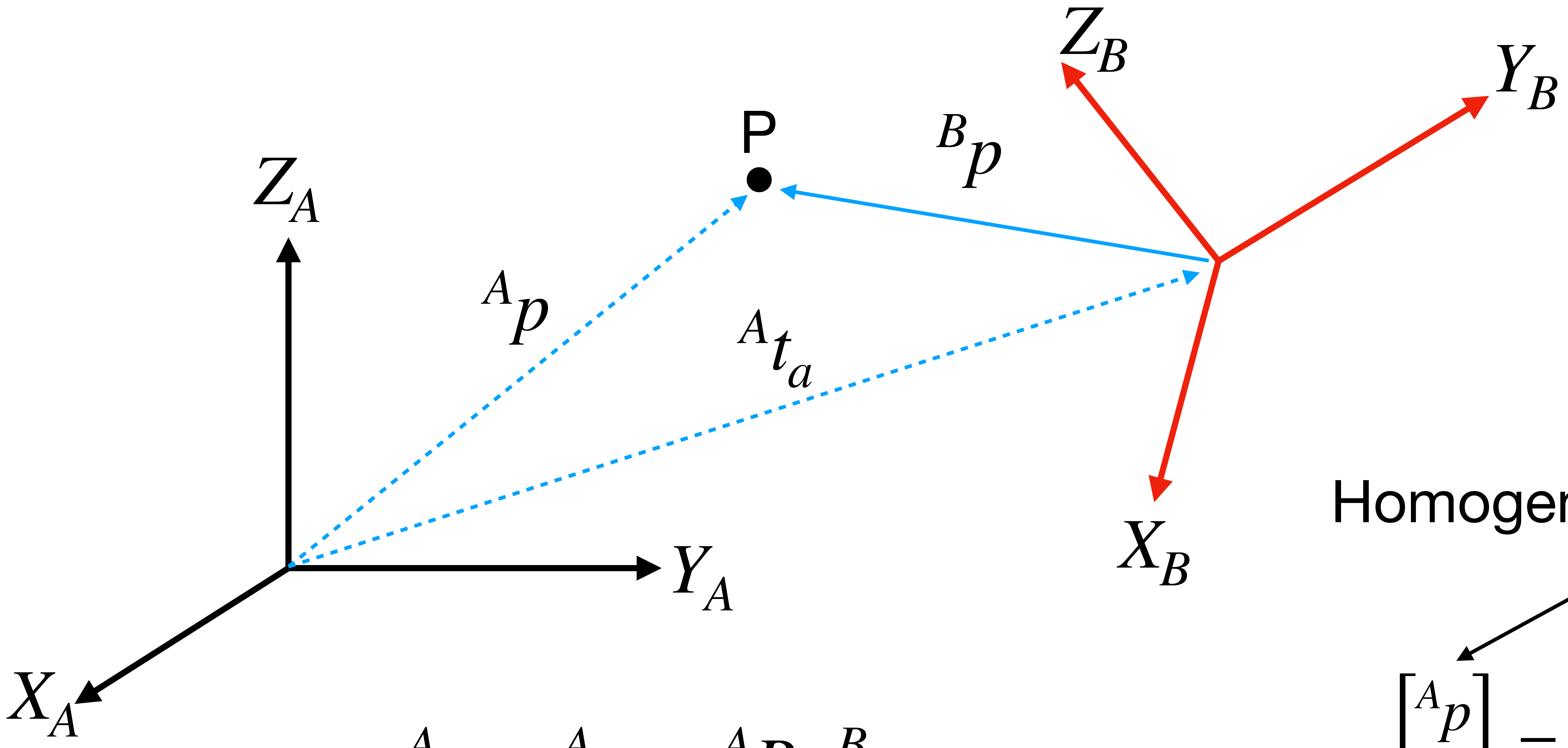
$$\dot{R} \triangleq S(\omega)R$$

where $\boldsymbol{\omega} \in \mathbb{R}^3$ is the angular velocity. By substitution:

$$\begin{aligned} \dot{\mathbf{R}}\mathbf{R}^{\mathrm{T}} &= -\mathbf{R}\dot{\mathbf{R}}^{\mathrm{T}} \\ \mathrm{S}(\boldsymbol{\omega})\mathbf{R}\mathbf{R}^{\mathrm{T}} &= -\mathbf{R}(\mathrm{S}(\boldsymbol{\omega})\mathbf{R})^{\mathrm{T}} \\ \mathrm{S}(\boldsymbol{\omega}) &= \mathbf{R}\mathbf{R}^{\mathrm{T}}\mathrm{S}(\boldsymbol{\omega}) \\ \mathrm{S}(\boldsymbol{\omega}) &= \mathrm{S}(\boldsymbol{\omega}) \ ✓ \end{aligned}$$



Homogeneous Transformations



$$^Ap = {}^At_B + {}^AR_B\,{}^Bp$$

First, translate the frame by t then rotate it.

Homogenous coordinates

$$\begin{bmatrix} ^Ap \\ 1 \end{bmatrix} = \begin{bmatrix} ^AR_B & ^At_B \\ 0 & 1 \end{bmatrix} \begin{bmatrix} ^Bp \\ 1 \end{bmatrix}$$

Homogenous Transformation Matrix

Homogeneous Transformations

They represent the position and orientation (pose) of a frame with respect to another frame.

$$T = \begin{bmatrix} r_{11} & r_{12} & r_{13} & t_x \\ r_{21} & r_{22} & r_{23} & t_y \\ r_{31} & r_{32} & r_{33} & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} R & t \\ 0 & 1 \end{bmatrix}$$

Mathematically, they belong to a Special Euclidean $SE(3) : \mathbb{R}^3 \times SO(3)$

Homogeneous Transformations

Pure Transformations

$$T = \begin{bmatrix} R & 0 \\ 0 & 1 \end{bmatrix}$$

Rotation

$$T = \begin{bmatrix} I & t \\ 0 & 1 \end{bmatrix}$$

Translation

$$\text{Rot}_x(\theta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\text{Rot}_y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

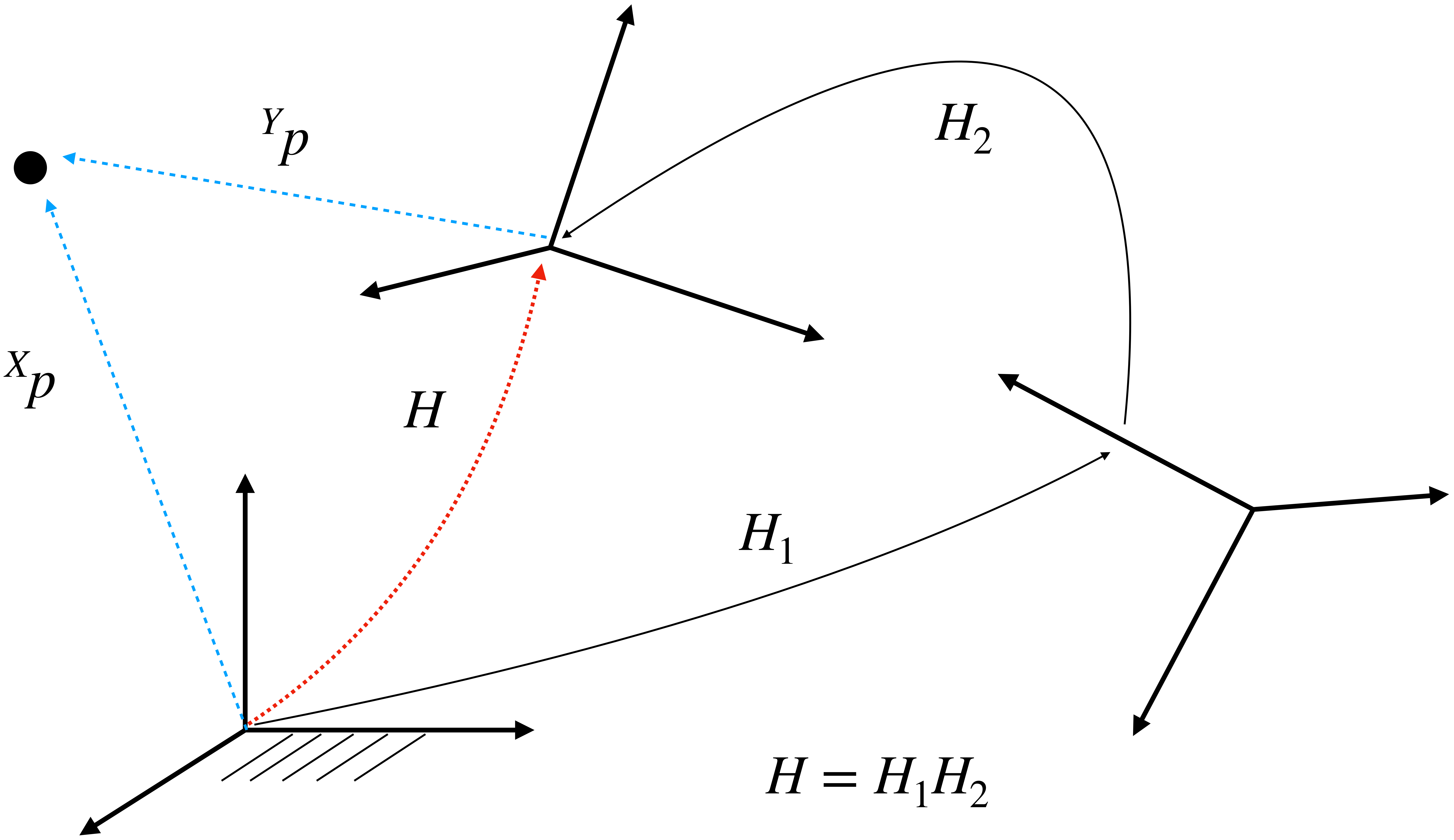
$$\text{Rot}_z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\text{Trans}_z(d) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\text{Trans}_x(d) = \begin{bmatrix} 1 & 0 & 0 & d \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\text{Trans}_y(d) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & d \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Homogeneous Transformations



Homogeneous Transformations

1. When a transformation is applied with respect to the fixed frame: A pre-multiplication is used.

2. When a transformation is applied with respect to the mobile (current new) frame: A post-multiplication is used.

Question: A frame {A} is rotated 90 degrees about x, and then translated a vector (6, -2, 10) with respect to the fixed (initial) frame. Find the homogeneous transformation that describes {B} with respect to {A}?

Solve:

Homogeneous Transformations

Question: Find the homogeneous transformation matrix that represents a rotation of an angle a about the x axis, followed by a translation of b units along the new x axis, followed by a translation of d units along the new z axis followed by a rotation of an angle J about the new z axis?

Solve:

Homogeneous Transformations

1. They represent the pose (position+orientation) of a frame (rigid body) with respect to another frame

2. They change the reference frame in which a point is represented (using a linear relation):

$$^{A}\tilde{p} = {}^{A}T_{B}\ {}^{B}\tilde{p}$$

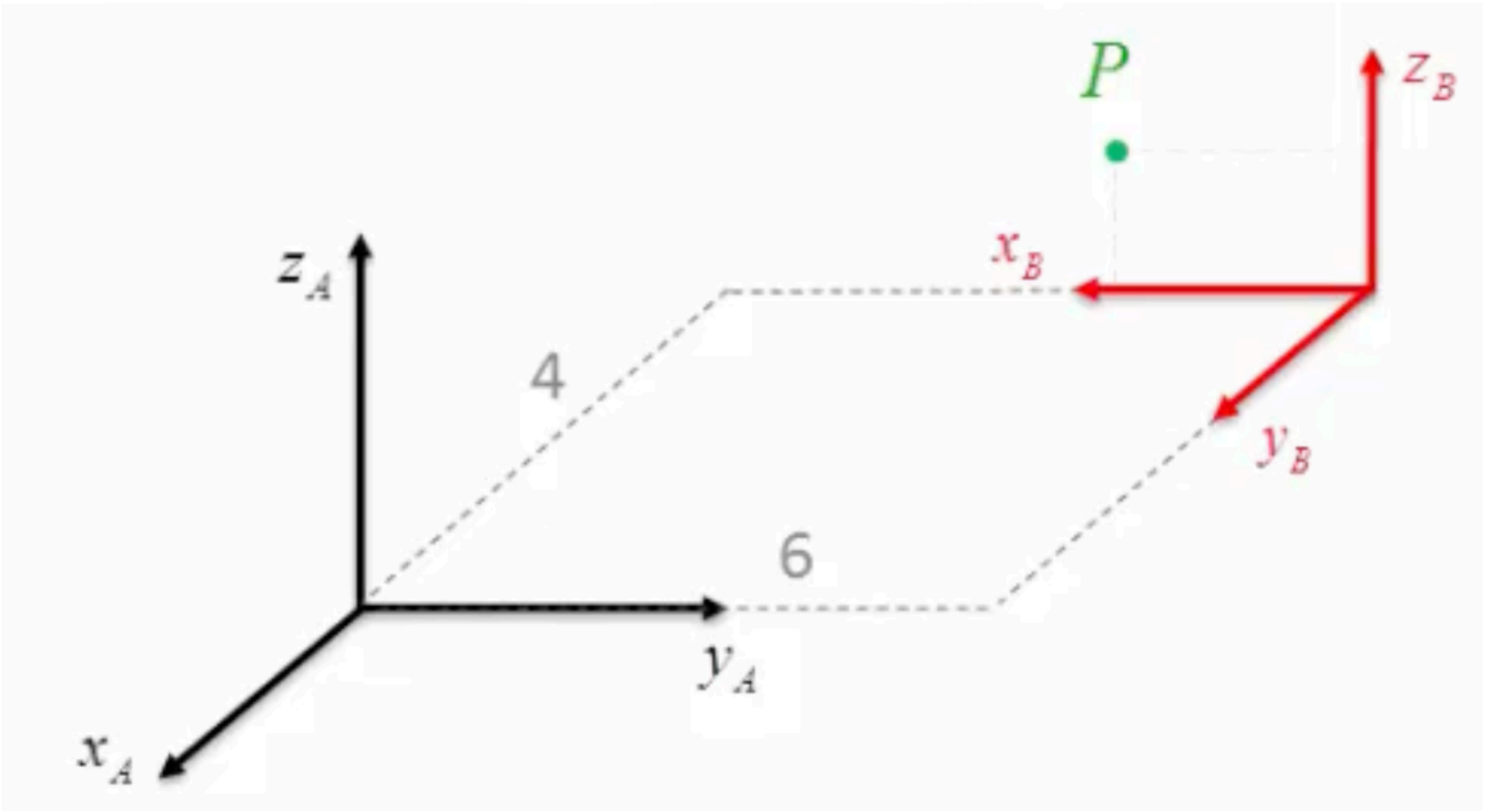
   - Note: the point must be represented using homogeneous coordinates (its notation uses ~)

3. They apply a transformation (rotation + translation) to a point in the same reference frame

Homogeneous Transformations

Question: Consider frame {A} and {B}. Point P in frame {B} is given by (2, 0, 1), find its coordinates with respect to frame {A} using homogeneous transformation matrix?

Solve:



Homogeneous Transformations

Question: A frame {A} is rotated 90 degree about x, and then it is translated a vector (6, -2, 10) with respect to the fixed (initial) frame. Consider a point p = (-5, 2, -12) with respect to the new frame {B}. Determine the coordinate of that point with respect to the initial frame.

Solve:

Not commutative $T_1T_2 \neq T_2T_1$

Homogeneous Transformations

- Inverse of a homogeneous transformation:

$$T = \begin{bmatrix} R & \mathbf{t} \\ \mathbf{0} & 1 \end{bmatrix} \qquad T^{-1} = \begin{bmatrix} R^T & -R^T\mathbf{t} \\ \mathbf{0} & 1 \end{bmatrix}$$

- Why?

$${}^{A}p = {}^{A}t_{B} + {}^{A}R_{B}\,{}^{B}p \xrightarrow{\text{Solving for } {}^{B}p} {}^{B}p = {}^{A}R_{B}^{T}\,{}^{A}p - {}^{A}R_{B}^{T}\,{}^{A}t_{B}$$

$${}^{A}\tilde{p} = {}^{A}T_{B}\;{}^{B}\tilde{p} \longrightarrow {}^{B}\tilde{p} = {}^{A}T_{B}^{-1}\;{}^{A}\tilde{p} = {}^{B}T_{A}\;{}^{A}\tilde{p}$$

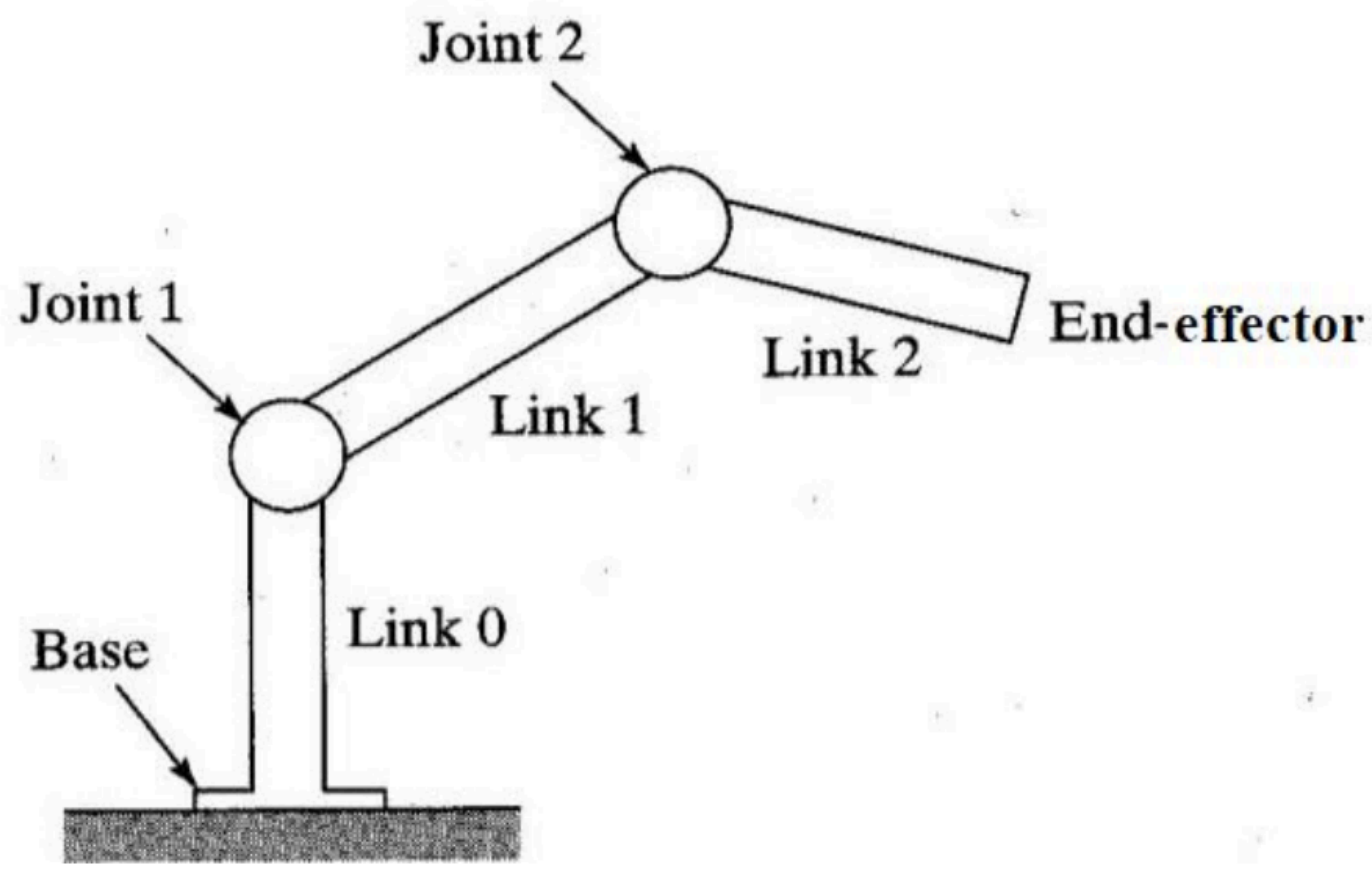
- Product of homogeneous transformations:

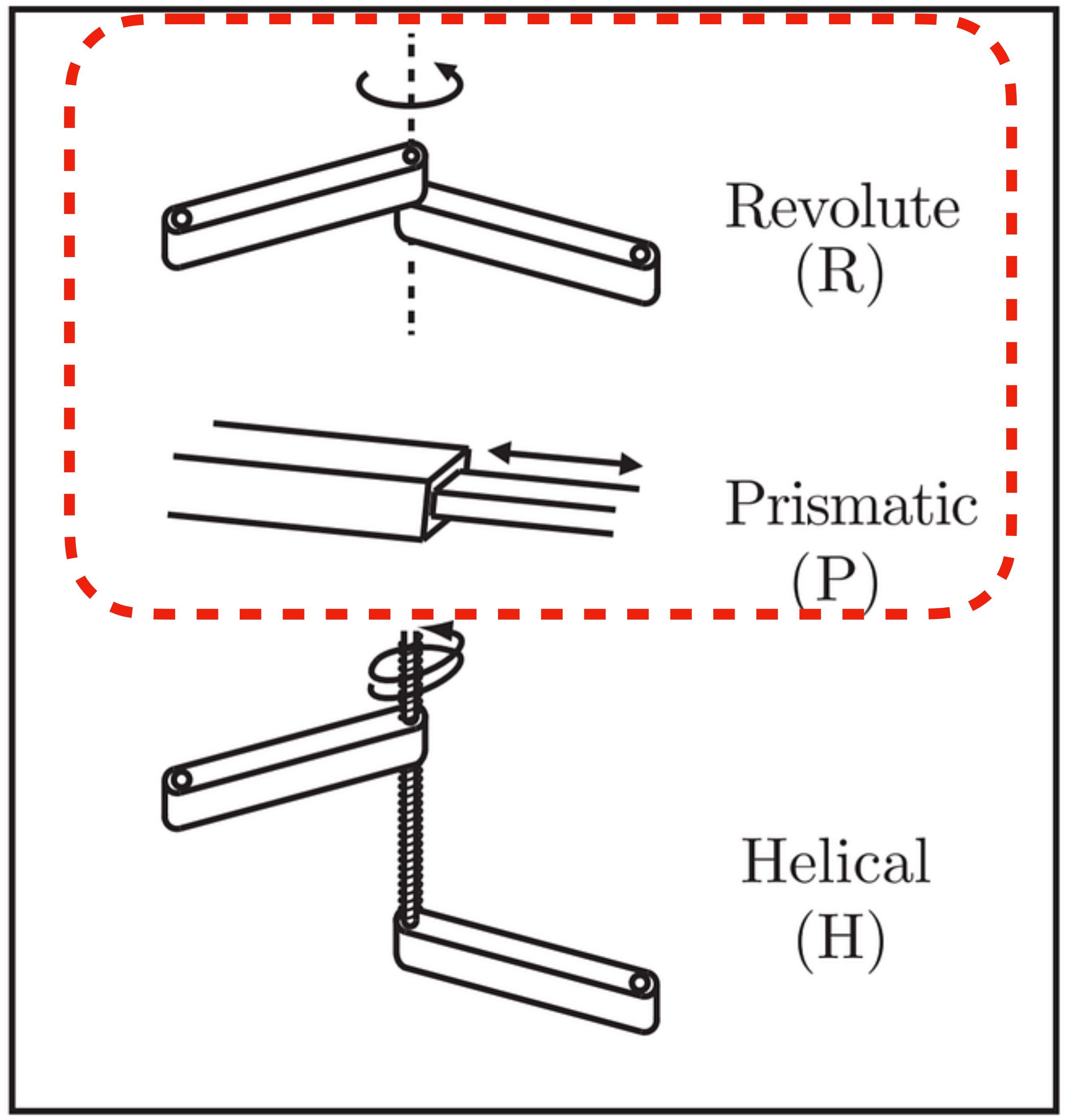
$$T_1 = \begin{bmatrix} R_1 & \mathbf{t}_1 \\ \mathbf{0} & 1 \end{bmatrix} \quad T_2 = \begin{bmatrix} R_2 & \mathbf{t}_2 \\ \mathbf{0} & 1 \end{bmatrix} \Rightarrow T_1T_2 = \begin{bmatrix} R_1R_2 & R_1\mathbf{t}_2 + \mathbf{t}_1 \\ \mathbf{0} & 1 \end{bmatrix}$$

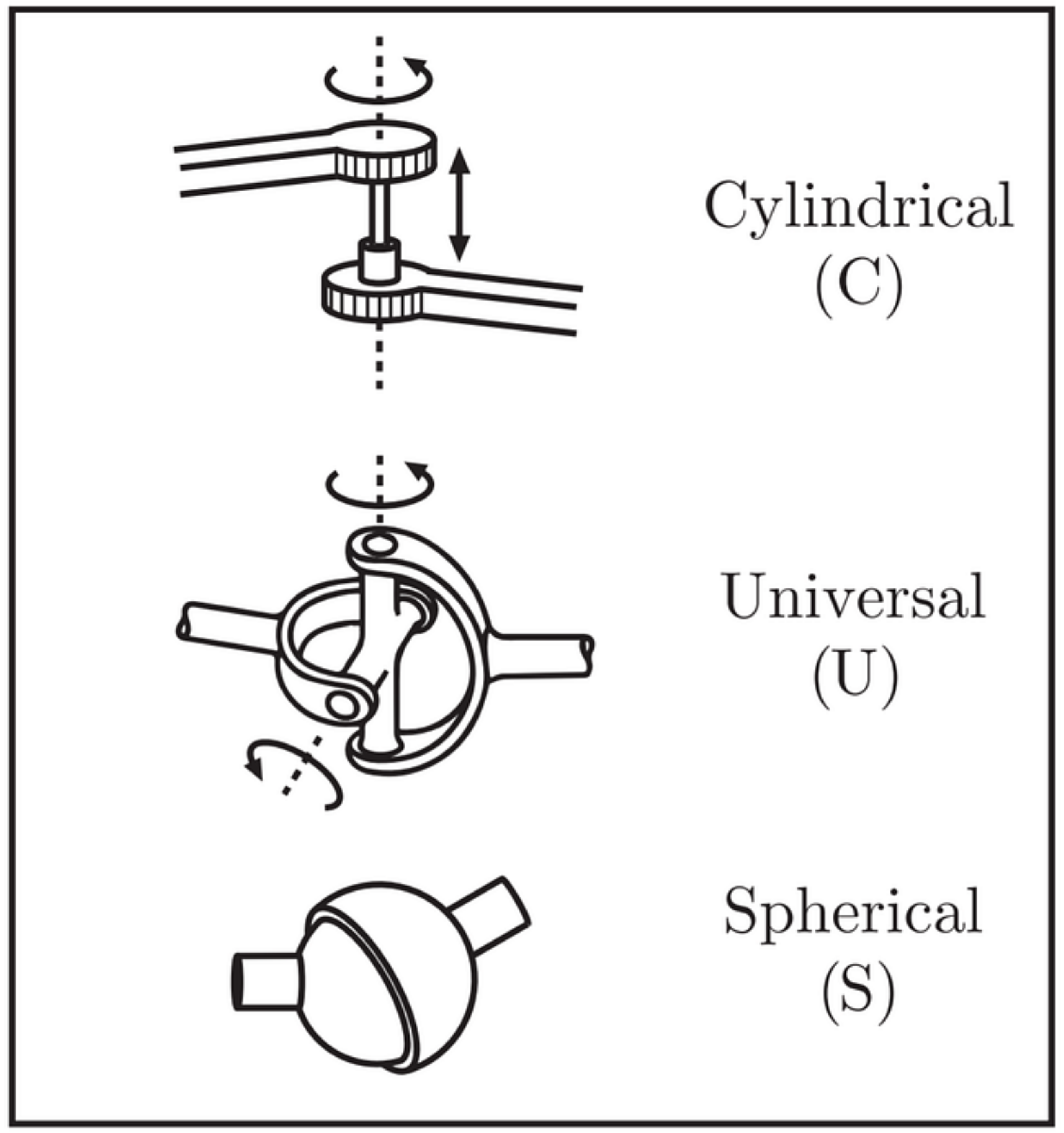
It is not commutative

Forward Kinematics

- Forward kinematics of a robot arm: compute the position and orientation of its end-effector from its joint coordinates θ

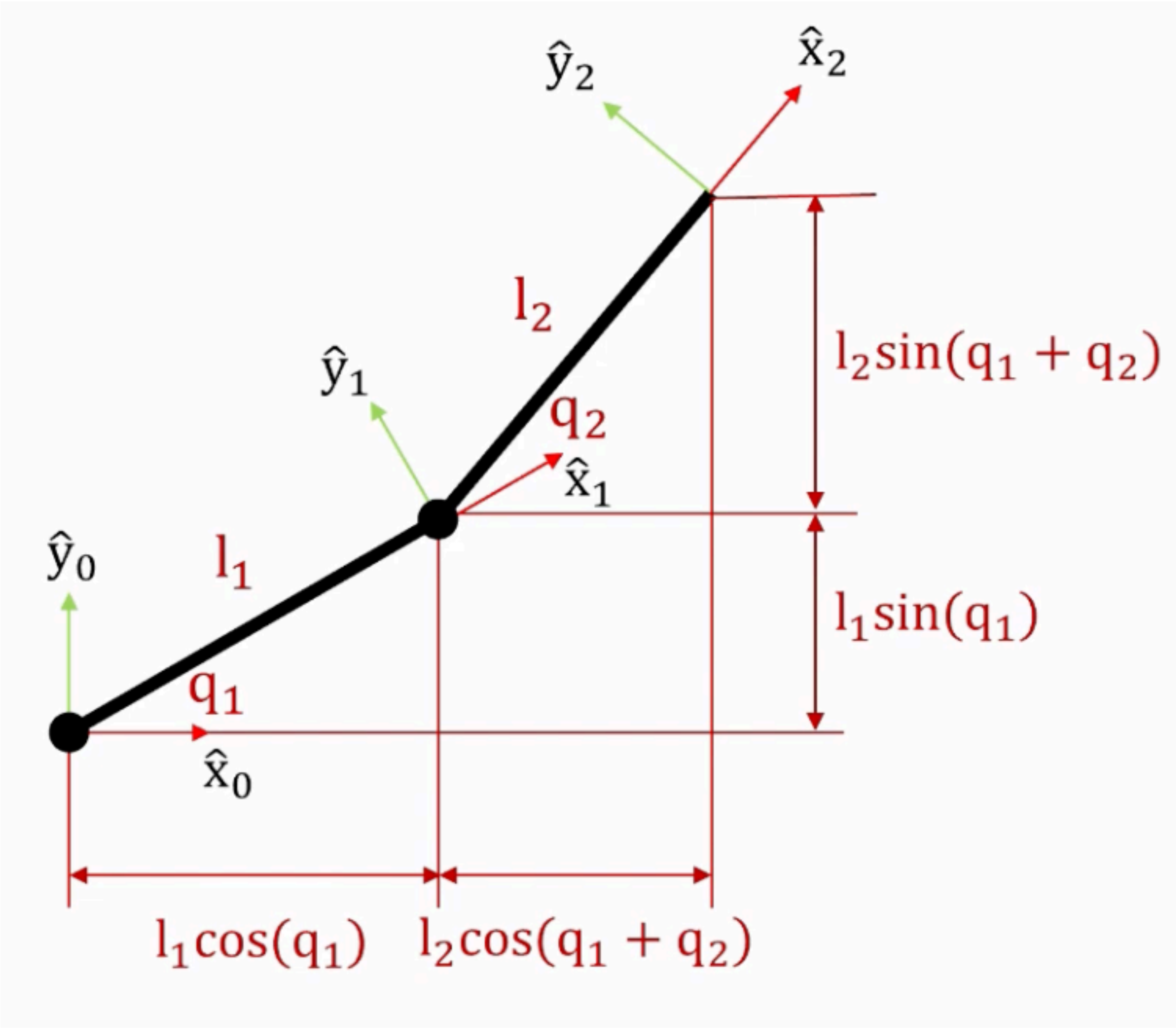






Live Demo

Forward Kinematics of a 2DoF, Planar Manipulator



Task space:

$$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$$

Joint space:

$$\mathbf{q} = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \in \mathbb{R}^2$$

Forward Kinematics:

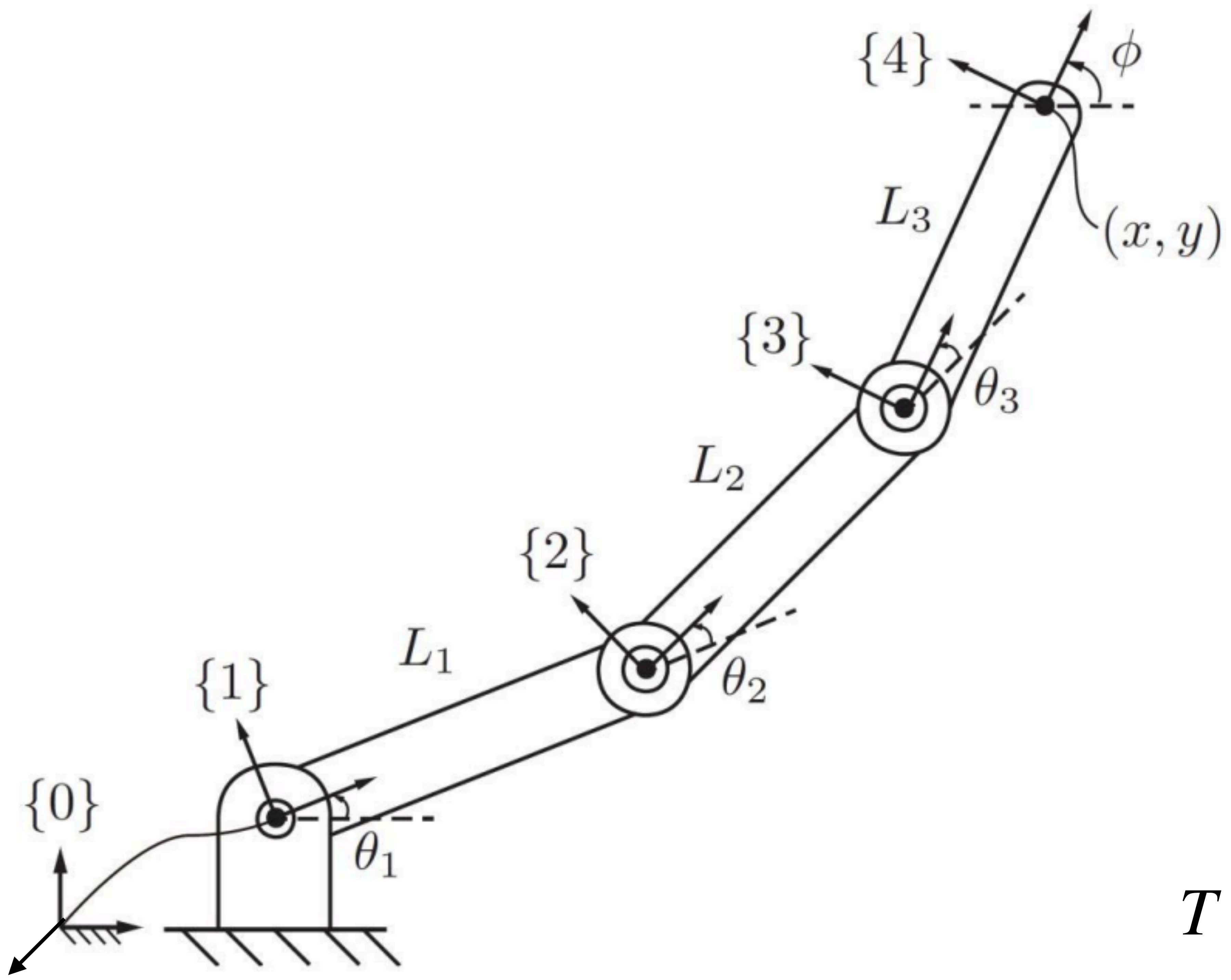
$$x = f(q)$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} l_1 \cos(q_1) + l_2 \cos(q_1 + q_2) \\ l_1 \sin(q_1) + l_2 \sin(q_1 + q_2) \end{bmatrix}$$

Orientations:

$$\begin{bmatrix} x \\ y \\ \psi \end{bmatrix} = \begin{bmatrix} l_1 \cos(q_1) + l_2 \cos(q_1 + q_2) \\ l_1 \sin(q_1) + l_2 \sin(q_1 + q_2) \\ q_1 + q_2 \end{bmatrix}$$

Forward Kinematics of a 3DoF Manipulator



1. Attaching frames to links
2. Using homogeneous transformations

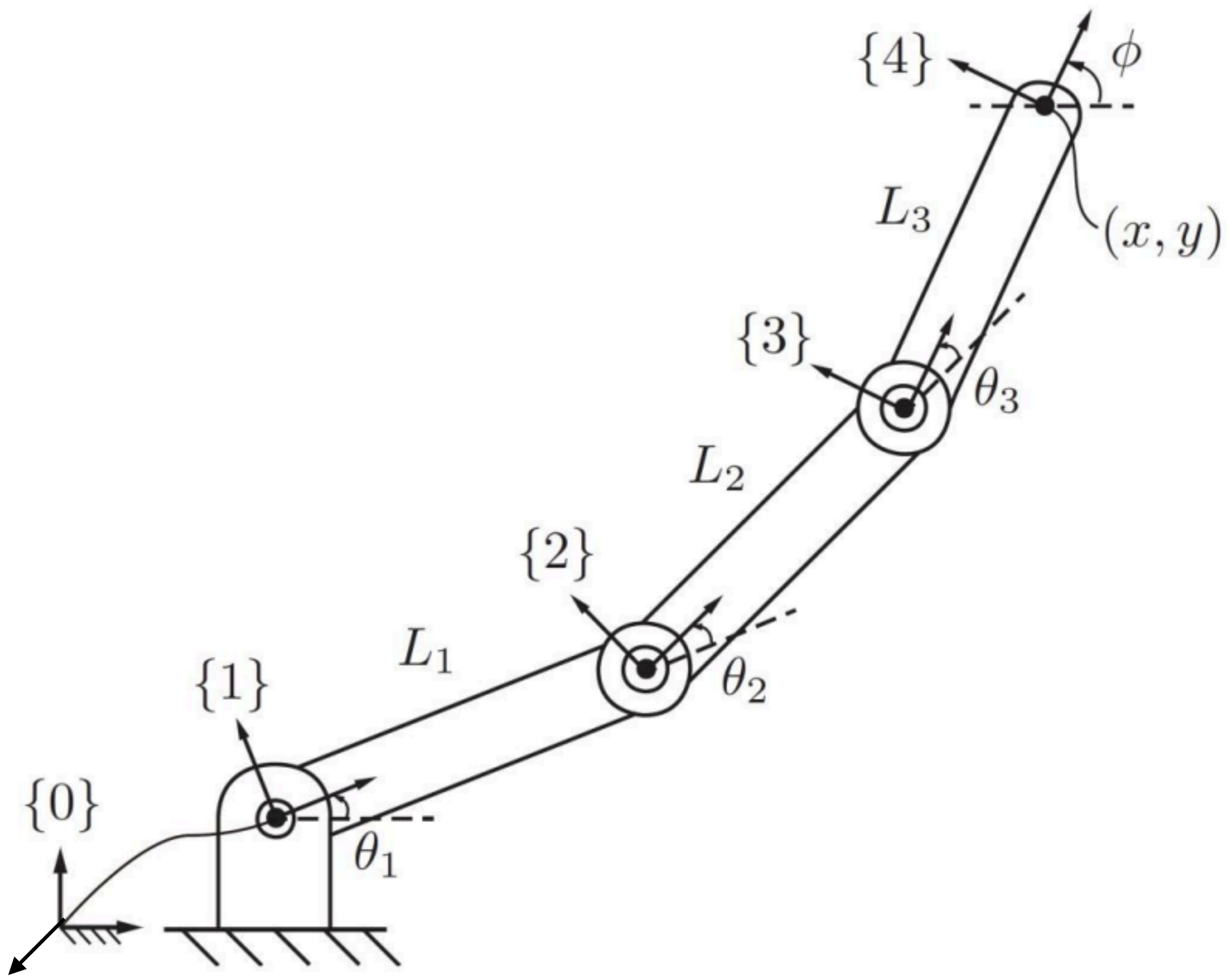
$$T_{04} = T_{01}\,T_{12}\,T_{23}\,T_{34}$$

- $T_{01} = R_z(\theta_1)$
- $T_{12} = R_z(\theta_2)\,\mathrm{Trans}_x(L_1)$
- $T_{23} = R_z(\theta_3)\,\mathrm{Trans}_x(L_2)$
- $T_{34} = \mathrm{Trans}_x(L_3)$

$$R_z(\theta_1) = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{p}_{01} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$T = \begin{bmatrix} R & \mathbf{p} \\ \mathbf{0} & 1 \end{bmatrix} \qquad R = R_z(\theta_1), \quad \mathbf{p} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad T_{01} = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Forward Kinematics of a 3DoF Manipulator



$$T_{01} = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad T_{12} = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & 0 & L_1 \\ \sin\theta_2 & \cos\theta_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_{23} = \begin{bmatrix} \cos\theta_3 & -\sin\theta_3 & 0 & L_2 \\ \sin\theta_3 & \cos\theta_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad T_{34} = \begin{bmatrix} 1 & 0 & 0 & L_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Final homogeneous transform

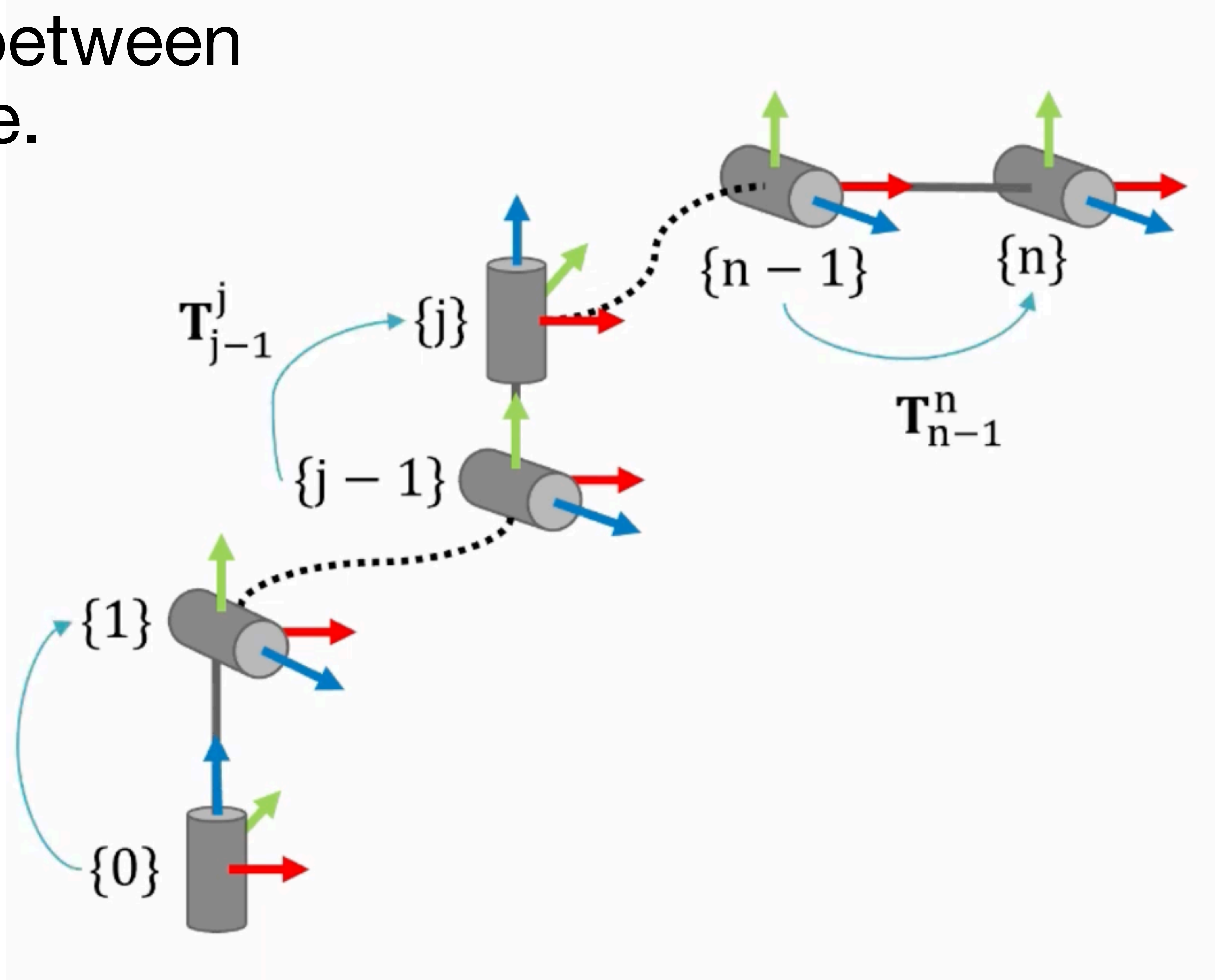
$$T_{04} = \begin{bmatrix} \cos\theta_{123} & -\sin\theta_{123} & 0 & x \\ \sin\theta_{123} & \cos\theta_{123} & 0 & y \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Forward Kinematics Using Transformation Matrices

We can concatenate transformation matrices between joint frames to determine the end-effector pose.

$$T_0^n = T_0^1 \times T_1^2 \times T_2^3 \times \cdots \times T_{n-1}^n$$

$$= \prod_{j=1}^{n} T_{j-1}^j$$



Need to describe T as a function of simple geometry.

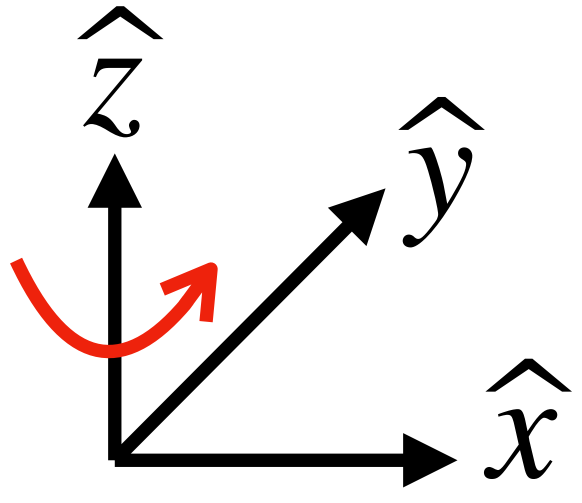
Denavit-Hartenberg (DH) Paramters

It is defined as four parameters associated with the DH convention for attaching references frame to the links of a spatial kinematic chain, or robot manipulator.

Minimum of 4 parameters, applied in sequence

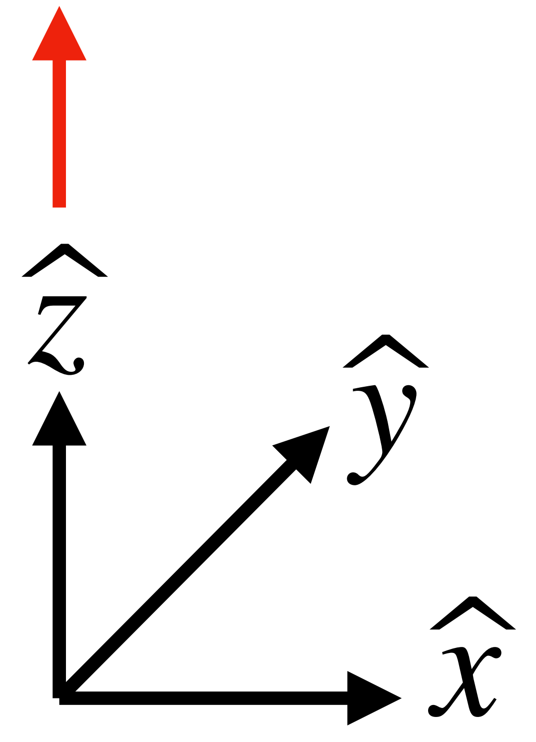
1. Rotate about z-axis by θ

$$\mathbf{T}_{R_z}(\theta) = \begin{bmatrix} \mathbf{R}_z(\theta) & \mathbf{0}_{3\times1} \\ \mathbf{0}_{1\times3} & 1 \end{bmatrix}$$



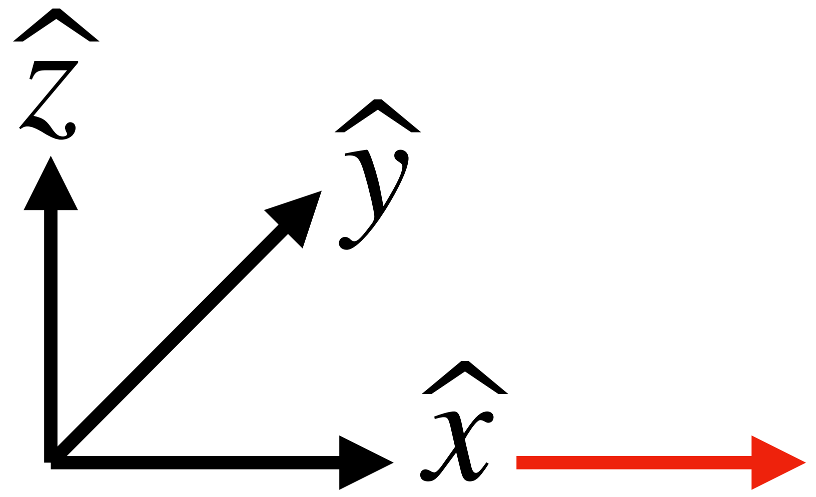
2. Translate across z-axis by d

$$\mathbf{T}_z(d) = \begin{bmatrix} & 0 \\ \mathbf{I} & 0 \\ & d \\ \mathbf{0}_{1\times3} & 1 \end{bmatrix}$$



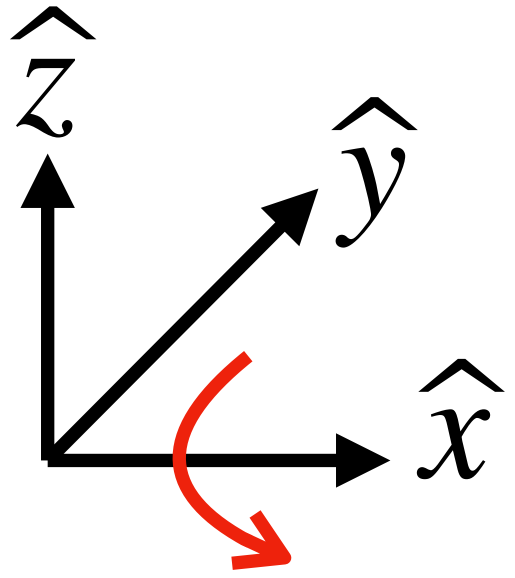
3. Translate across x-axis by a

$$\mathbf{T}_x(a) = \begin{bmatrix} & a \\ \mathbf{I} & 0 \\ & 0 \\ \mathbf{0}_{1\times3} & 1 \end{bmatrix}$$



4. Rotate about x-axis by α

$$\mathbf{T}_{R_x}(\alpha) = \begin{bmatrix} \mathbf{R}_x(\alpha) & \mathbf{0}_{3\times1} \\ \mathbf{0}_{1\times3} & 1 \end{bmatrix}$$



For joint {j-1} to {j}

$$\mathbf{T}^{j}_{j-1} = \mathbf{T}_{Rz}(\theta_j)\mathbf{T}_z(d_j)\mathbf{T}_x(a_j)\mathbf{T}_{Rx}(\alpha_j)$$

Why DH parameters?

Minimum number of parameters to describe FK with guarantee for a serial chain with n 1-DoF joints.

Universal nomenclature

1. Knowing DH parameters gives complete knowledge of kinematics.
2. Easily understood by anyone and common practice.

Good for analysis (Jacobians, Singularities, Dynamics).

Computationally efficient for differential kinematics, dynamics

Rules for DH Parameters

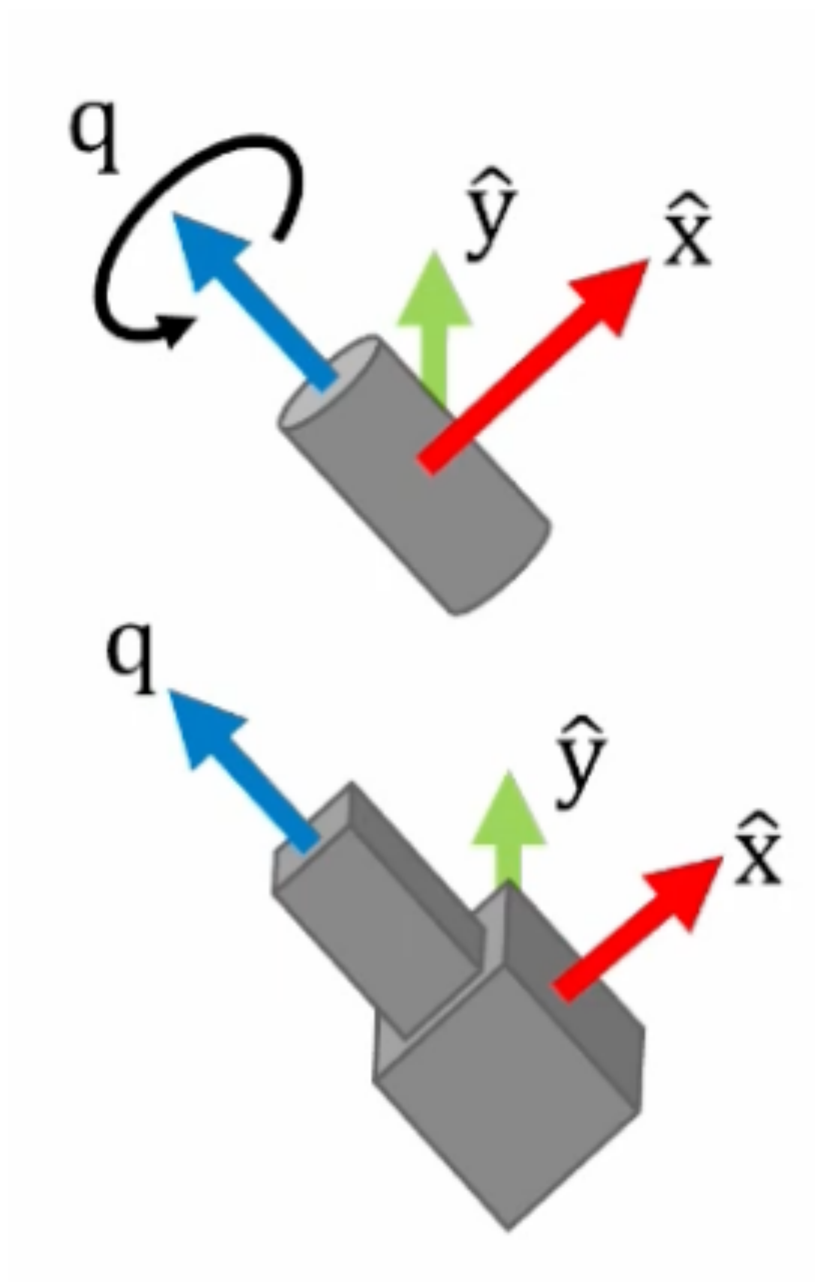
1. Actuate about z-axis

Rotate about zero for revolute joints.

$$\mathbf{T}_{Rz}(q) = \begin{bmatrix} \mathbf{R}_z(q) & \mathbf{0}_{3\times1} \\ \mathbf{0}_{1\times3} & 1 \end{bmatrix}$$

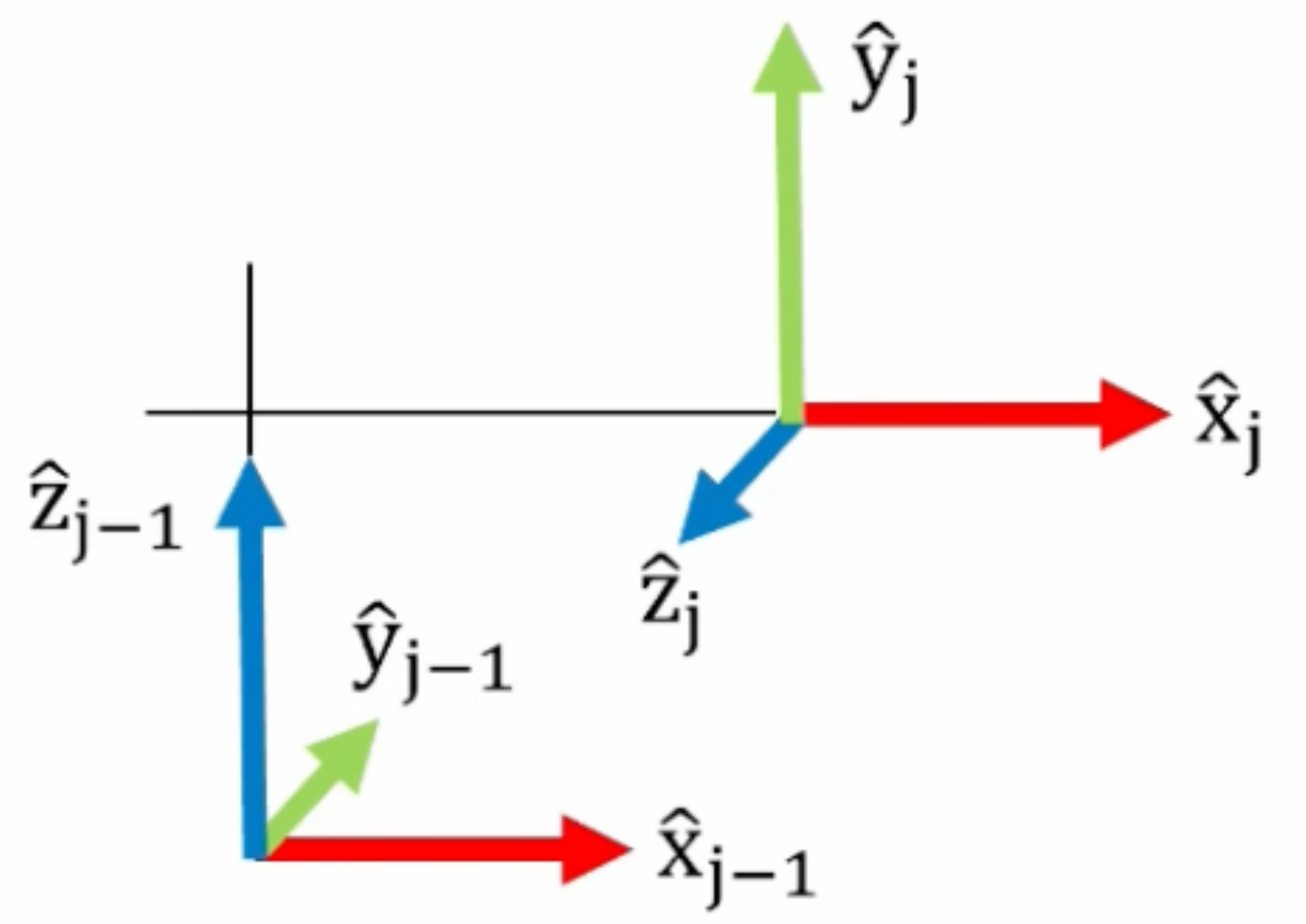
Translate about z for prismatic joints.

$$\mathbf{T}_z(q) = \begin{bmatrix} & & & 0 \\ & \mathbf{I} & & 0 \\ & & & q \\ & \mathbf{0}_{1\times3} & & 1 \end{bmatrix}$$



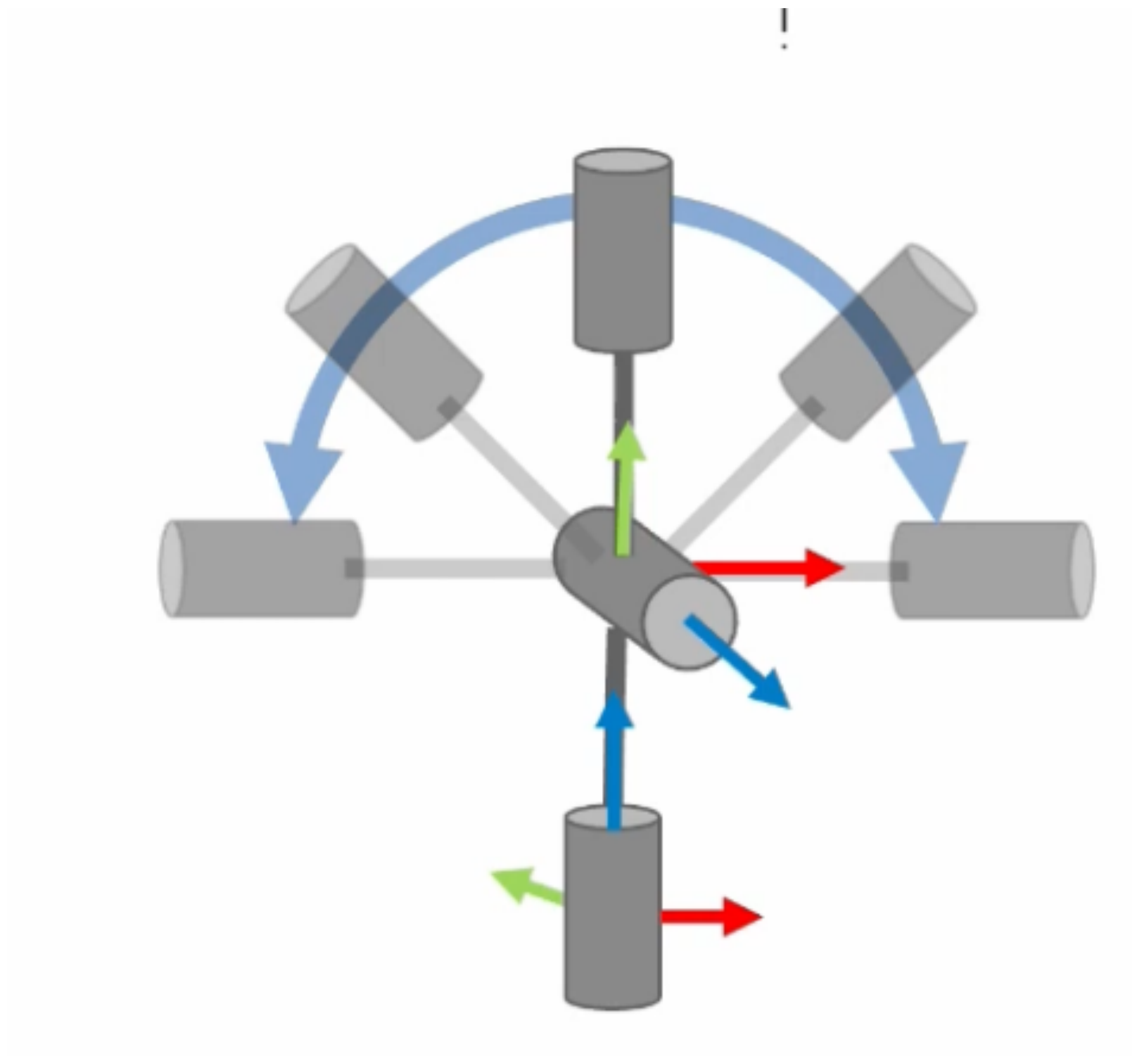
2. Axis Z(j-1) is perpendicular to, and intersects, Xj.

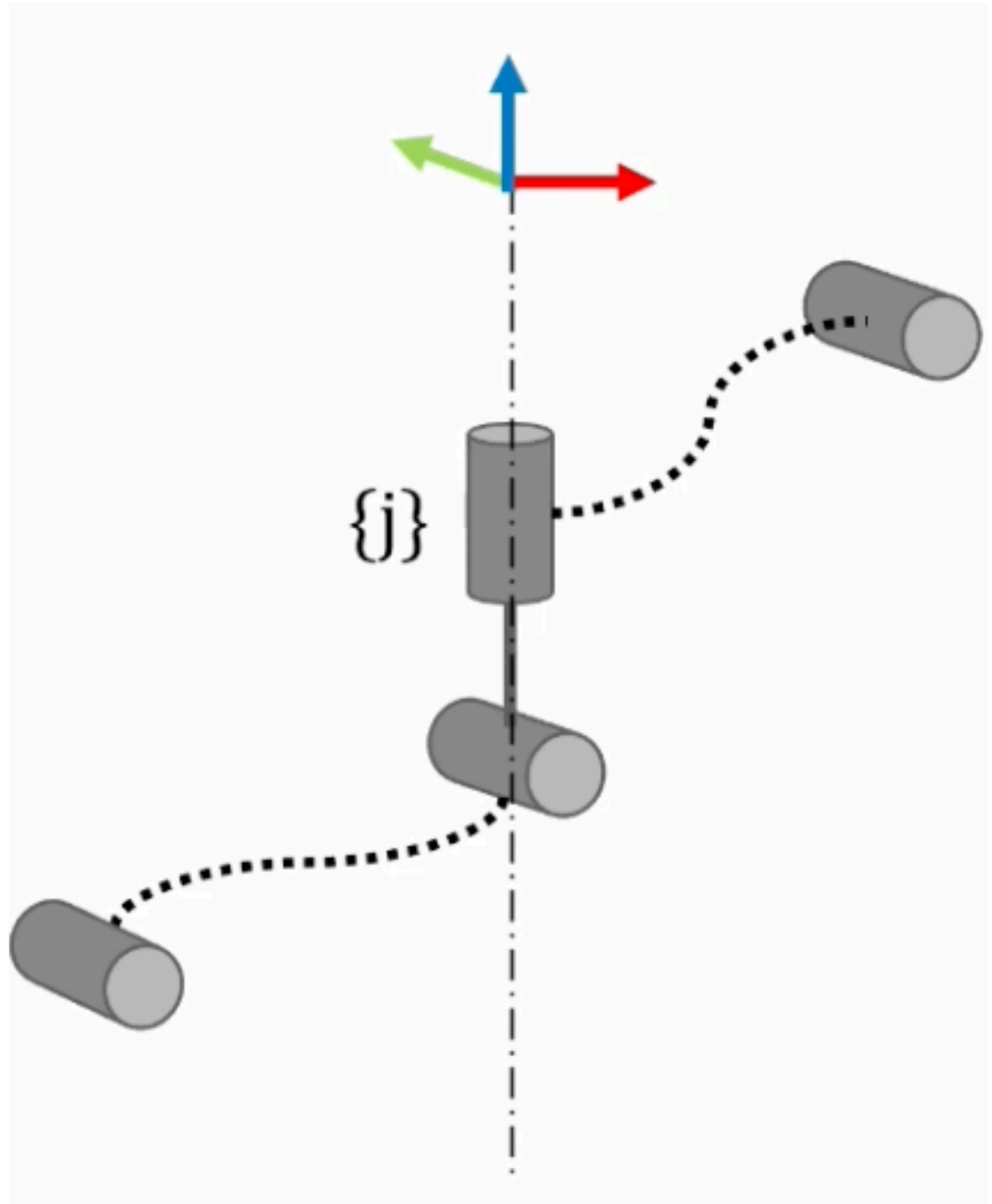
3. The y-axis is solved implicitly: y_j= z_j X x_j.



Rules for DH Parameters

4. Joint frame does not need to be physically coincide with the actual joint.





5. The robot arm can be arranged in any configuration that suits the DH parameters.

Rules for DH Parameters

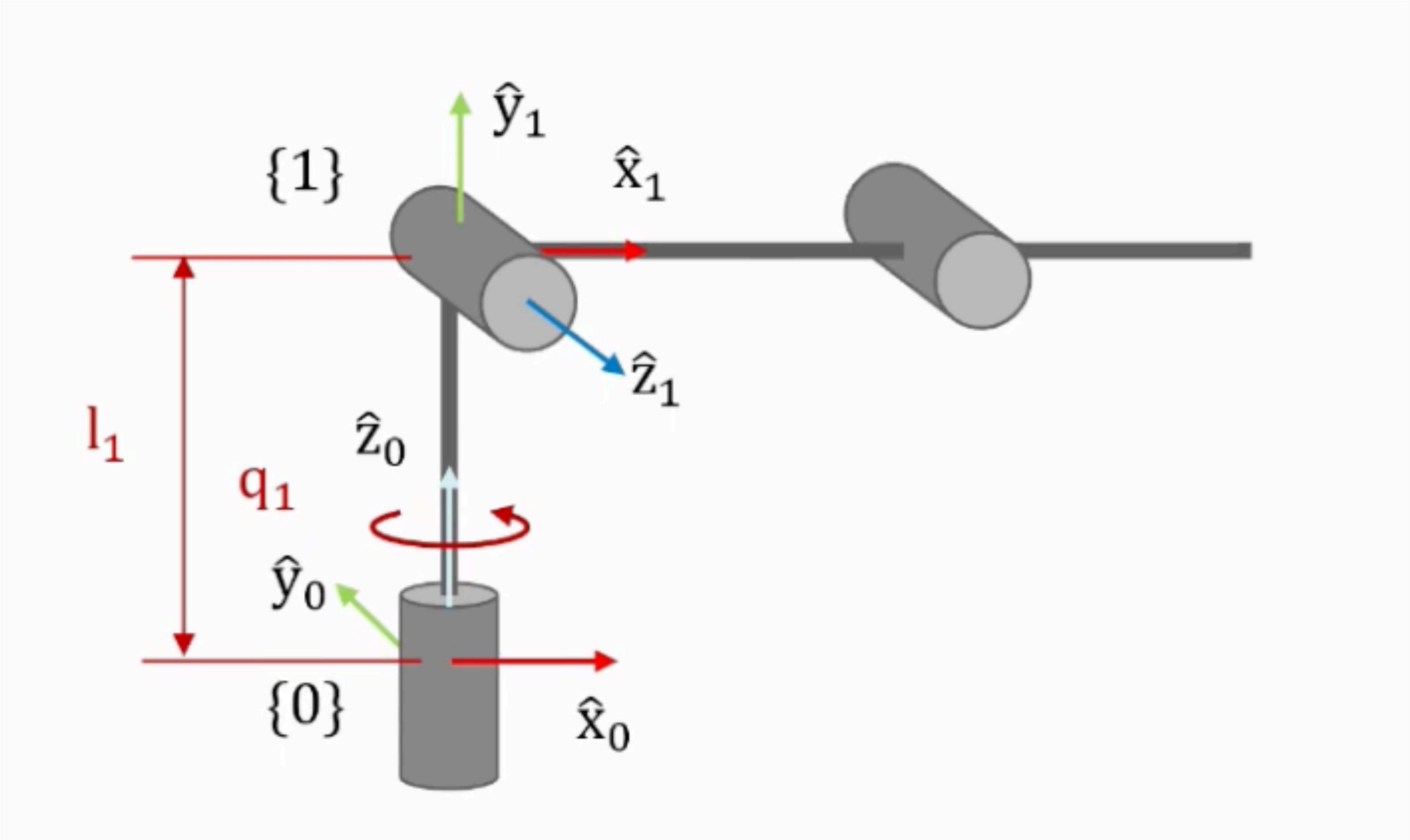
To get FK at a particular joint configuration q, substitute the joint value into the z-component of the transform chain.

$$\mathbf{T}_{j-1}^{j} = \begin{cases} \mathbf{T}_{R_z}(q_j)\,\mathbf{T}_z(d_j)\,\mathbf{T}_x(a_j)\,\mathbf{T}_{R_x}(\alpha_j), & \text{for revolute} \\ [6pt]\mathbf{T}_{R_z}(\theta_j)\,\mathbf{T}_z(q_j)\,\mathbf{T}_x(a_j)\,\mathbf{T}_{R_x}(\alpha_j), & \text{for prismatic} \end{cases}$$

Alternative sequences can be used for example: $\mathbf{T}_{j-1}^{j} = \mathbf{T}_{Rz}(d_j)\mathbf{T}_z(\theta_j)\mathbf{T}_x(\alpha_j)\mathbf{T}_{Rx}(a_j)$

The order is important! Matrices are not commutative: AB does not equal to BA.

Example: FK of 3DoF Manipulator with DH



Reference

https://www.youtube.com/watch?v=JBNmPq8eg8w

https://yuxng.github.io/Courses/CS6341Fall2025/lecture_08_forward_kinematics_DH.pdf

https://hades.mech.northwestern.edu/images/7/7f/MR.pdf