UNIVERSITY of WASHINGTON

Spatial Algebra & Kinematics

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CSE 571

Things to cover today:

- Relative Pose & Translation
- Rotation matrices in 2D
- Rotation matrices in 3D
- Homogeneous transformation matrices
- Forward kinematics
- DH Parameters

Objective of the class

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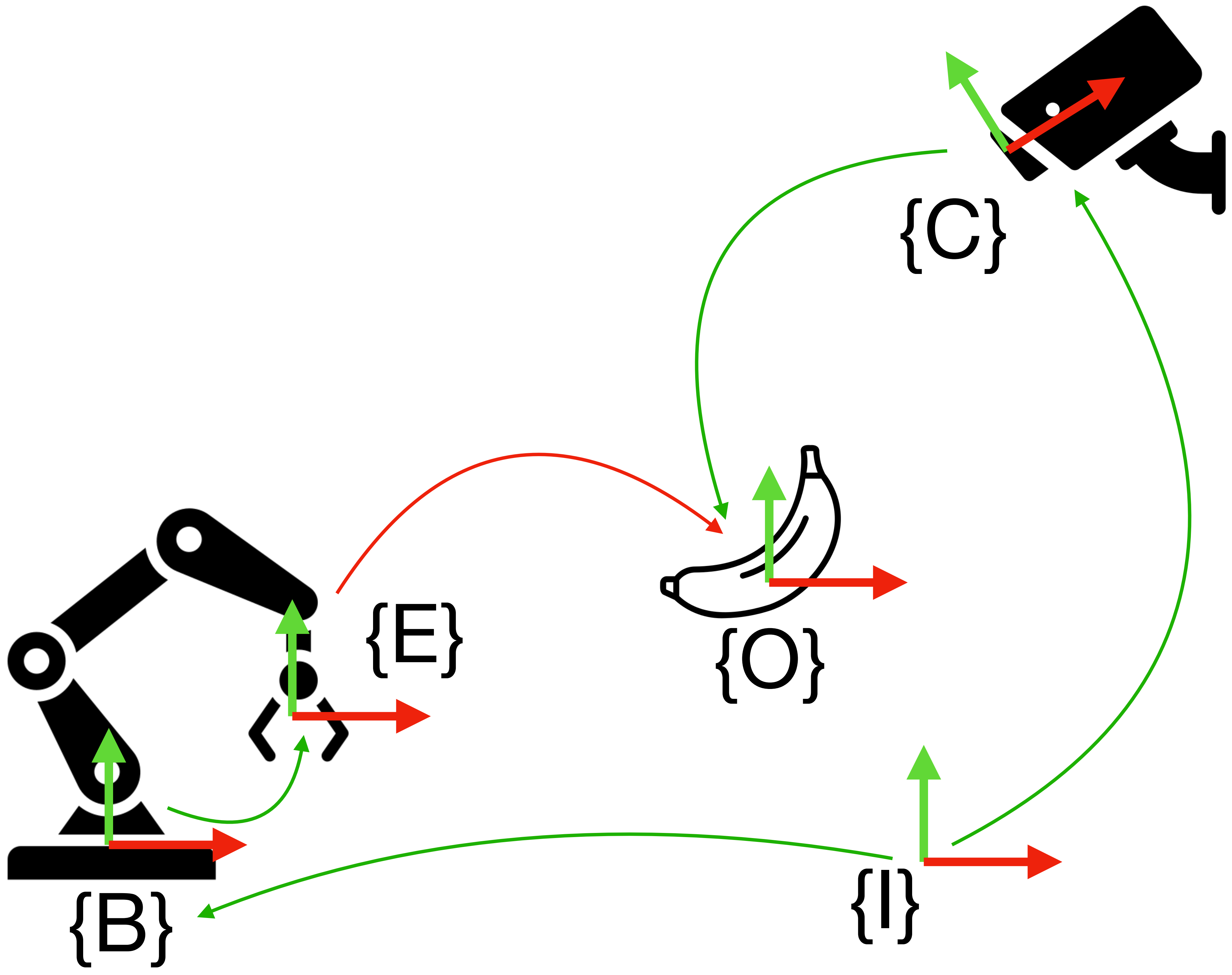
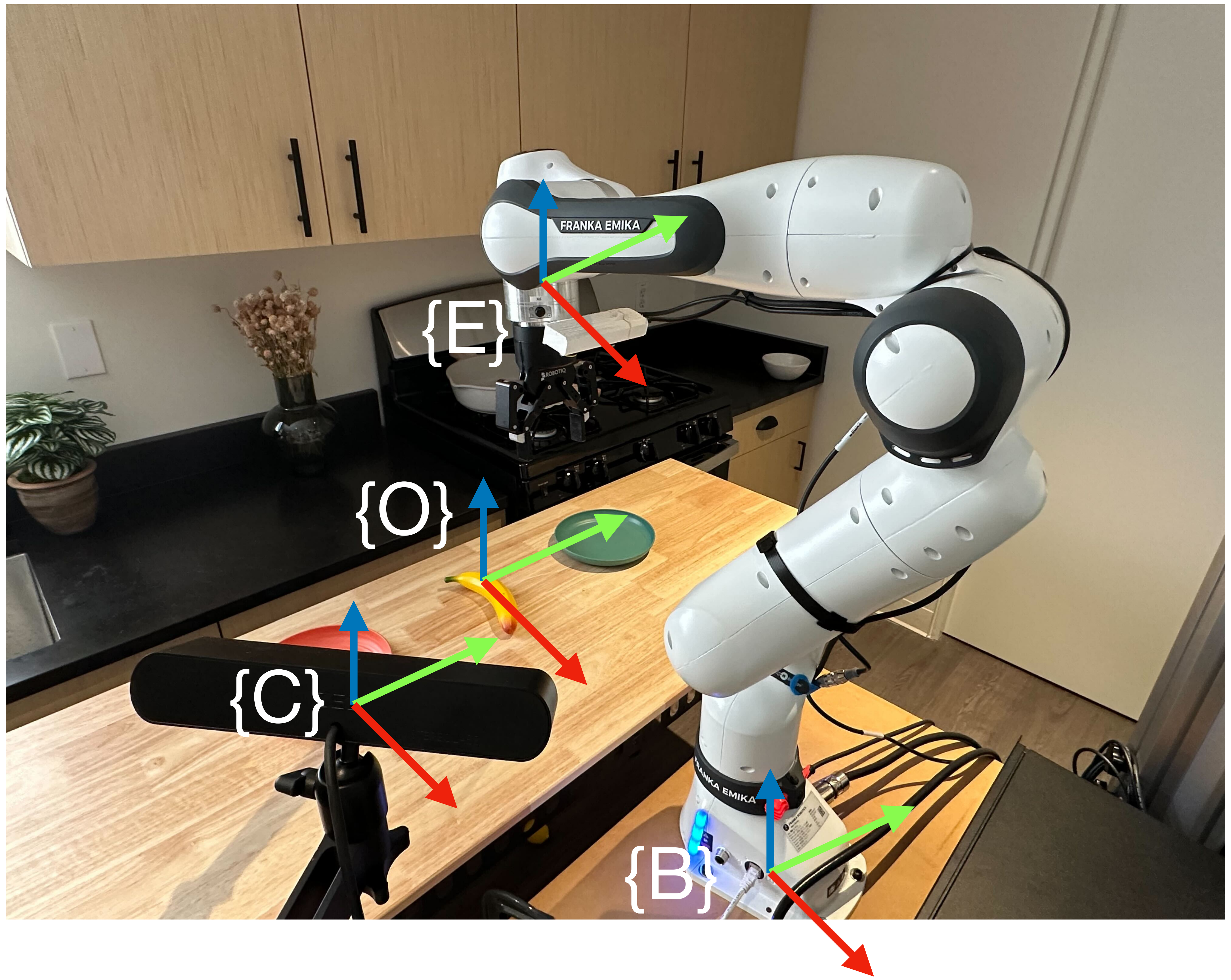




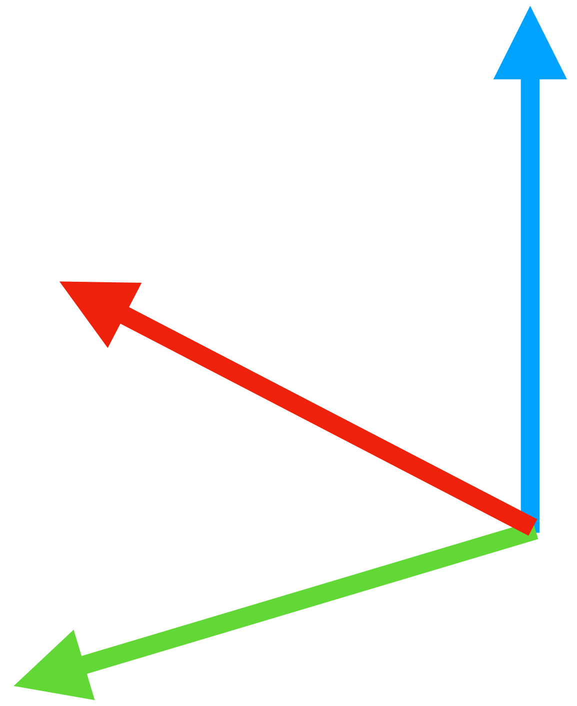
What is $\mathbf{R}_E^O$?

Reference Frames

> Position and orientation in space is relative

> We need both to fully describe 3D

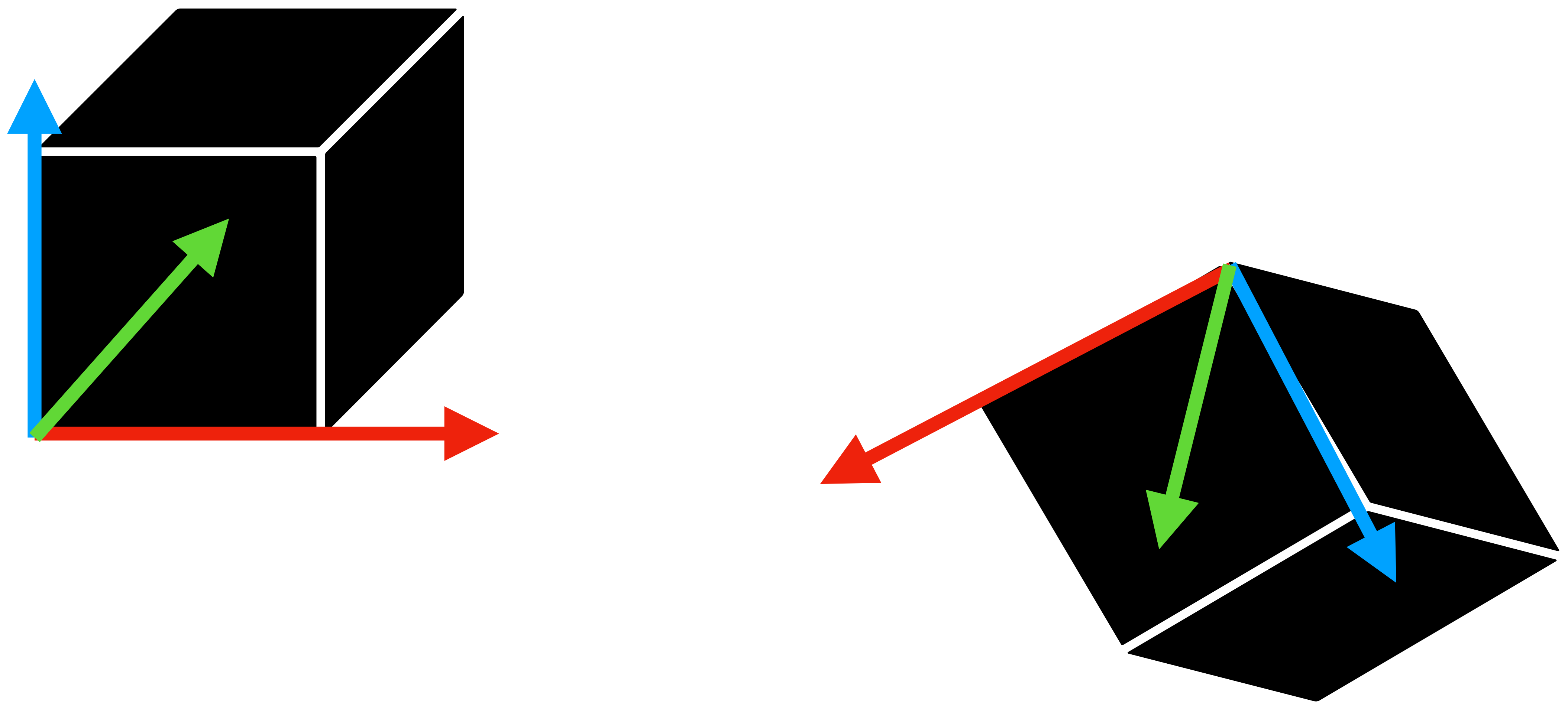
> Pose = Position + Orientation

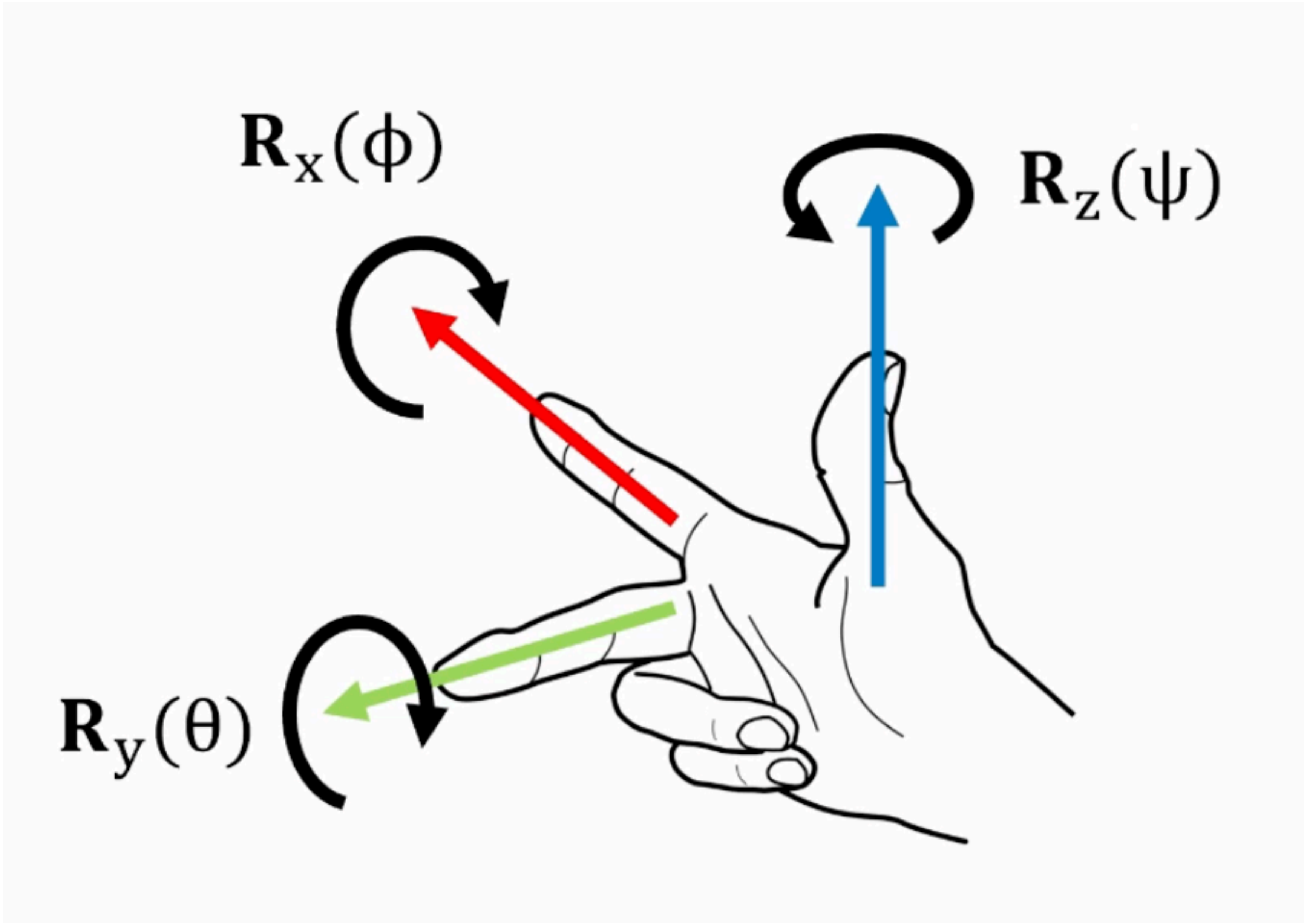


RGB = XYZ

RGB = RPY

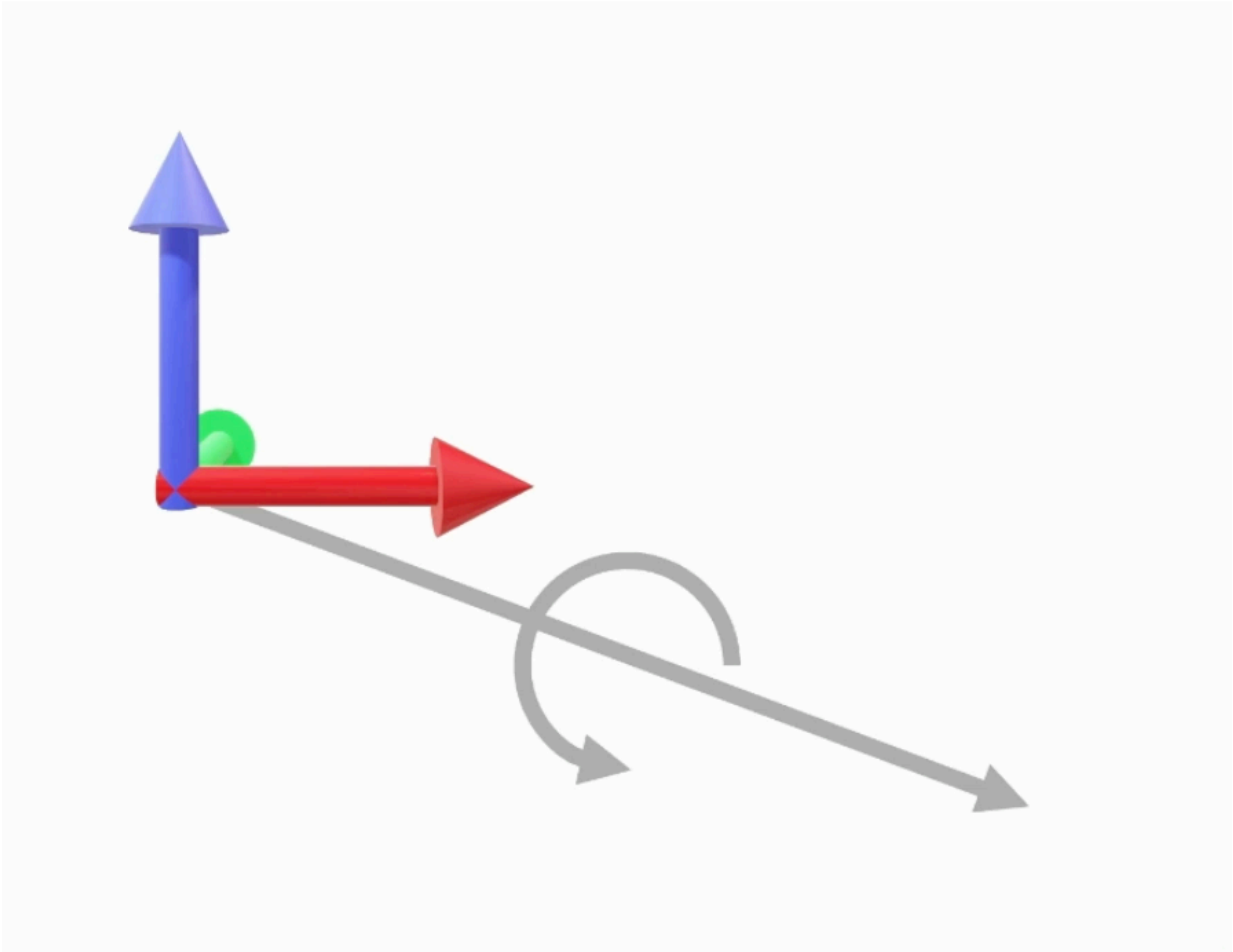
Vehicle coordinate frame

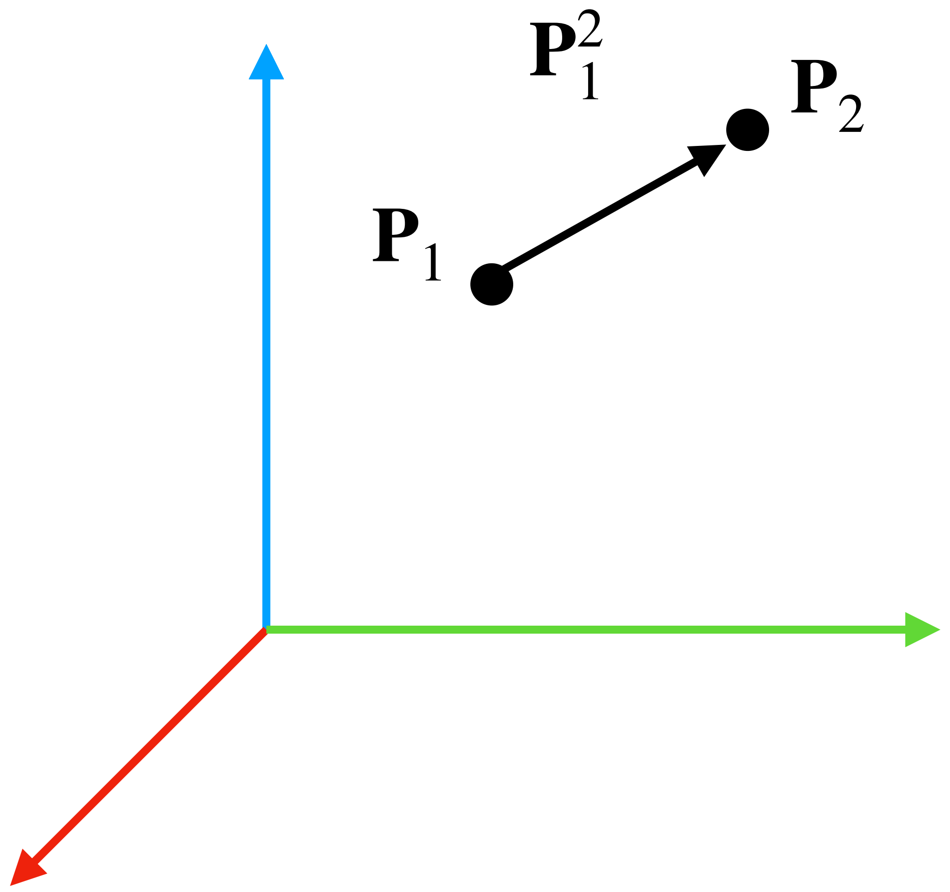




Transformation between reference frames

> Moving between reference frames involves both a translation and rotation.





Two points in 3D Euclidean space:

$$\mathbf{p}_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \in \mathbb{R}^3 \qquad \mathbf{p}_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} \in \mathbb{R}^3$$

Translation between points?

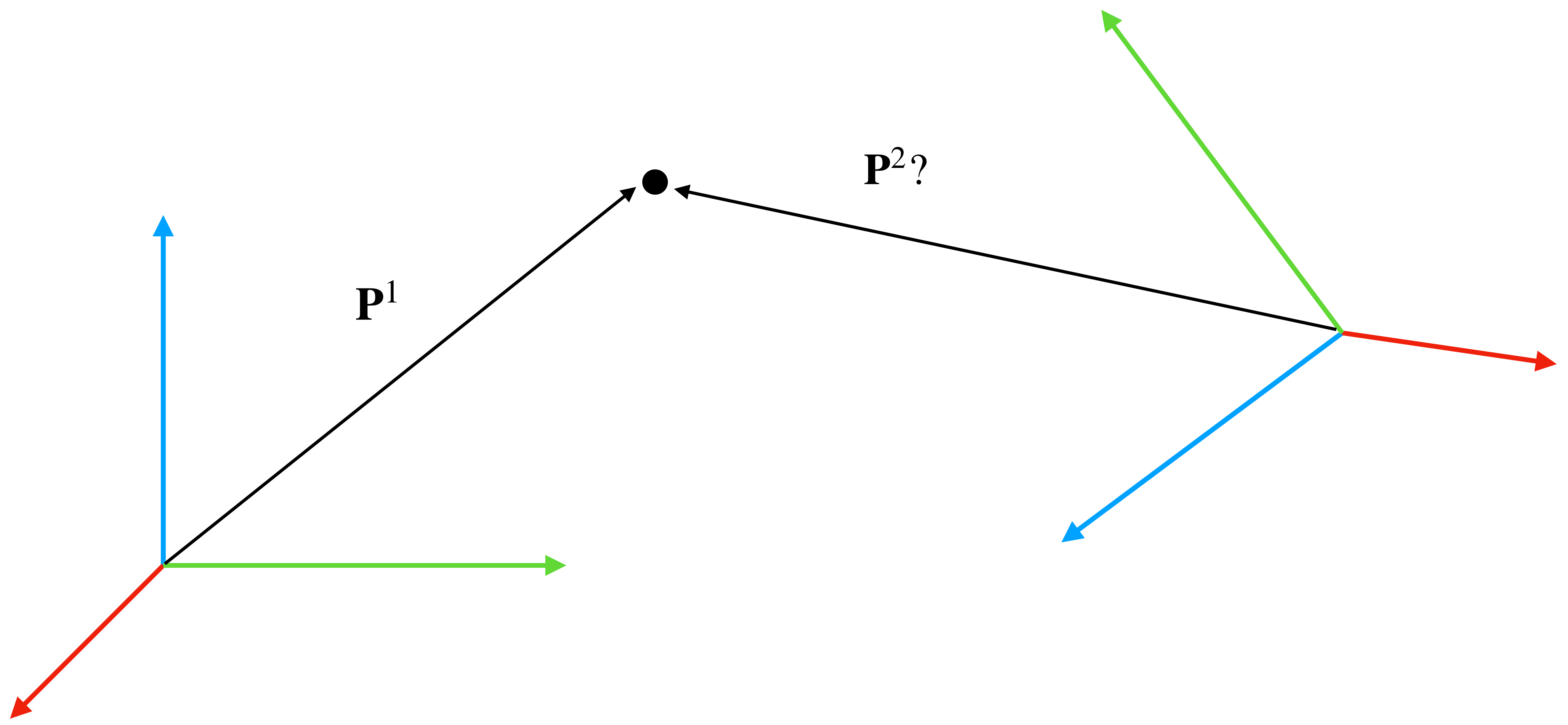
$$\mathbf{p}_1^2 = \mathbf{p}_2 - \mathbf{p}_1 = \begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \\ z_2 - z_1 \end{bmatrix} \in \mathbb{R}^3$$

Transformation between reference frames

> The question to ask is two reference frames with different position and orientation.

> What is the distance from frame {2} to the point? What is the orientation?

> Need to consider the relative orientation between reference frames.



Transformation between reference frames

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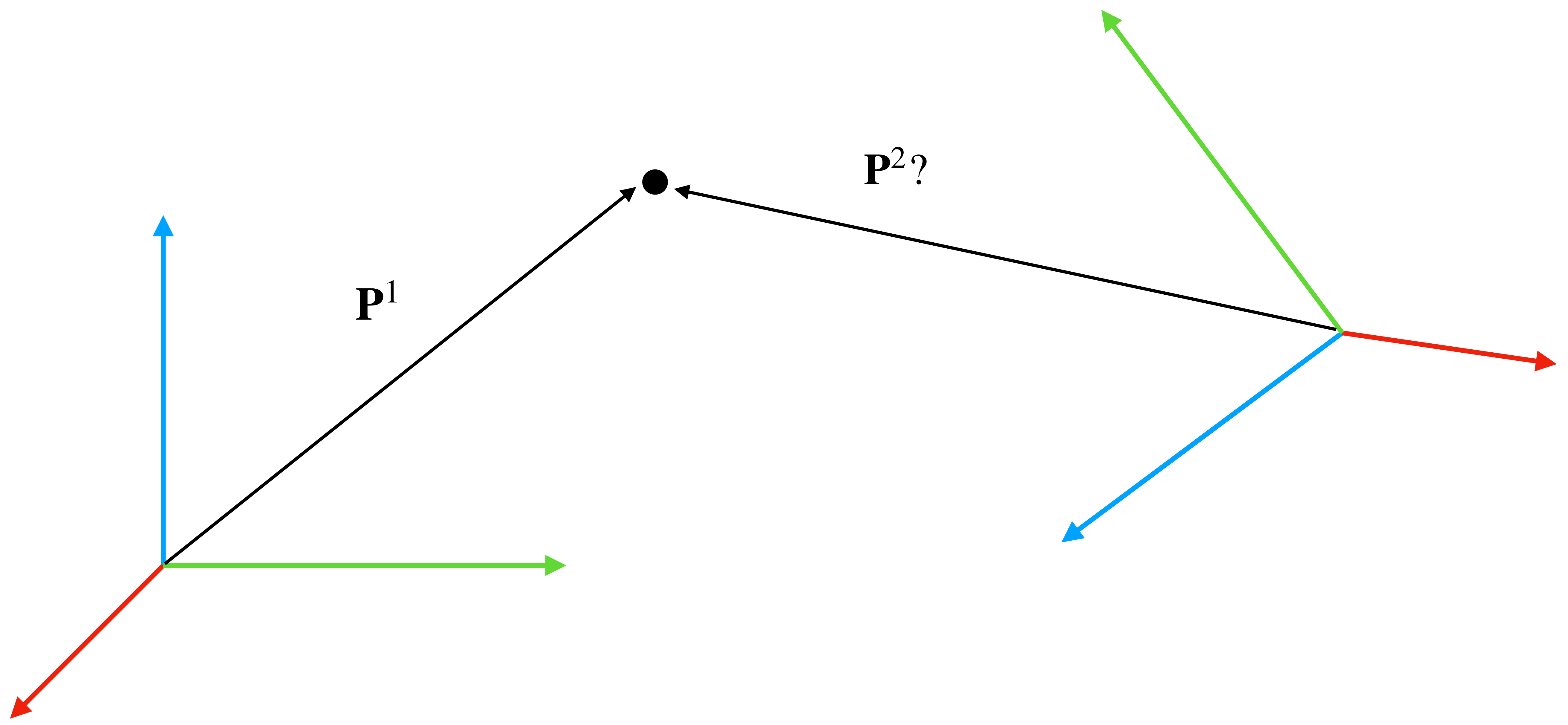
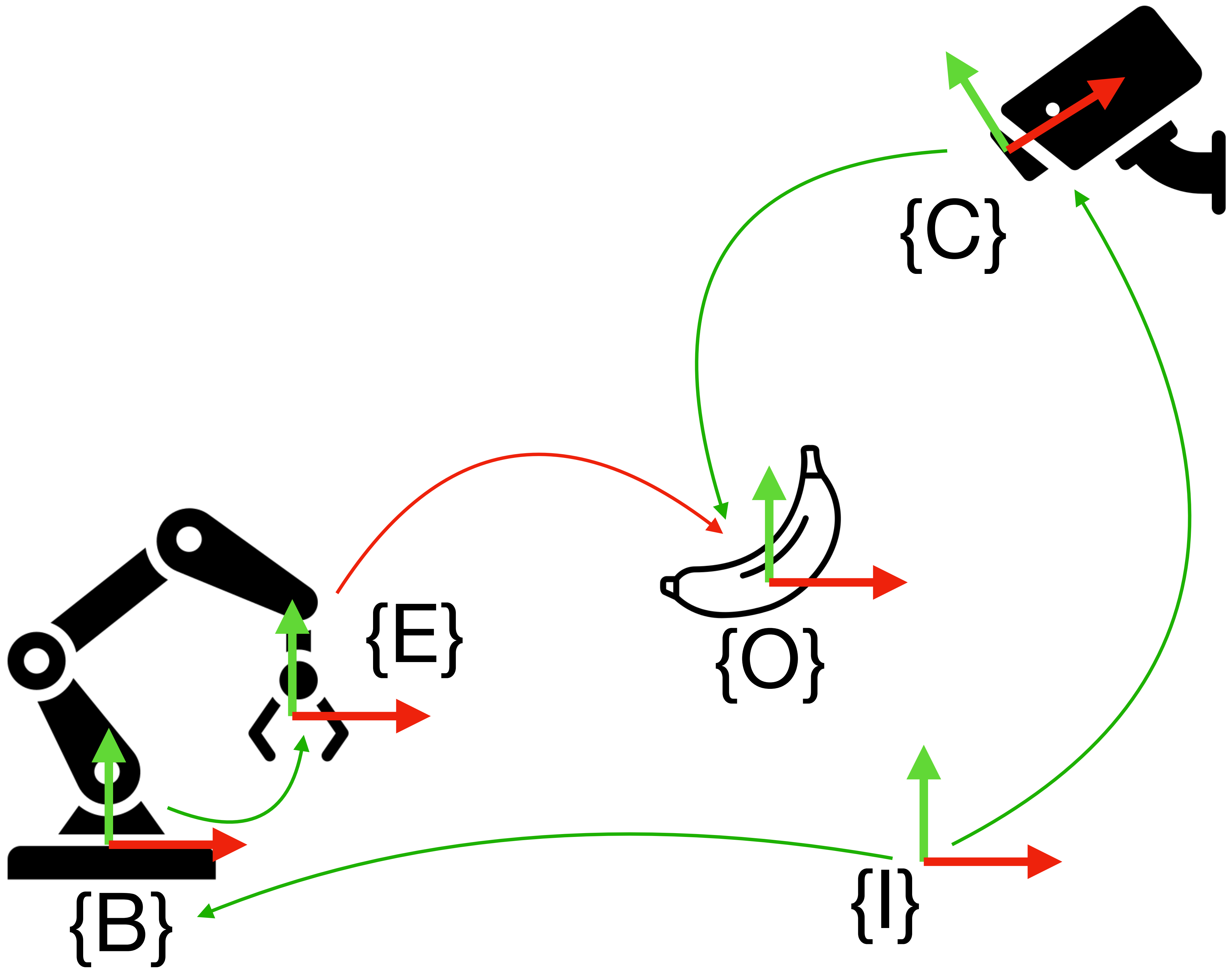


Rotation matrices in 2D

The goal: how can we describe the relative orientation between reference frames?

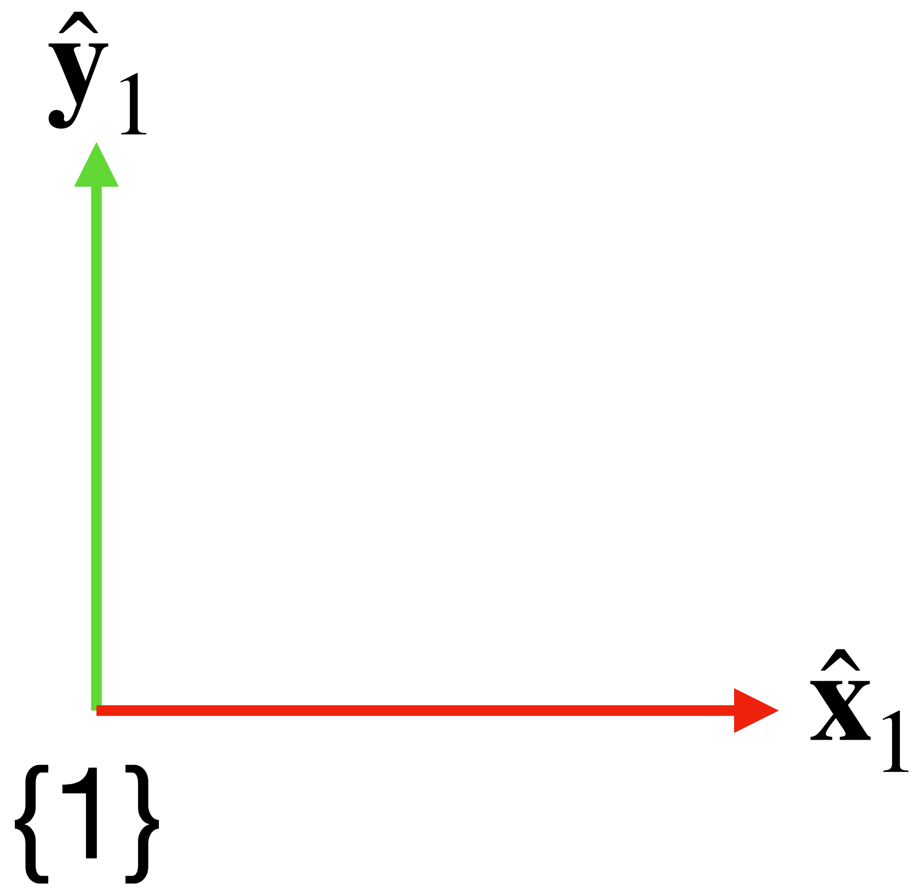


Given the following rotation:

- $\mathbf{R}_I^C$ -> Inertial to camera
- $\mathbf{R}_C^O$ -> Camera to Object
- $\mathbf{R}_B^I$ -> Intertial to Base
- $\mathbf{R}_B^E$ -> Base to End-effector

What is $\mathbf{R}_E^O$?

Constructing a Reference Frame in 2D



$\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ are unit vectors:

$$\|\hat{\mathbf{x}}\| = 1$$

$$\|\hat{\mathbf{y}}\| = 1$$

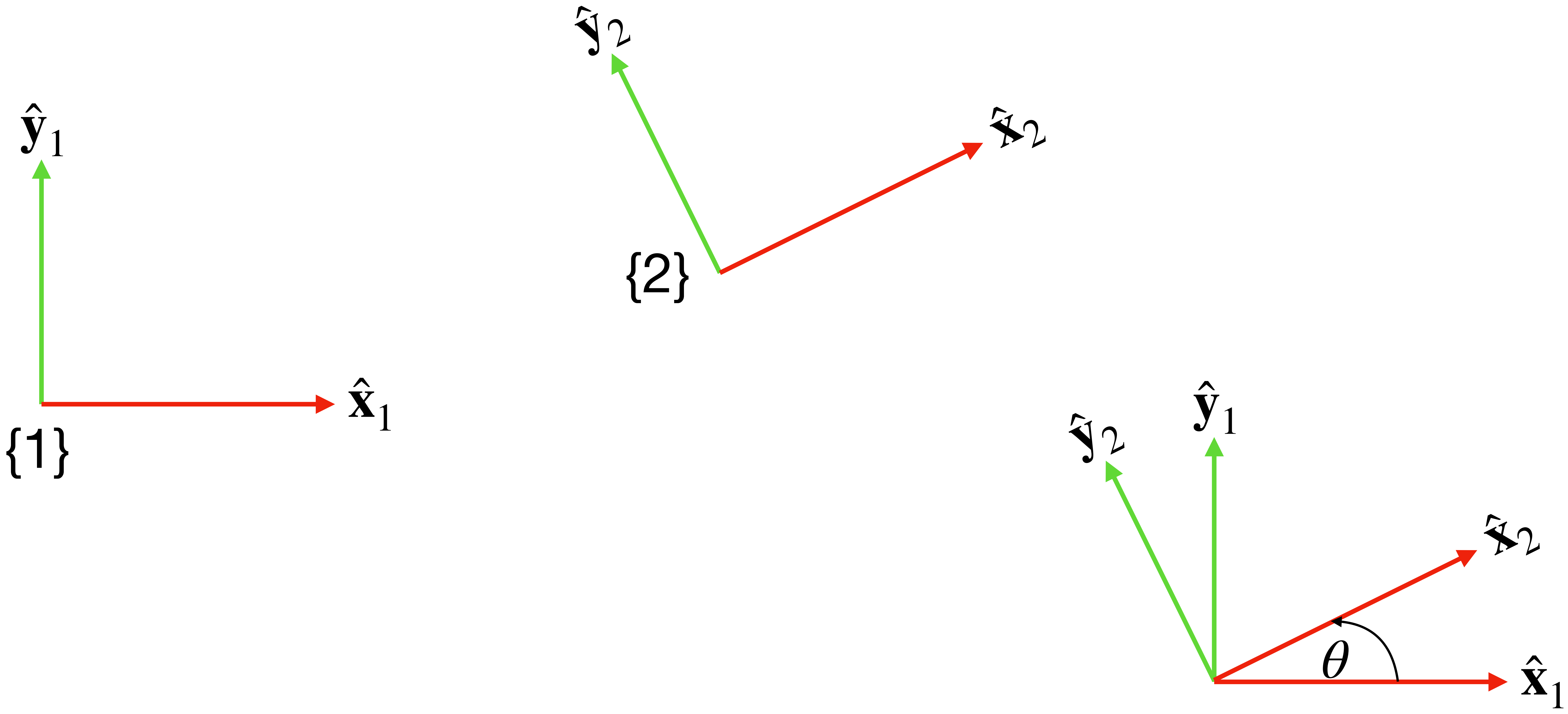
$\hat{\mathbf{x}}$ is orthogonal to $\hat{\mathbf{y}}$.

$$\hat{\mathbf{x}}^\top \hat{\mathbf{y}} = 0$$

$\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ are orthonormal.

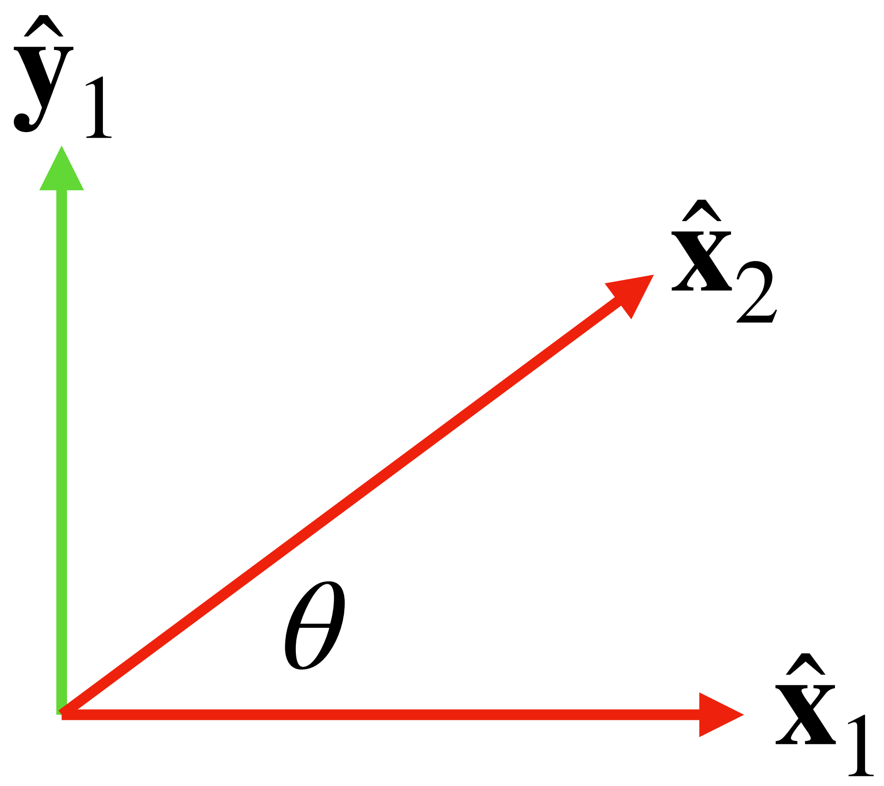
Orthogonal + normalized

Rotation between 2 Reference Frames in 2D



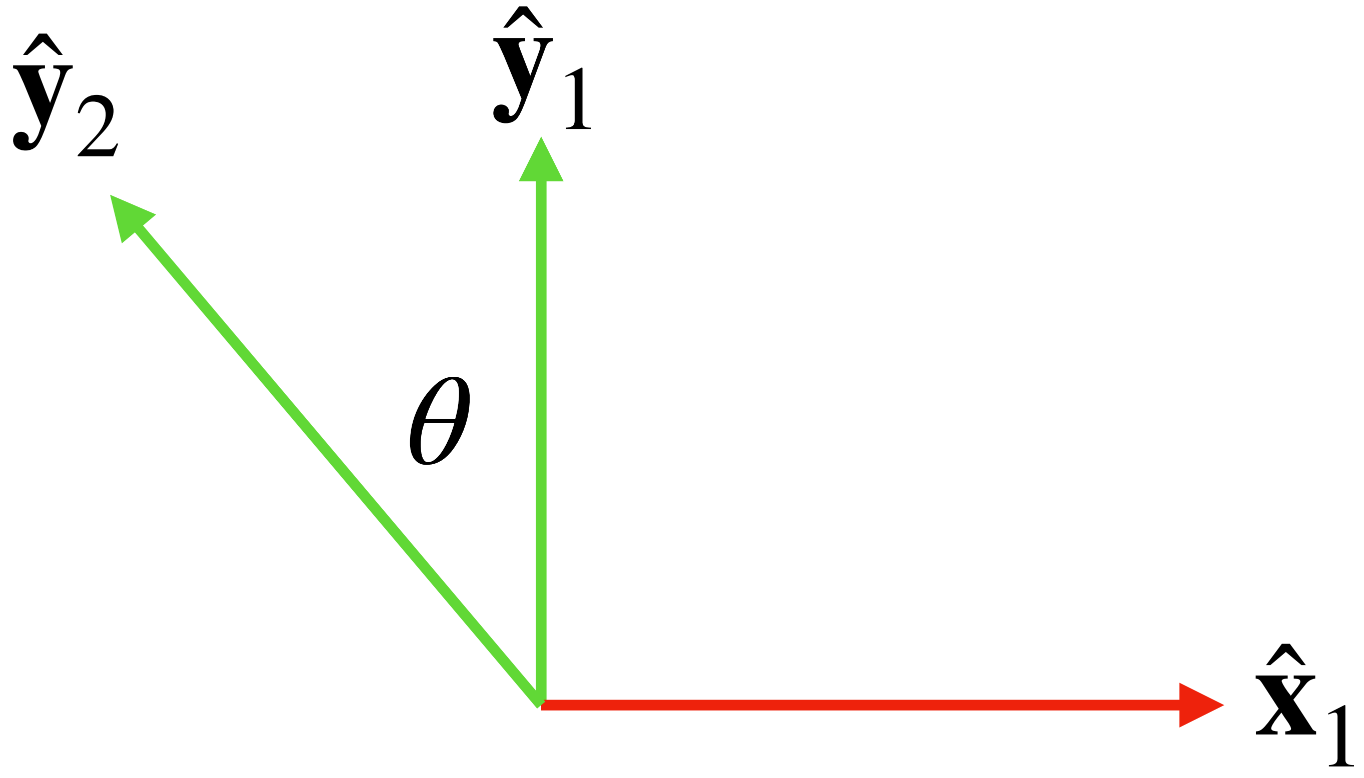
How can we describe {2} with respect to {1} mathematically?

Rotation between 2 Reference Frames in 2D



Express axes of frame {1} as functions of axes of frame {2}:

$$\begin{bmatrix} \hat{x}_1 \\ \hat{y}_1 \end{bmatrix} = \begin{bmatrix} \hat{x}_2 \cos(\theta) - \hat{y}_2 \sin(\theta) \\ \hat{x}_2 \sin(\theta) + \hat{y}_2 \cos(\theta) \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} \hat{x}_2 \\ \hat{y}_2 \end{bmatrix}$$



Define the Rotation Matrix from {1} to {2} as:

$$\mathbf{R}_1^2(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

The transpose of a rotation matrix is equivalent to its inverse

Multiply the rotation matrix by its transpose:

$$\mathbf{RR}^{\mathrm{T}} = \begin{bmatrix} \cos(\psi) & -\sin(\psi) \\ \sin(\psi) & \cos(\psi) \end{bmatrix} \begin{bmatrix} \cos(\psi) & \sin(\psi) \\ -\sin(\psi) & \cos(\psi) \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2(\psi) + \sin^2(\psi) & \cos(\psi)\sin(\psi) - \cos(\psi)\sin(\psi) \\ \sin(\psi)\cos(\psi) - \sin(\psi)\cos(\psi) & \sin^2(\psi) + \cos^2(\psi) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{I}$$

We also know that:

$$\mathbf{RR}^{-1} = \mathbf{I}$$

$$\therefore \mathbf{R}^{-1} = \mathbf{R}^{\mathrm{T}}$$

The Rotation Matrix is **orthogonal**.

The transpose of a rotation matrix is equivalent to its inverse

Rotation from {1} to {2}:

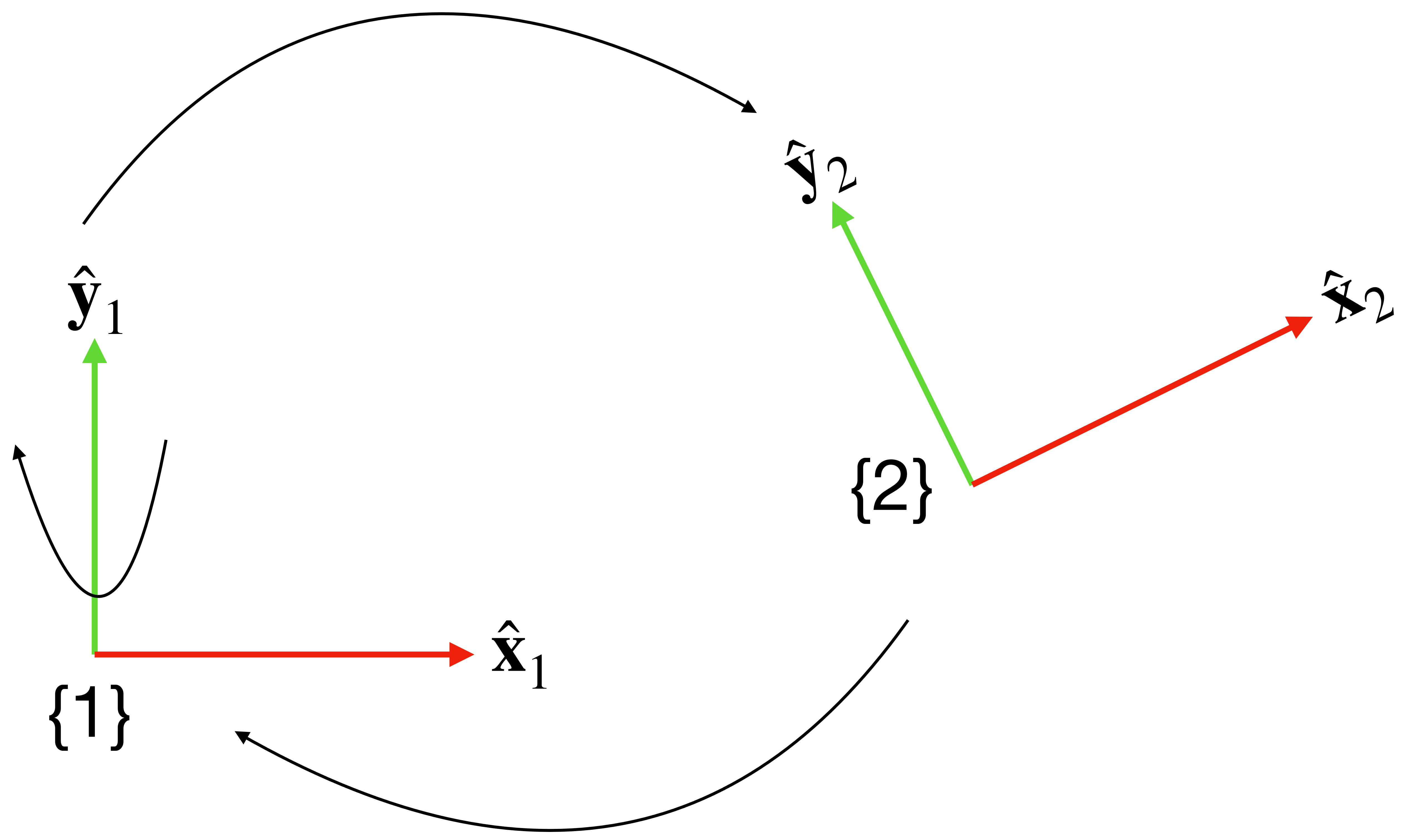
$$\mathbf{R}_1^2$$

Rotation from {2} to {1}:

$$\mathbf{R}_2^1 = \left(\mathbf{R}_1^2\right)^{\mathrm{T}}$$

Rotation from {1} to {1}:

$$\mathbf{R}_1^1 = \mathbf{R}_1^2\mathbf{R}_2^1 = \mathbf{R}_1^2\left(\mathbf{R}_1^2\right)^{\mathrm{T}}$$



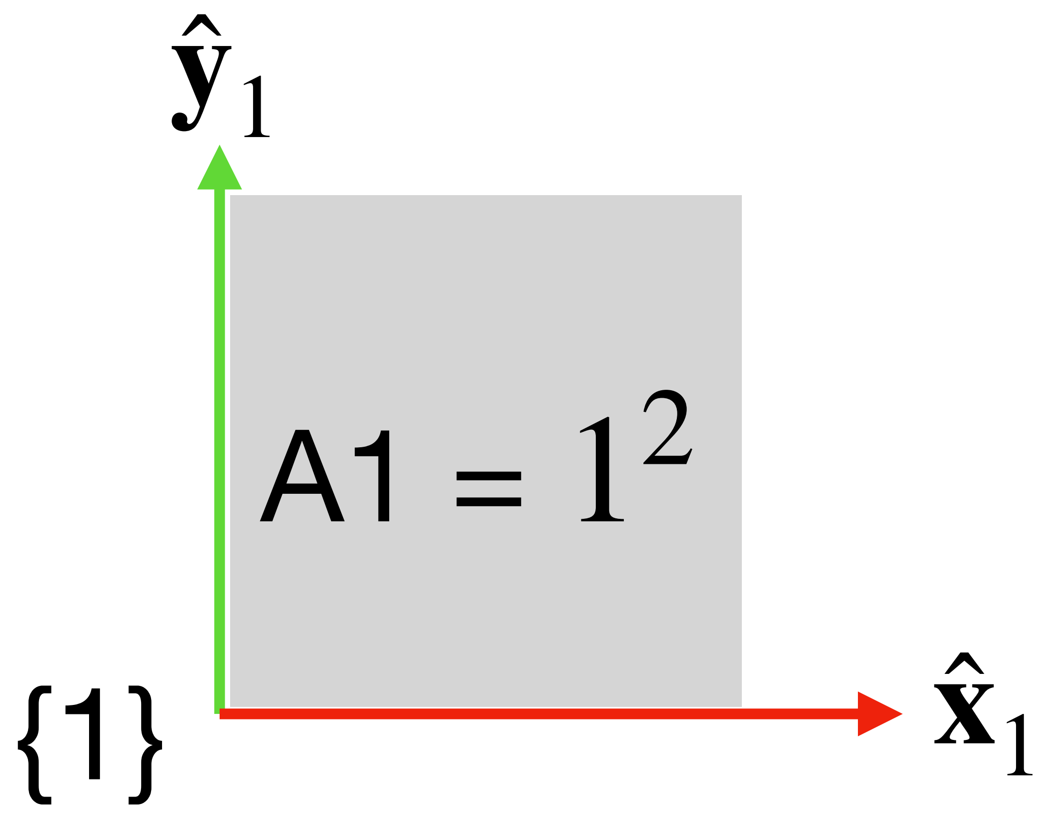
The Determinant of a Rotation Matrix is 1

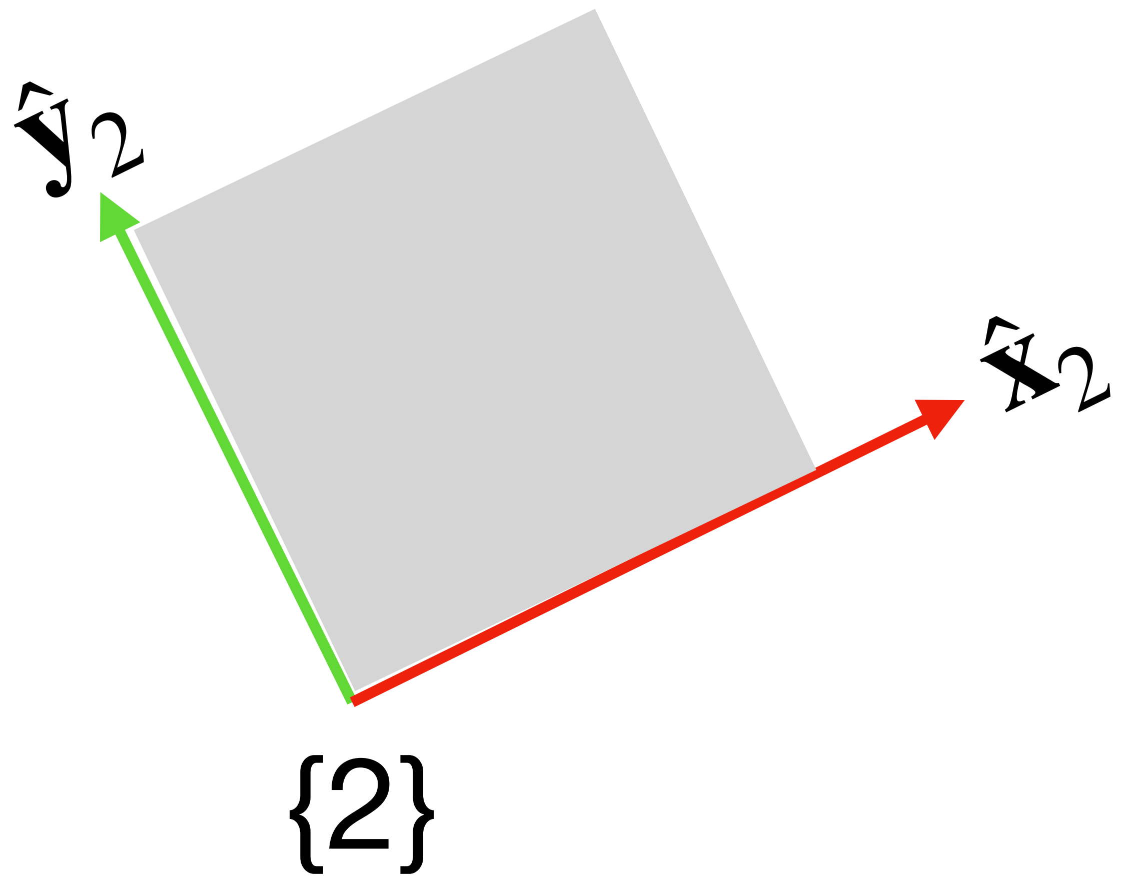
$$\det(\mathbf{R}) = \begin{vmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{vmatrix} = \cos(\theta)\cos(\theta) - \sin(\theta)(-\sin(\theta)) = \cos^2(\theta) + \sin^2(\theta) = 1$$

This means the area (or volume for 3D) bounded by the axes remains constant.

i.e scaled by 1

A2 = det(R)A1 = 1^2





A Rotation Matrix is in the Special Orthogonal Group (SO)

$$\mathbf{R}\mathbf{R}^\top = \mathbf{R}^\top\mathbf{R} = \mathbf{I}$$
$$\|\mathbf{R}\| = 1$$
$$\det(\mathbf{R}) = 1$$

If R is an n x n matrix with the above properties, then R is SO(n).

Implications to Invariance Under Rotation

Velocity for observer {A}:

$$^{A}\mathbf{v}$$

Velocity for observer {B}:

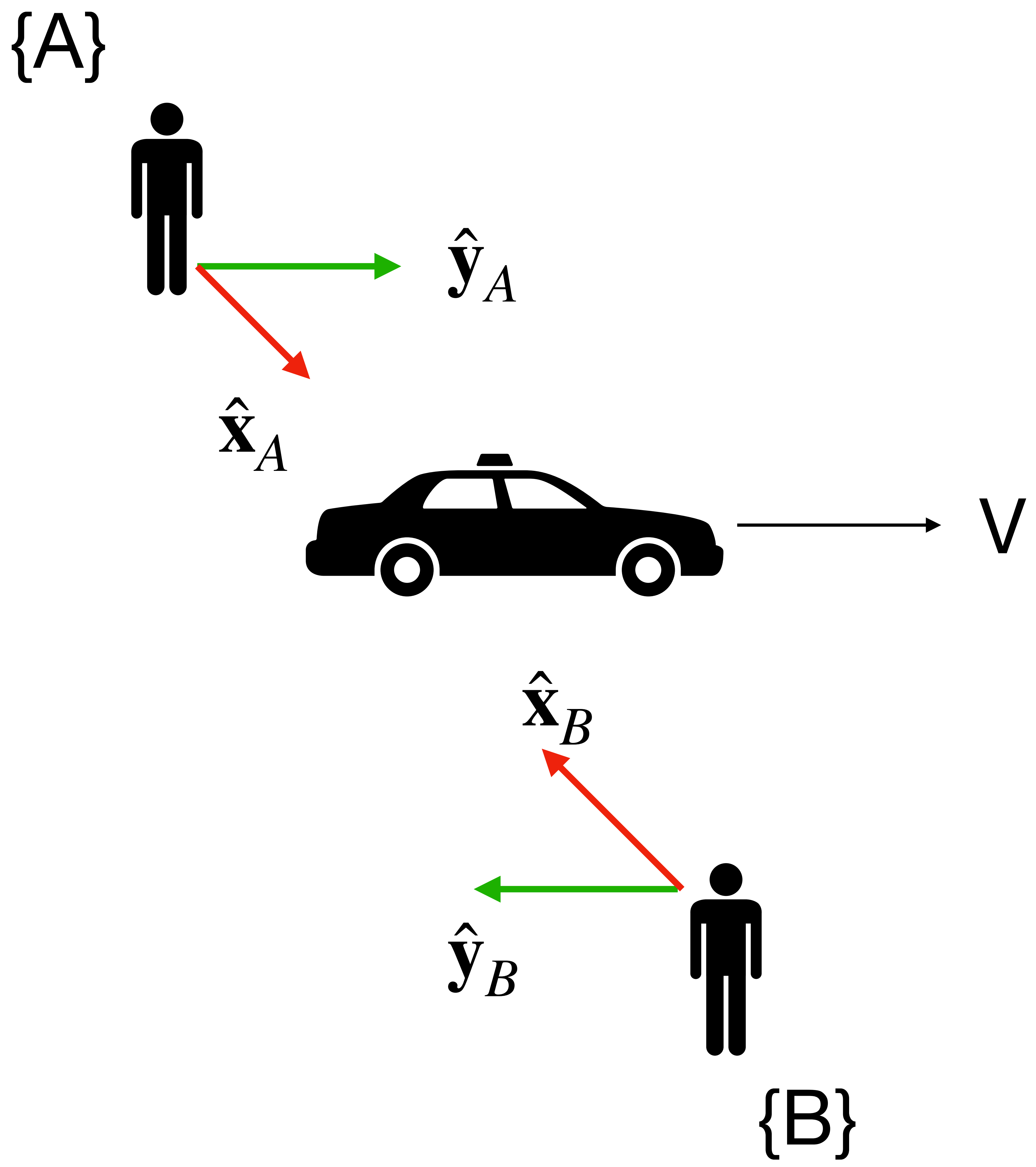
$$^{B}\mathbf{v} = \mathbf{R}_{B}^{A} \cdot {}^{A}\mathbf{v}$$

Direction of the vectors is different:

$$^{A}\mathbf{v} \neq {}^{B}\mathbf{v}$$

But the magnitude will be the same:

$$\|^{A}\mathbf{v}\| = \|^{B}\mathbf{v}\|$$

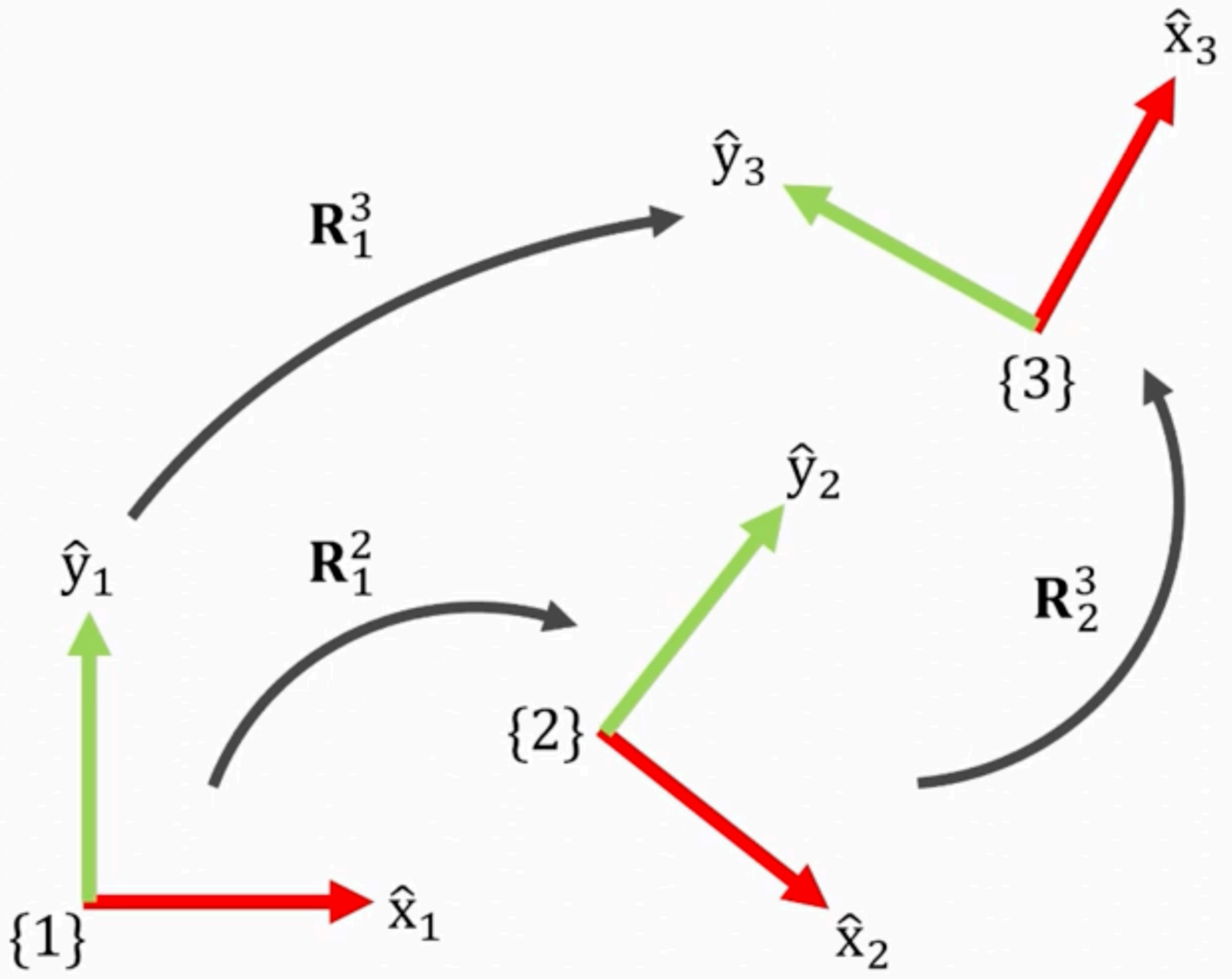


$$\mathbf{R}_1^3 = \mathbf{R}_1^2\mathbf{R}_2^3$$

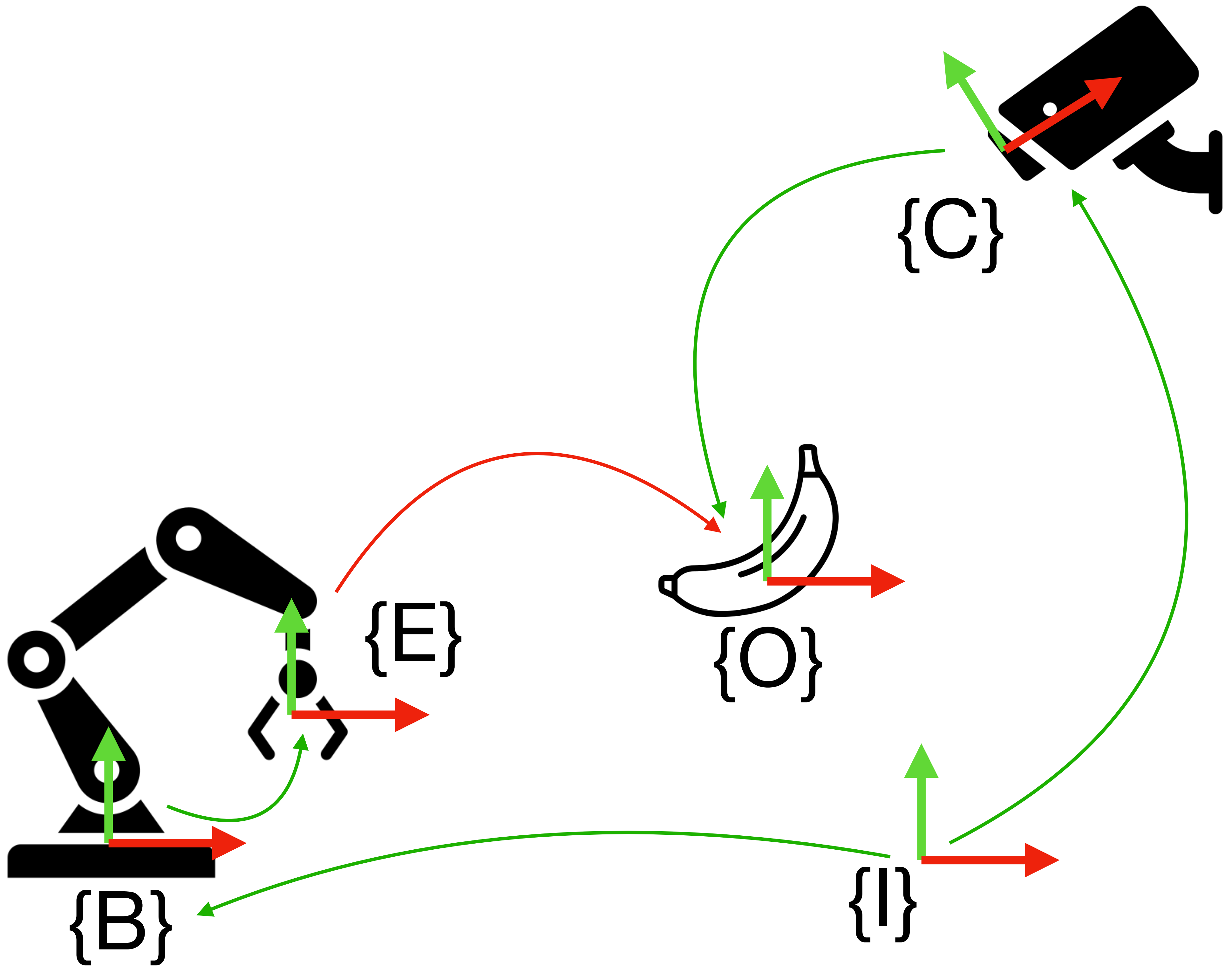
$$\begin{aligned}\mathbf{R}_1^3\left(\mathbf{R}_1^3\right)^{\mathrm{T}} &= \left(\mathbf{R}_1^2\mathbf{R}_2^3\right)^{\mathrm{T}}\left(\mathbf{R}_1^2\mathbf{R}_2^3\right)\\ &= \left(\mathbf{R}_2^3\right)^{\mathrm{T}}\left(\mathbf{R}_1^2\right)^{\mathrm{T}}\mathbf{R}_1^2\mathbf{R}_2^3\\ &= \left(\mathbf{R}_2^3\right)^{\mathrm{T}}\mathbf{R}_2^3\\ &= \mathbf{I}\end{aligned}$$

$$\begin{aligned}\det\left(\mathbf{R}_1^3\right) &= \det\left(\mathbf{R}_1^2\mathbf{R}_2^3\right)\\ &= \det\left(\mathbf{R}_1^2\right)\det\left(\mathbf{R}_2^3\right)\\ &= 1 \times 1\\ &= 1\end{aligned}$$

$\mathbf{R}_1^3 \in \mathbb{SO}$ is another rotation matrix!



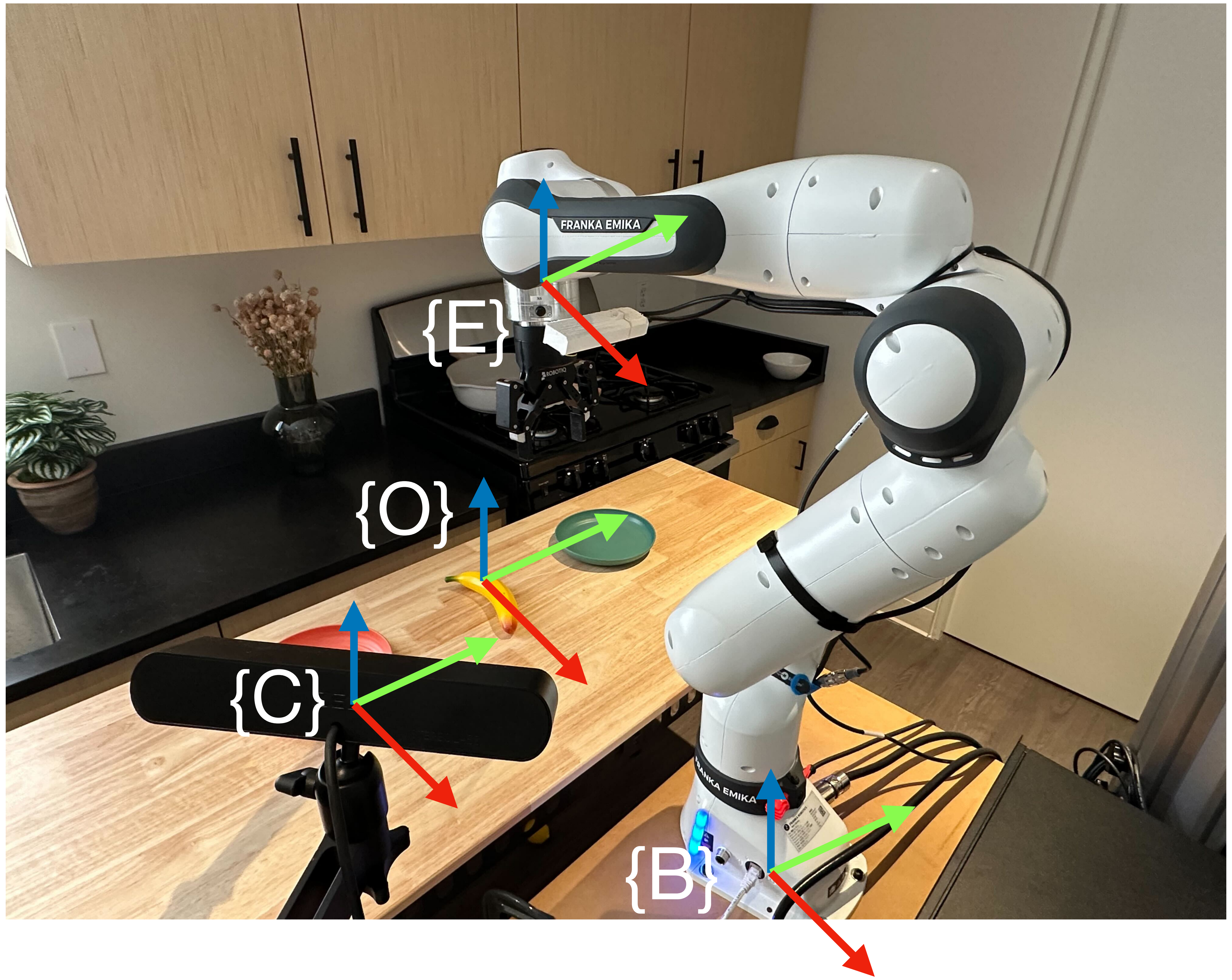
How should the robot orient its End-effector?



$$\mathbf{R}_E^O = \mathbf{R}_E^B \mathbf{R}_B^I \mathbf{R}_I^C \mathbf{R}_C^O$$

Rotation matrices in 3D

The goal: how can we describe the relative orientation between reference frames?

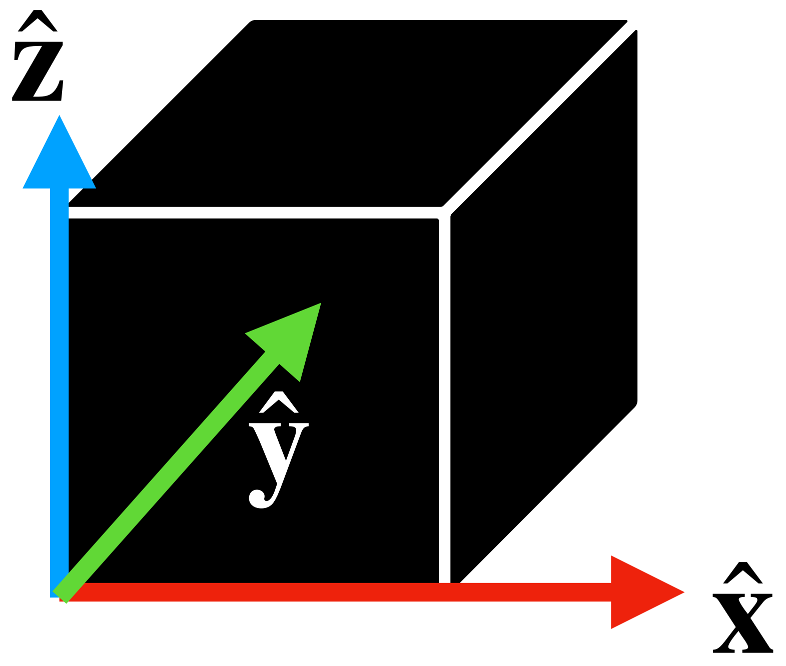


What is $\mathbf{R}_E^O$?

3D Rotation Matrix (Properties)

Add a 3rd dimension to the rotation matrix:

$$\mathbf{R} = \begin{bmatrix}\hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}}\end{bmatrix} \in SO(3)$$



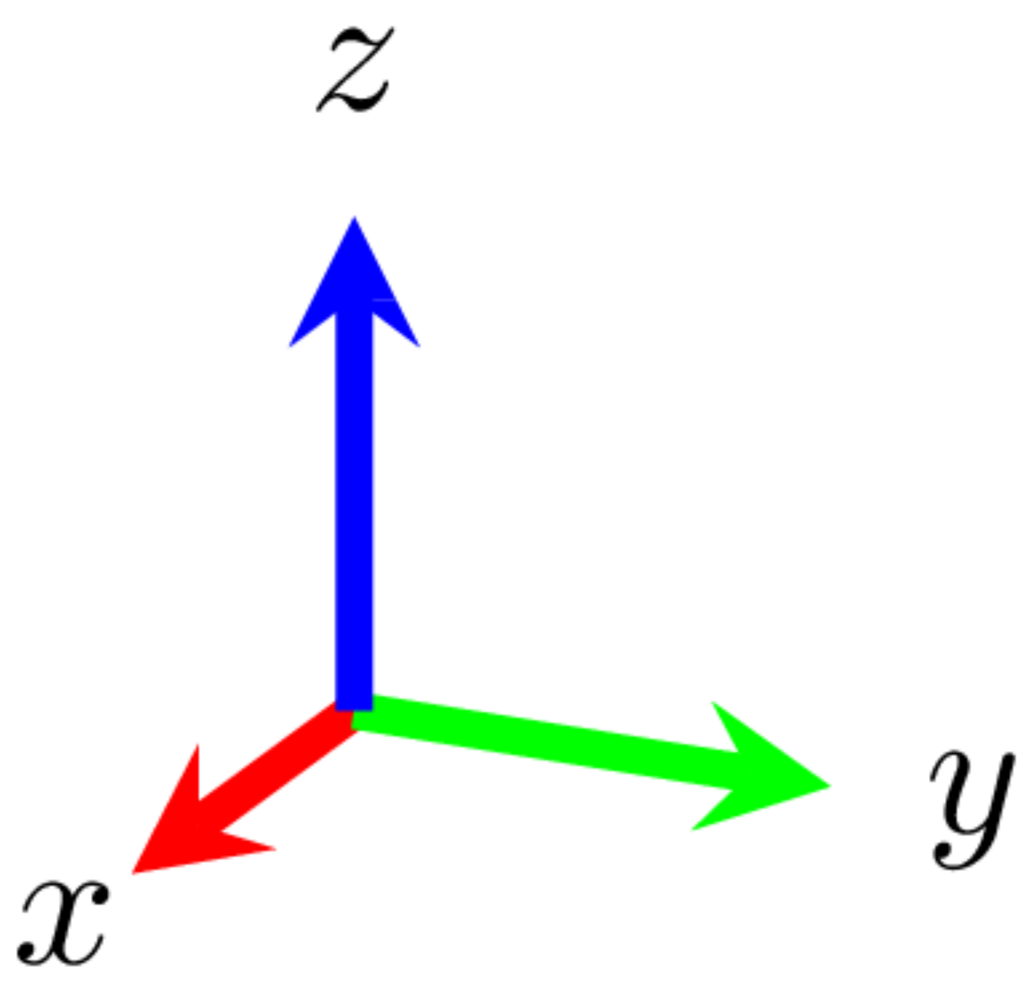
Each column is a 3D unit vector

$$\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}} \in \mathbb{R}^3$$

$$\| \hat{\mathbf{x}} \| = \| \hat{\mathbf{y}} \| = \| \hat{\mathbf{z}} \| = 1$$

All columns are orthonormal

$$\hat{\mathbf{x}}^\top \hat{\mathbf{y}} = \hat{\mathbf{x}}^\top \hat{\mathbf{z}} = \hat{\mathbf{y}}^\top \hat{\mathbf{z}} = 0$$



3D Rotation Matrix (Properties)

Add a 3rd dimension to the rotation matrix:

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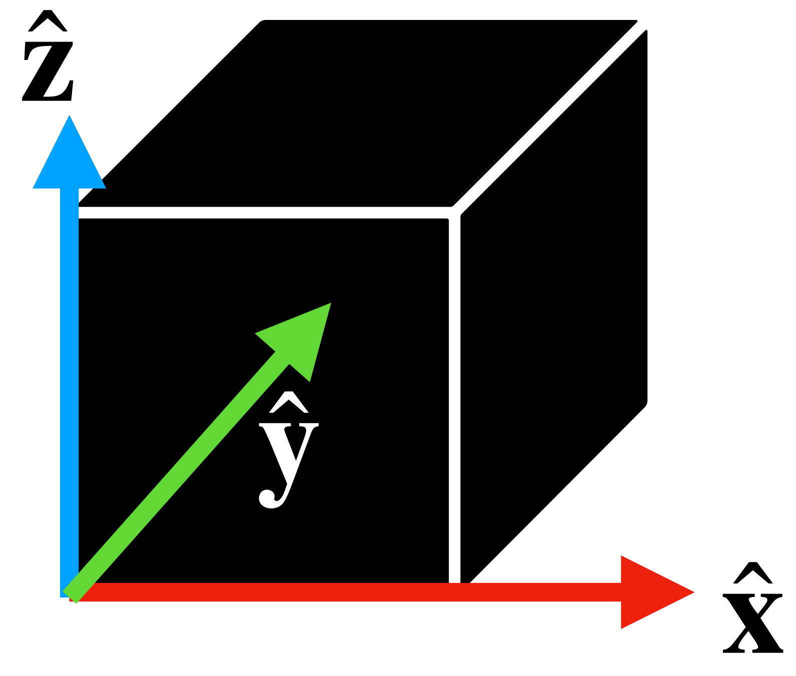

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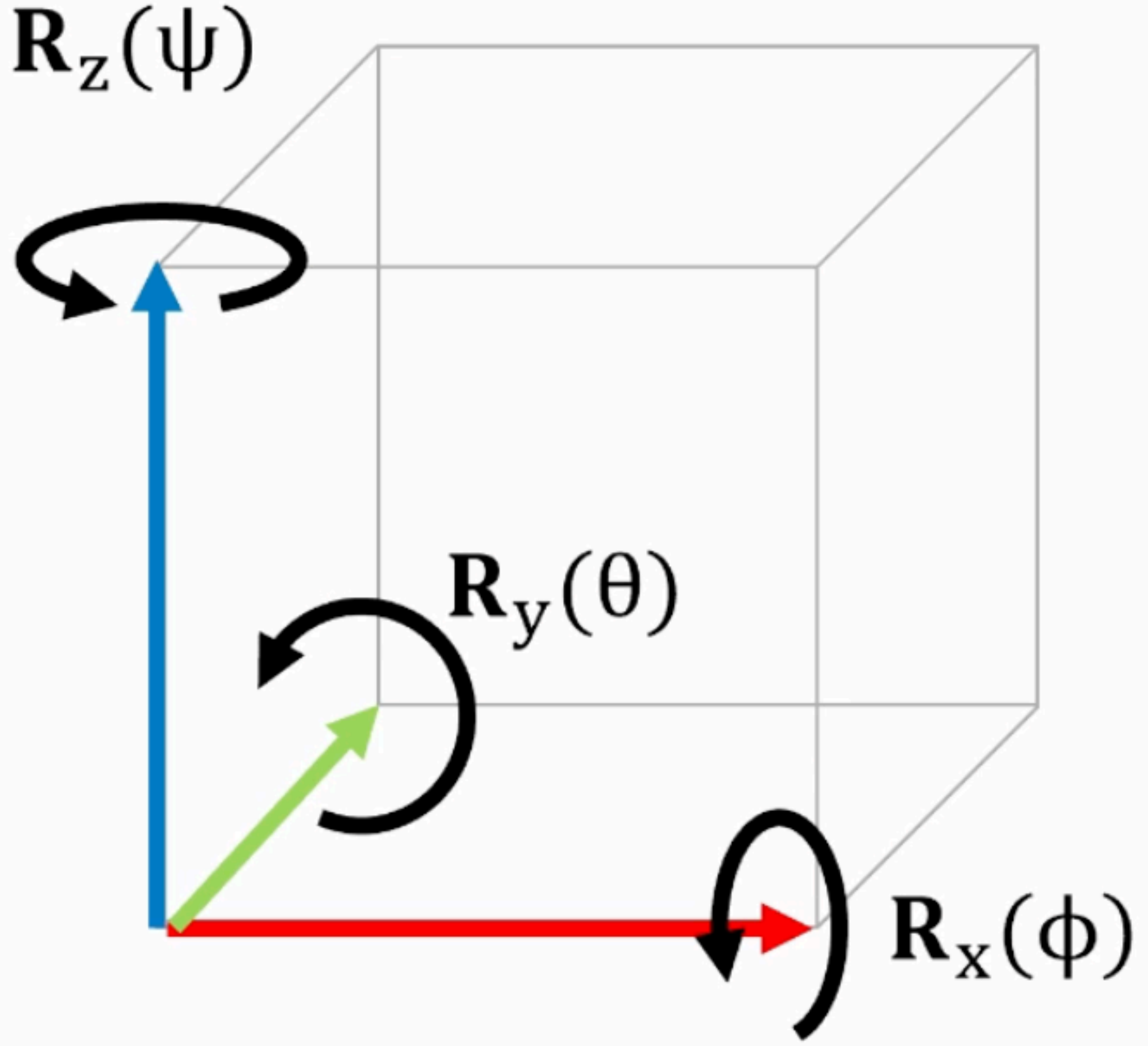
$$\hat{\mathbf{x}}^\top \hat{\mathbf{y}} = \hat{\mathbf{x}}^\top \hat{\mathbf{z}} = \hat{\mathbf{y}}^\top \hat{\mathbf{z}} = 0$$

Elementary Rotations

$$\mathbf{R}_x(\phi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\phi) & -\sin(\phi) \\ 0 & \sin(\phi) & \cos(\phi) \end{bmatrix}$$

$$\mathbf{R}_y(\theta) = \begin{bmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{bmatrix}$$

$$\mathbf{R}_z(\psi) = \begin{bmatrix} \cos(\psi) & -\sin(\psi) & 0 \\ \sin(\psi) & \cos(\psi) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



Build any rotation from 3 sequential rotations

-In 3D, any orientation can be represented as at most three rotations about coordinate axes.

-You cannot rotate about the same axis twice in a row.

-In total, there are 3x2x2=12 total combinations:

1. First axis: 3 choices

2. Second axis: 2 choices (cannot equal first)

3. Third axis: 2 choices (cannot equal second)

Euler Angles:

Same axis twice

$3 \times 2 \times 1 = 6$ combinations

XYX	XYZ	XZX	XZY
YXY	YXZ	YZX	YZY
ZXY	ZXZ	ZYX	ZYZ

Cardan Angles:

All 3 axes

$3 \times 2 \times 1 = 6$ combinations

XYX	XYZ	XZX	XZY
YXY	YXZ	YZX	YZY
ZXY	ZXZ	ZYX	ZYZ

Order of Rotation matters!

Matrices are not commutative

$$R_X R_Y R_Z \neq R_Z R_Y R_X$$

Proof:

$$(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$$

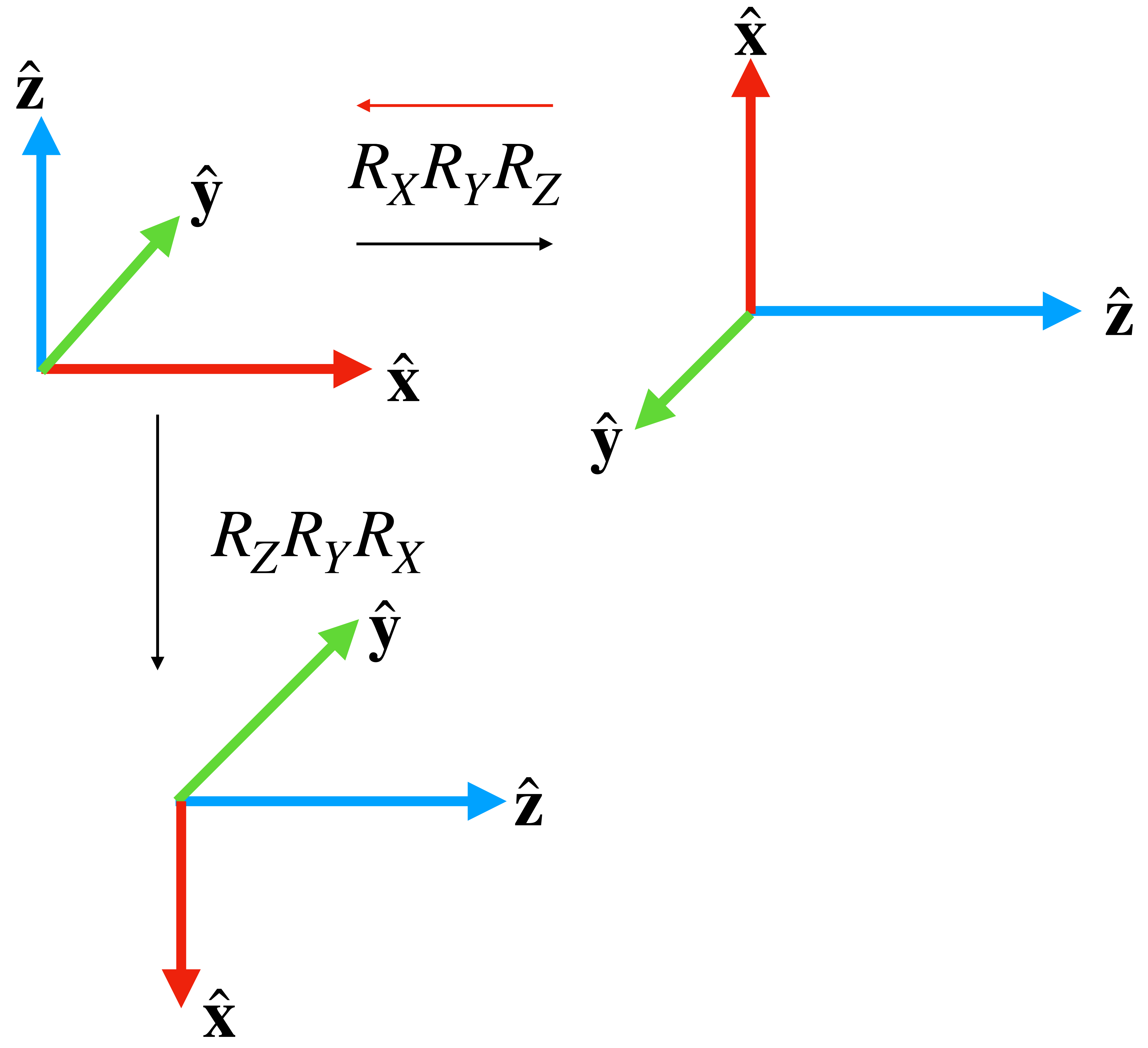
Such that:

$$\begin{aligned}\mathbf{AB}(\mathbf{AB})^{-1} &= \mathbf{ABB}^{-1}\mathbf{A}^{-1} \\ &= \mathbf{AA}^{-1} \\ &= \mathbf{I}\end{aligned}$$

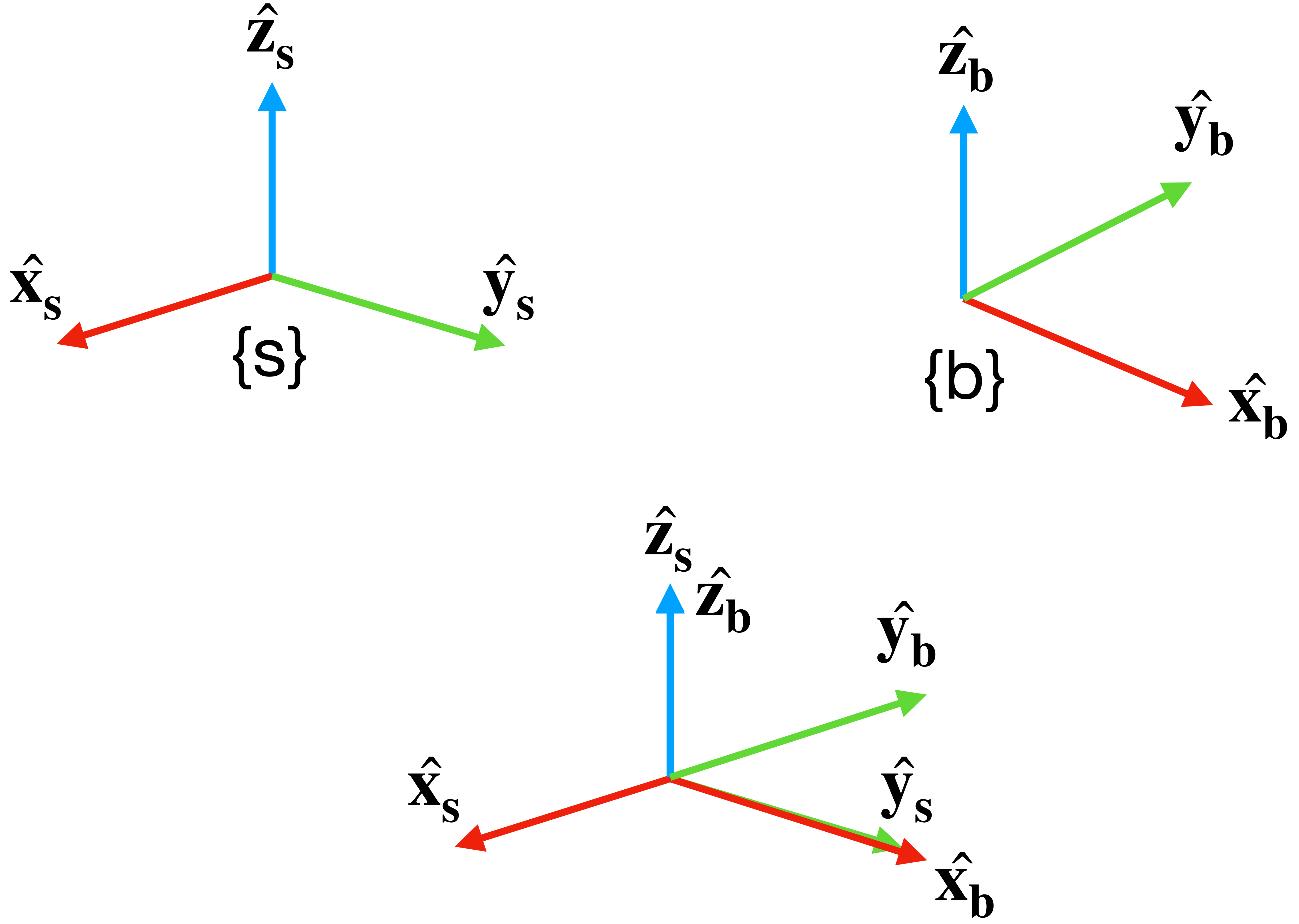
But,

$$\begin{aligned}\mathbf{BA}(\mathbf{AB})^{-1} &= \mathbf{BAB}^{-1}\mathbf{A}^{-1} \\ &\neq \mathbf{I}\end{aligned}$$

$$\therefore \mathbf{AB} \neq \mathbf{BA}$$



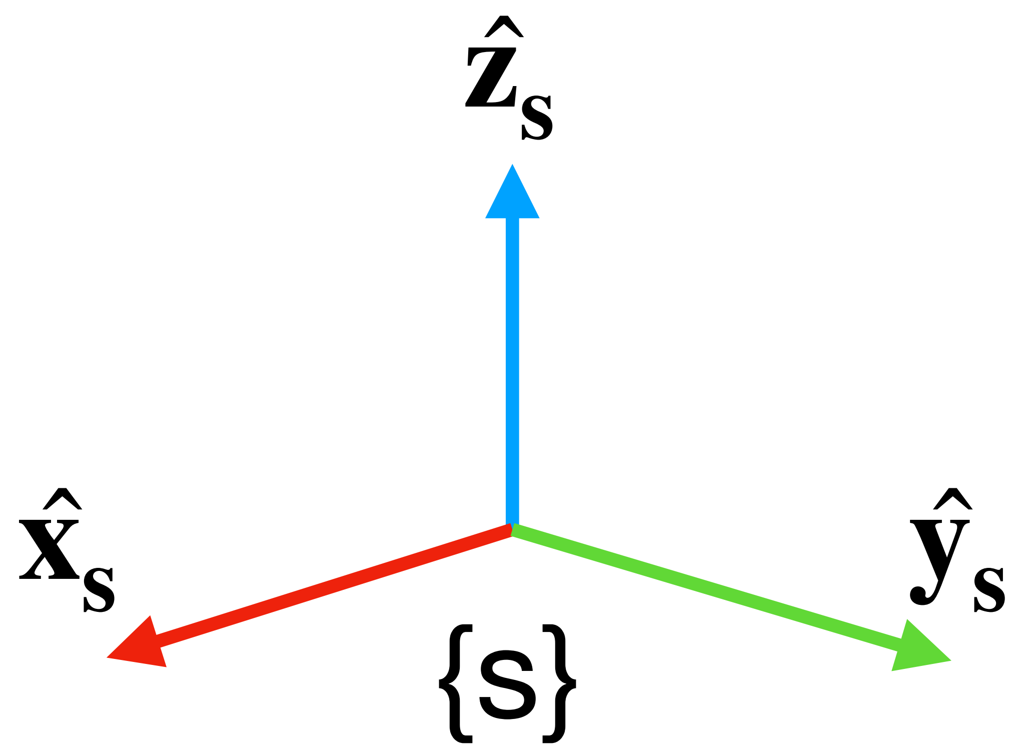
How to use Rotation Matrices

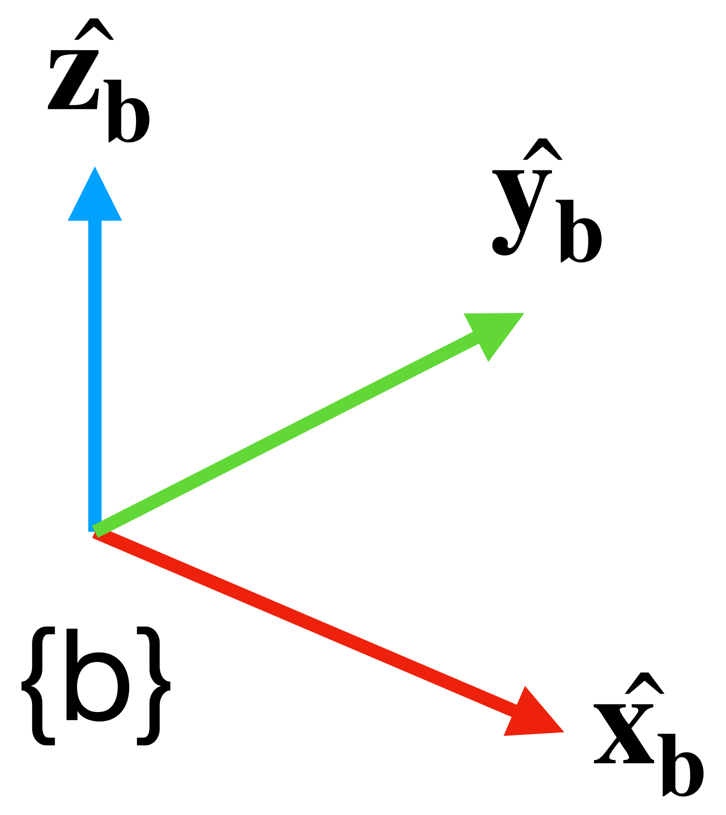


$$\hat{x}_b = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \hat{y}_b = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \quad \hat{z}_b = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$R_{sb} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

How to use Rotation Matrices





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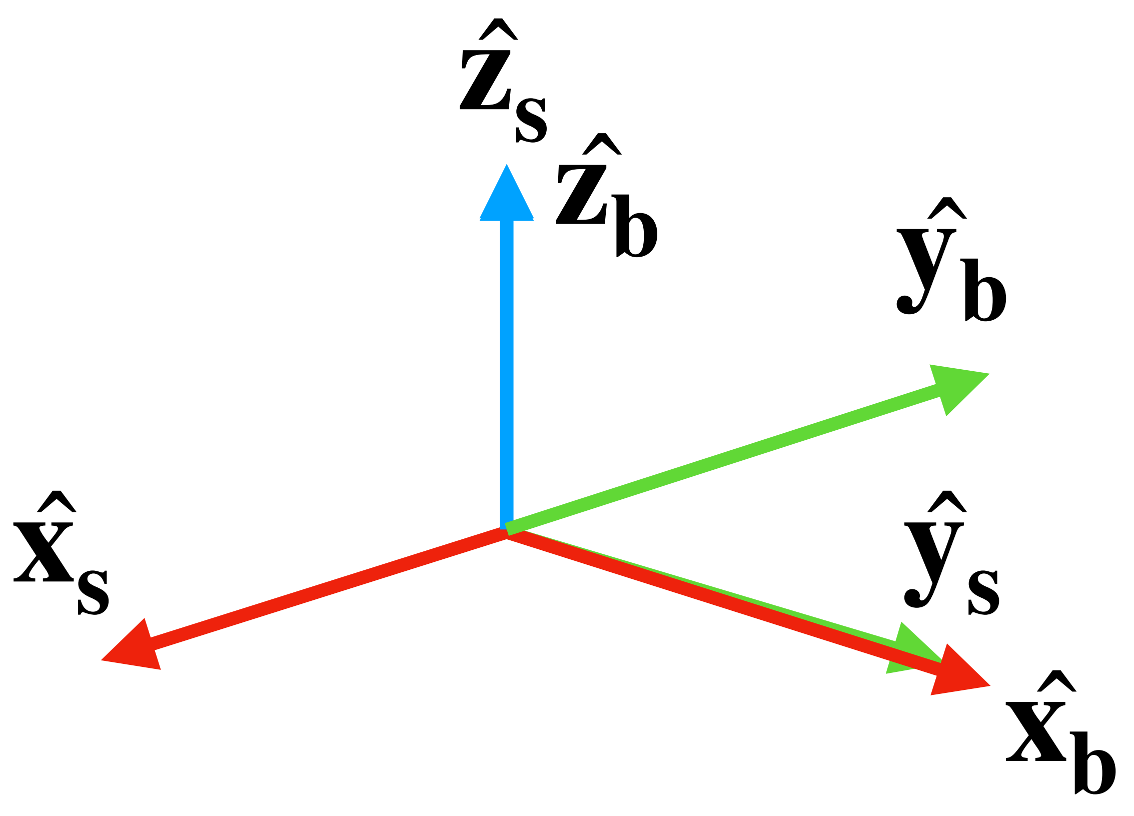
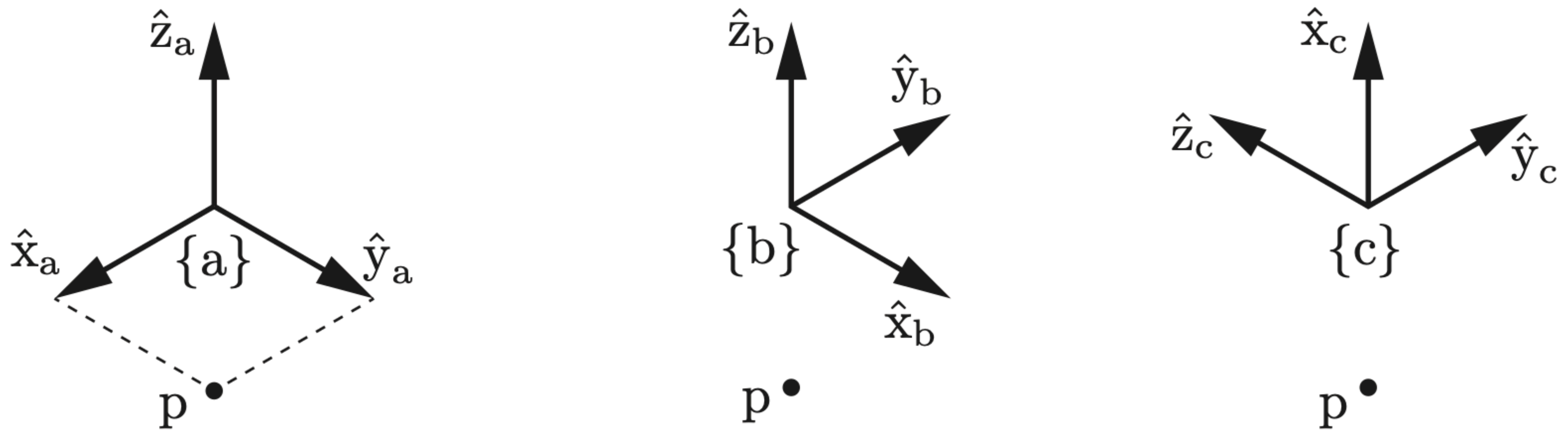


$$R_{sb} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Uses of Rotation Matrices - Represent an orientation

Imagine the three frames have the same origin



Rac frame {c} relative to frame {a}

$$R_{ac} = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix}$$

When we write Rc, we are implicitly referring to the orientation of frame {c} relative to the fixed frame {s}. Hence, it can also be Rsc.

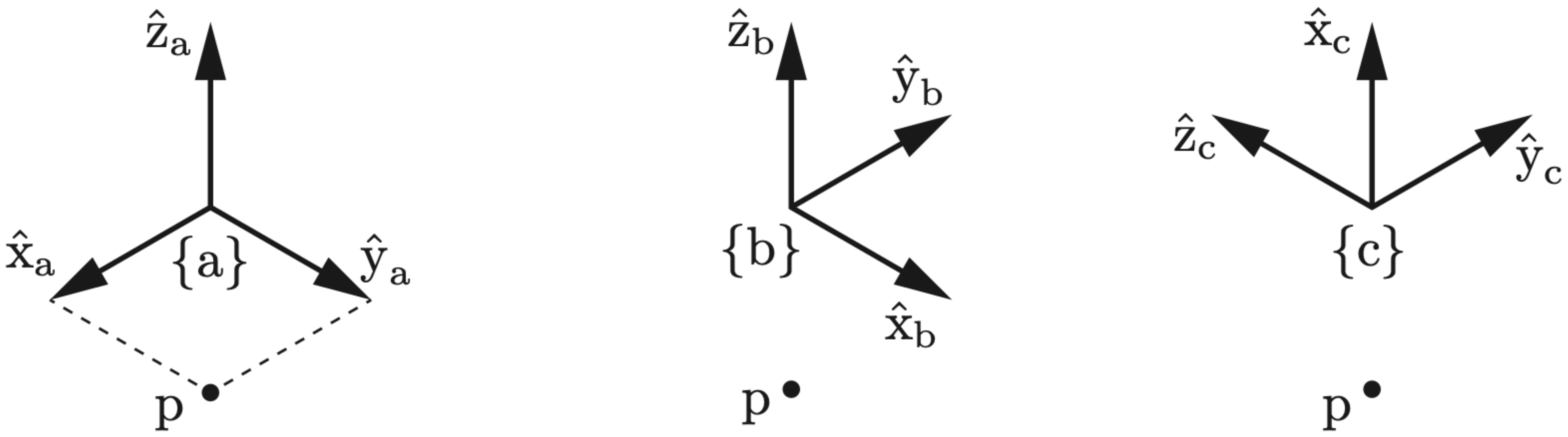
$$R_{ca} = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$$

Transpose this, switch the rows with column

$$R_{ac}R_{ca} = I \qquad R_{ac} = R_{ca}^{-1} \qquad R_{ac} = R_{ca}^{\mathrm{T}}$$

Uses of Rotation Matrices - Change reference frame

Imagine the three frames have the same origin



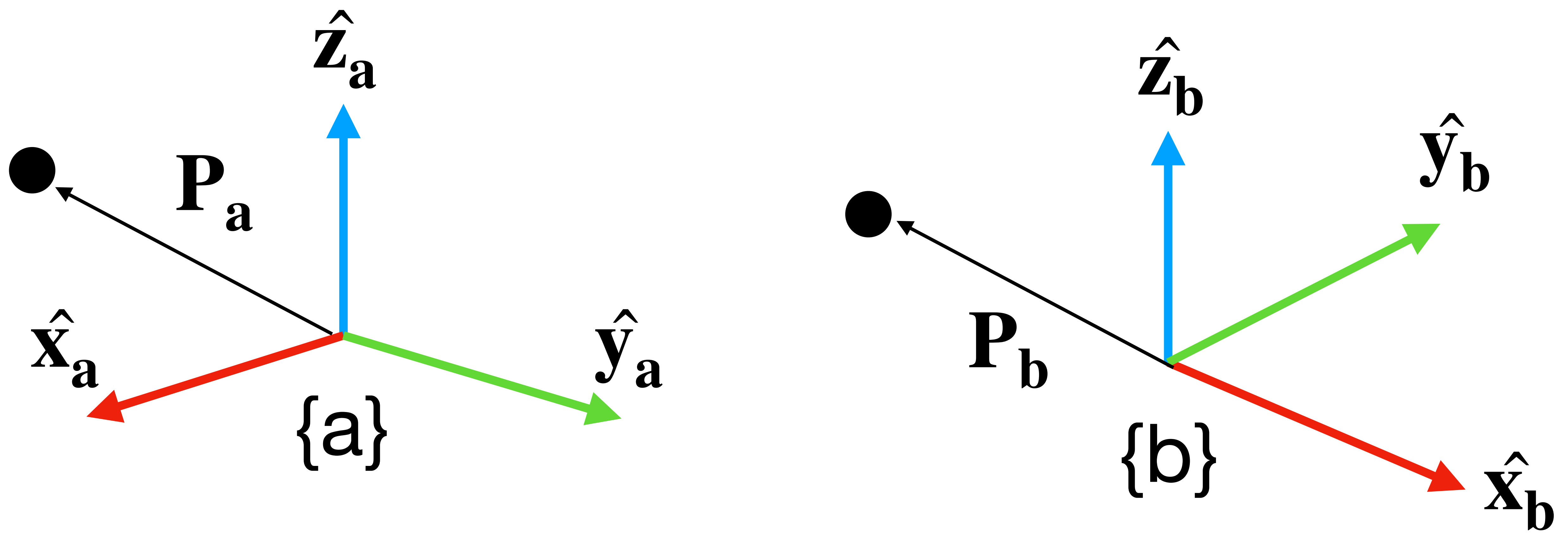
$$R_{bc} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

When we write Rc, we are implicitly referring to the orientation of frame {c} relative to the fixed frame {s}. Hence, it can also be Rsc.

$$R_{ac} = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$R_{ac} = R_{a\not{b}} R_{\not{b}c} = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix}$$

Uses of Rotation Matrices - Change reference frame



How can I get P_s from P_b given R_ab?

$$p_b = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$$

$$p_s = R_{s\not{b}} p_{\not{b}} = R_{sb} \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$$

Changing the Reference Frame

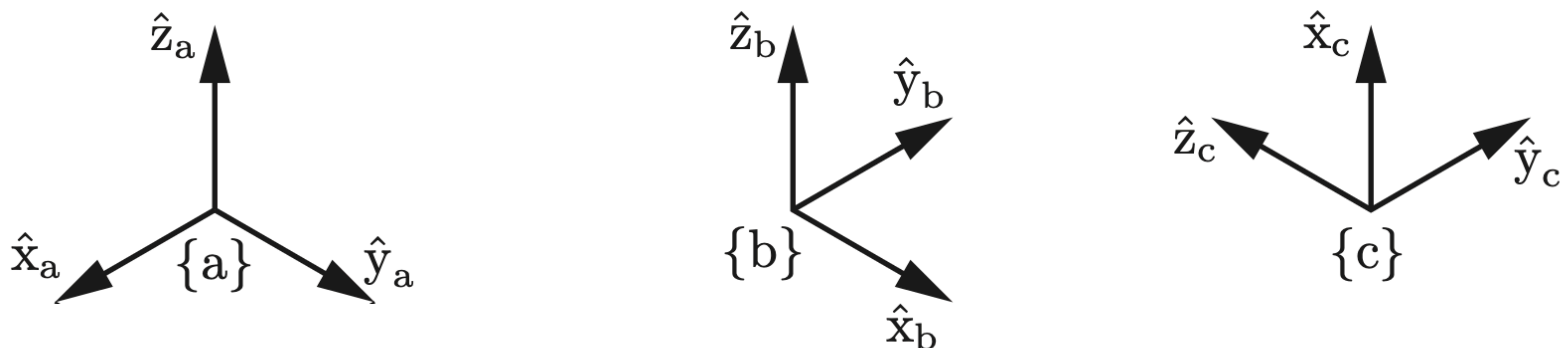
- Orientation of {b} in {a} R_{ab}
- Orientation of {c} in {b} R_{bc}
- Orientation of {c} in {a}

$$R_{ac} = R_{ab}R_{bc} \quad = change\_reference\_frame\_b\_to\_a(R_{bc})$$

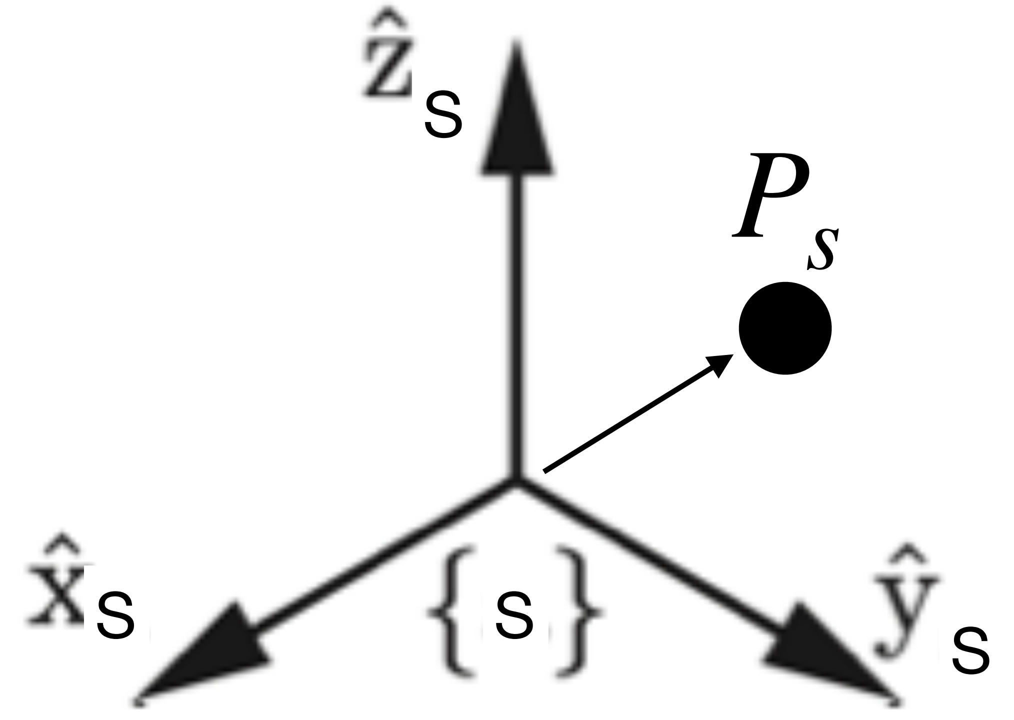
- Subscript cancel rule

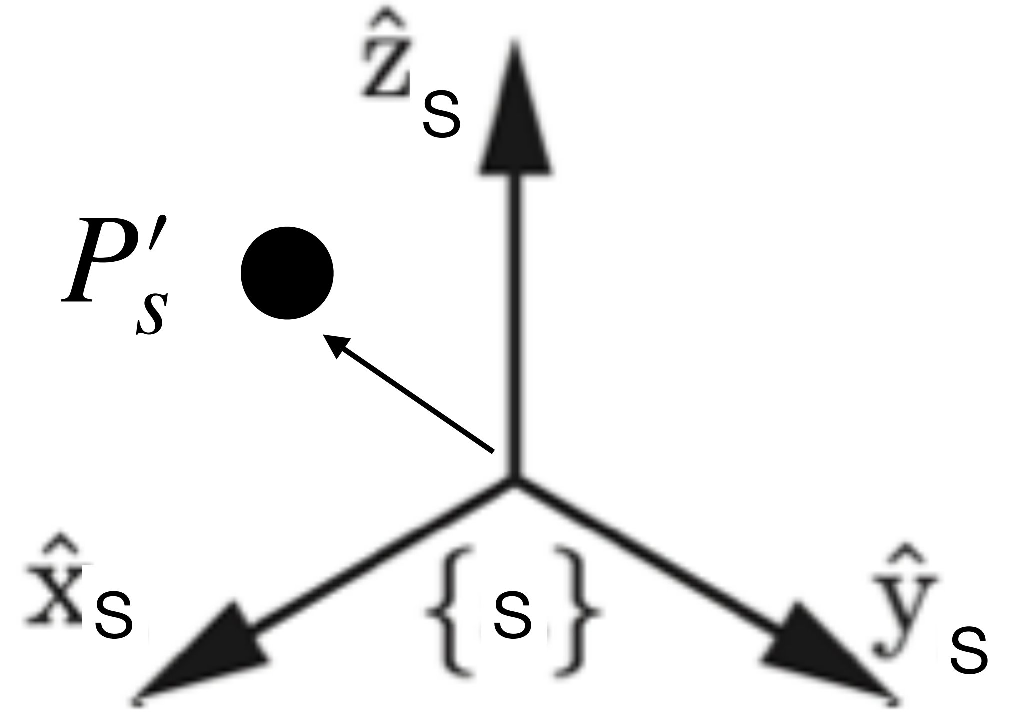
$$R_{ab}R_{bc} = R_{a\not{b}}R_{\not{b}c} = R_{ac} \qquad R_{ab}p_b = R_{a\not{b}}p_{\not{b}} = p_a$$

Rotate a vector or frame - Vector

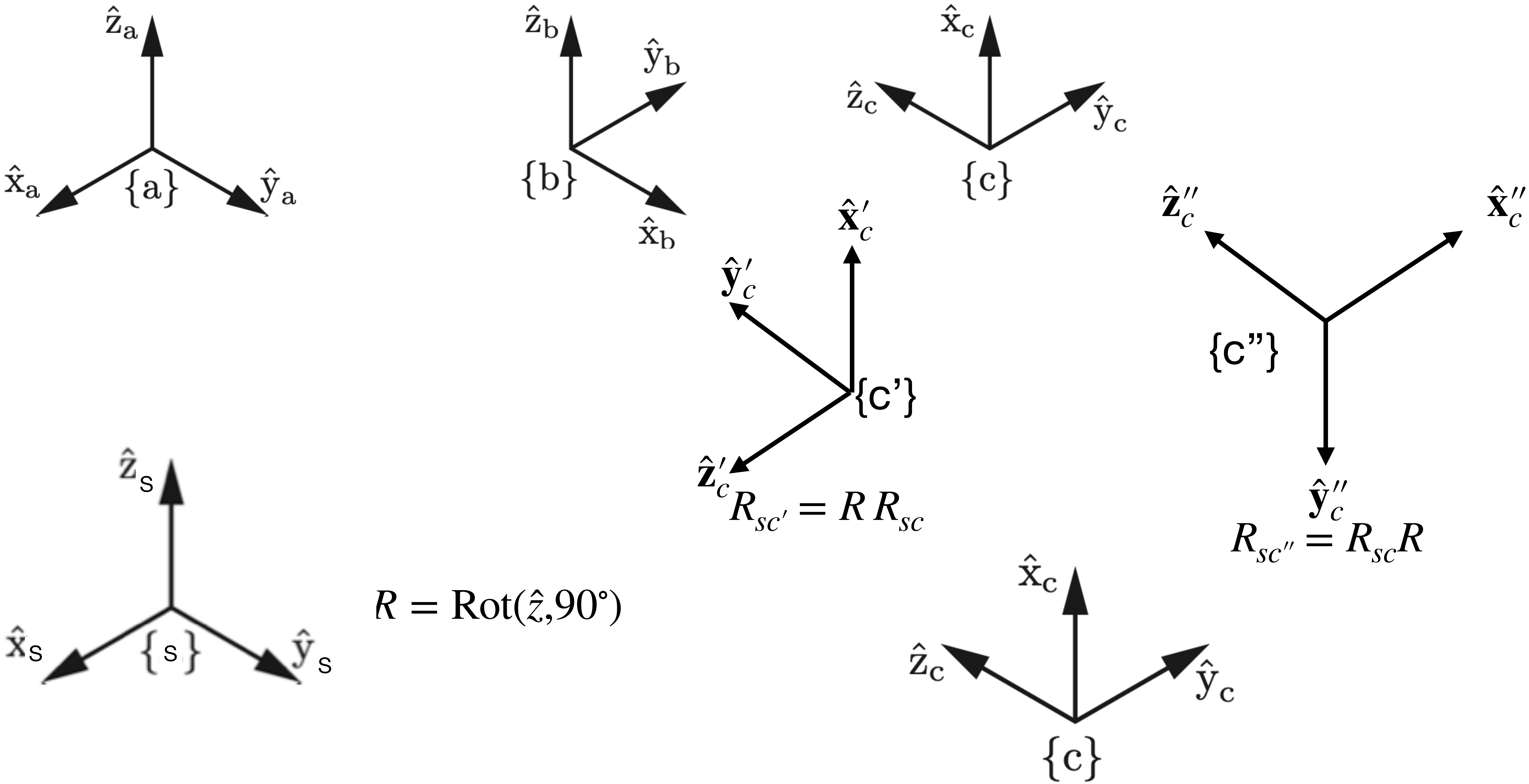


$$R_{sb} = R = \mathrm{Rot}(\hat{z}, 90°) \qquad P_s' = RP_s$$





Rotate a vector or frame - Frame



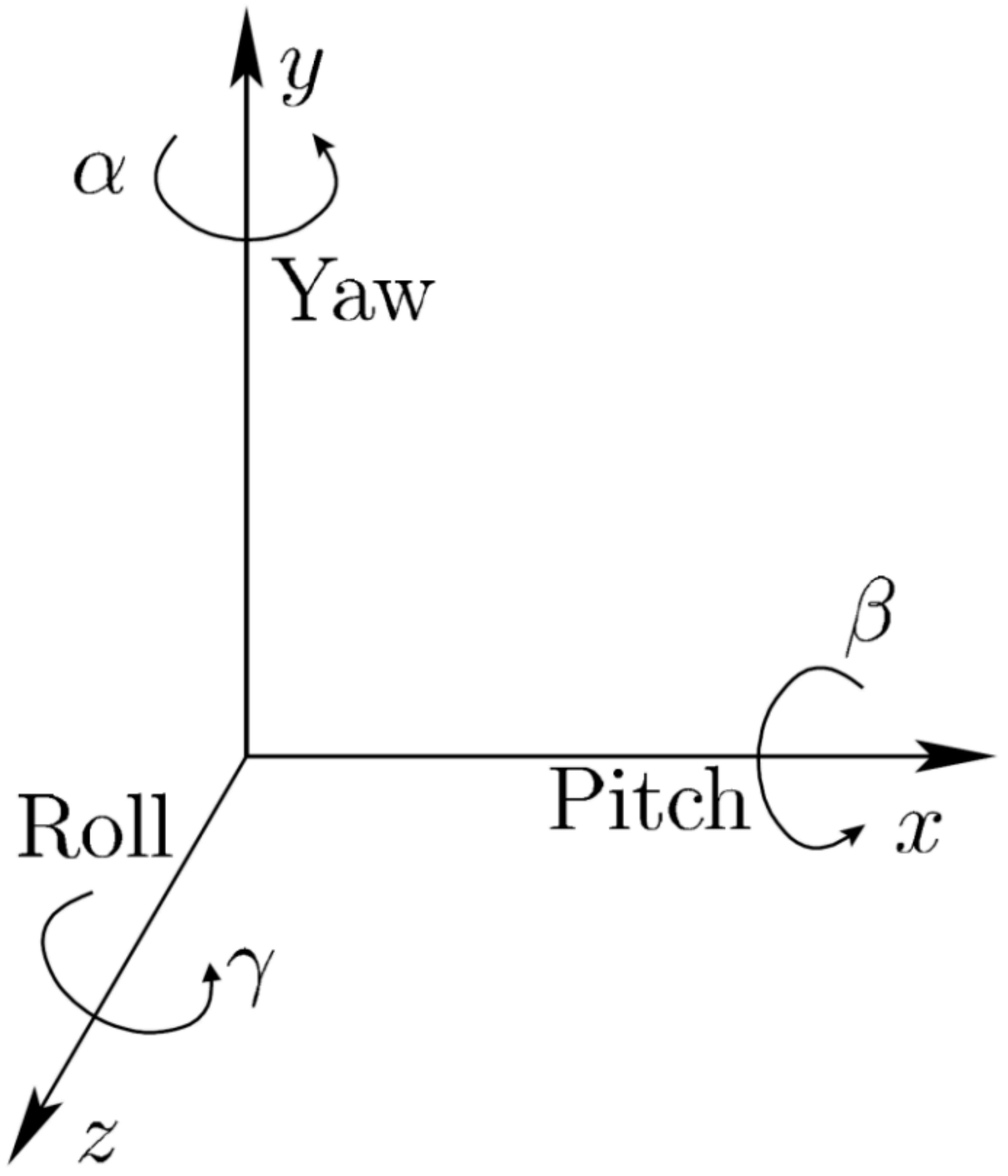
$$R_{sc'} = R\,R_{sc}$$

$$R_{sc''} = R_{sc}R$$

$$R = \mathrm{Rot}(\hat{z}, 90°)$$

Euler Angles: Yaw, Pitch, Roll

- Counterclockwise rotation

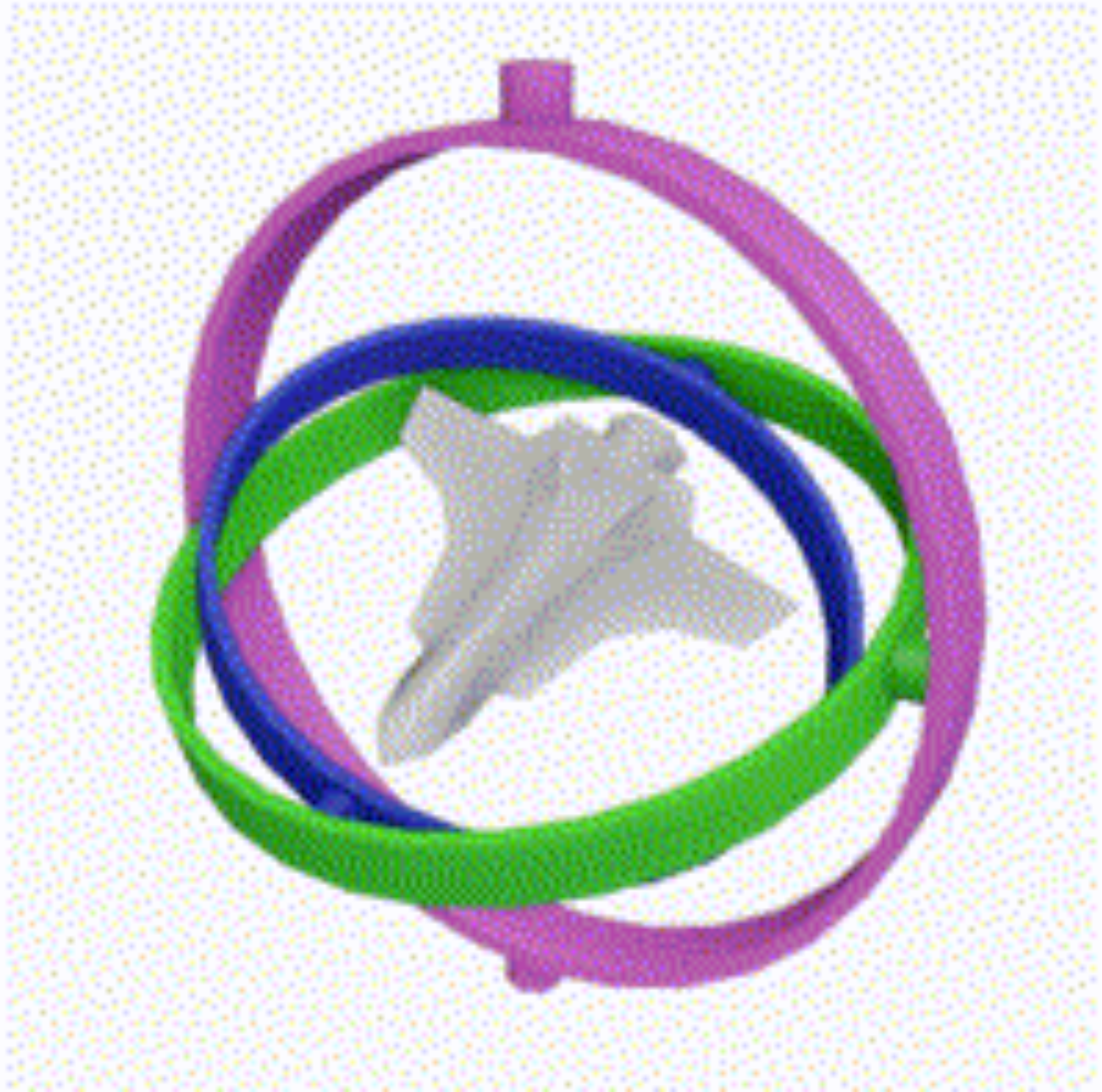


Roll
$$R_z(\gamma) = \begin{bmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Pitch
$$R_x(\beta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\beta & -\sin\beta \\ 0 & \sin\beta & \cos\beta \end{bmatrix}$$

Yaw
$$R_y(\alpha) = \begin{bmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{bmatrix}$$

Euler Angles: Yaw, Pitch, Roll



$$R = R_z(\psi)R_y(\theta)R_x(\phi)$$

$$R = \begin{bmatrix} c_\psi c_\theta & c_\psi s_\theta s_\phi - s_\psi c_\phi & c_\psi s_\theta c_\phi + s_\psi s_\phi \\ s_\psi c_\theta & s_\psi s_\theta s_\phi + c_\psi c_\phi & s_\psi s_\theta c_\phi - c_\psi s_\phi \\ -s_\theta & c_\theta s_\phi & c_\theta c_\phi \end{bmatrix}$$

where

$c = \cos$, $s = \sin$

Pitch comes from a single element. $R_{31} = -\sin\theta$ $\quad\theta = \arcsin(-R_{31})$

Yaw comes from the first column.

$$R_{11} = \cos\psi\cos\theta \qquad \arctan\left(\frac{R_{21}}{R_{11}}\right)$$
$$R_{21} = \sin\psi\cos\theta$$

Roll comes from the three row.

$$R_{32} = \cos\theta\sin\phi$$
$$R_{33} = \cos\theta\cos\phi \qquad \arctan\left(\frac{R_{32}}{R_{33}}\right)$$

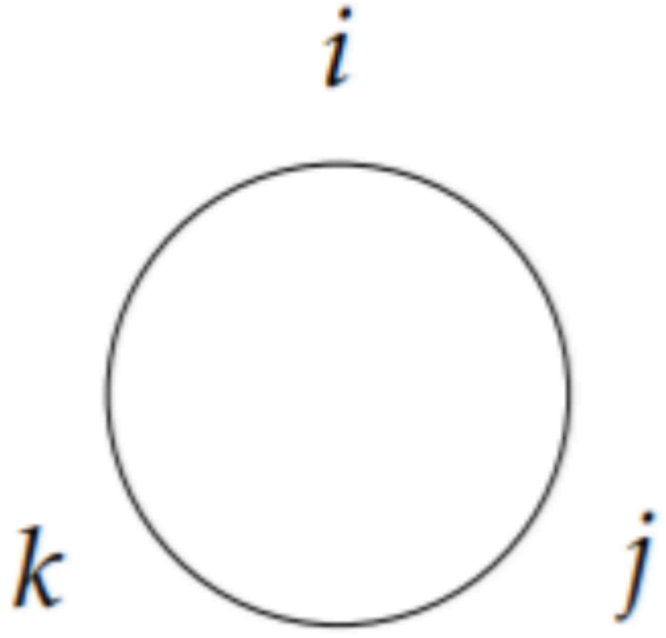
Quaternions

- Quaternions generalize complex numbers and can be used to represents 3D rotations

$$q = w + \underbrace{xi + yj + zk}$$

Scale (real part) — w Vector (imaginary part) — $xi + yj + zk$

- Properties
$$i^2 = j^2 = k^2 = -1$$
$$ij = k, ji = -k$$
$$jk = i, kj = -i$$
$$ki = j, ik = -j$$



Unit Quaternions as 3D Rotations

-For unit quaternions, axis-angle:

$$q = (w, \mathbf{v})$$

- w is the **scalar (real) part**
- $\mathbf{v} \in \mathbb{R}^3$ is the **vector (imaginary) part**

Sometimes you'll also see it written as

$$q = (w, \mathbf{v}) = (\cos\frac{\theta}{2}, \sin\frac{\theta}{2}\hat{\mathbf{n}})$$

$$q = w + v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k}.$$

Why Quaternions over Matrices/Euler Angles?

-No gimbal lock (unlike Euler angles)

-Compact (4 numbers vs 9)

-Stable interpolation

-Numerical stability: Avoids accumulating errors that break orthogonality in rotation matrices.