CSE-571
Robotics

Probabilistic Robotics

Probabilities
Bayes rule
Bayes filters
Probabilistic Robotics

Key idea: Explicit representation of uncertainty

(using the calculus of probability theory)

- Perception = state estimation
- Action = utility optimization
Discrete Random Variables

- $X$ denotes a random variable.
- $X$ can take on a countable number of values in \( \{x_1, x_2, \ldots, x_n\} \).
- $P(X=x_i)$, or $P(x_i)$, is the probability that the random variable $X$ takes on value $x_i$.
- $P(\cdot)$ is called probability mass function.
- E.g. $P(Room) = \langle 0.7, 0.2, 0.08, 0.02 \rangle$
Joint and Conditional Probability

• $P(X=x \text{ and } Y=y) = P(x,y)$

• If $X$ and $Y$ are independent then
  \[ P(x,y) = P(x) \, P(y) \]

• $P(x \mid y)$ is the probability of $x$ given $y$
  \[ P(x \mid y) = \frac{P(x,y)}{P(y)} \]
  \[ P(x,y) = P(x \mid y) \, P(y) \]

• If $X$ and $Y$ are independent then
  \[ P(x \mid y) = P(x) \]
Law of Total Probability, Marginals

**Discrete case**

\[
\sum_x P(x) = 1
\]

\[
P(x) = \sum_y P(x, y)
\]

\[
P(x) = \sum_y P(x \mid y)P(y)
\]

**Continuous case**

\[
\int p(x) \, dx = 1
\]

\[
p(x) = \int p(x, y) \, dy
\]

\[
p(x) = \int p(x \mid y)p(y) \, dy
\]
Events

- $P(+x, +y)$?
- $P(+x)$?
- $P(-y \text{ OR } +x)$?
- Independent?

$P(X,Y)$

<table>
<thead>
<tr>
<th>X</th>
<th>Y</th>
<th>P</th>
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<tbody>
<tr>
<td>+x</td>
<td>+y</td>
<td>0.2</td>
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<tr>
<td>+x</td>
<td>-y</td>
<td>0.3</td>
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<tr>
<td>-x</td>
<td>+y</td>
<td>0.4</td>
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<tr>
<td>-x</td>
<td>-y</td>
<td>0.1</td>
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Marginal Distributions

\[ P(X, Y) \]

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\[ P(x) = \sum_y P(x, y) \]

\[ P(y) = \sum_x P(x, y) \]
Conditional Probabilities

\[ P(X, Y) \]

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- \( P(+x \mid +y) ? \)
- \( P(-x \mid +y) ? \)
- \( P(-y \mid +x) ? \)
Bayes Formula

\[ P(x, y) = P(x \mid y)P(y) = P(y \mid x)P(x) \]

\[ \Rightarrow \]

\[ P(x \mid y) = \frac{P(y \mid x)P(x)}{P(y)} = \frac{\text{likelihood} \cdot \text{prior}}{\text{evidence}} \]

- Often causal knowledge is easier to obtain than diagnostic knowledge.
- Bayes rule allows us to use causal knowledge.
Simple Example of State Estimation

• Suppose a robot obtains measurement $z$
• What is $P(\text{open} | z)$?
Example

\[ P(z \mid \text{open}) = 0.6 \quad P(z \mid \neg \text{open}) = 0.3 \]

\[ P(\text{open}) = P(\neg \text{open}) = 0.5 \]

\[ P(\text{open} \mid z) = \frac{P(z \mid \text{open})P(\text{open})}{P(z \mid \text{open})p(\text{open}) + P(z \mid \neg \text{open})p(\neg \text{open})} \]

\[ P(\text{open} \mid z) = \frac{0.6 \cdot 0.5}{0.6 \cdot 0.5 + 0.3 \cdot 0.5} = \frac{2}{3} = 0.67 \]

- \( z \) raises the probability that the door is open.
Normalization

\[ P(x \mid y) = \frac{P(y \mid x) \ P(x)}{P(y)} = \eta \ P(y \mid x) \ P(x) \]

\[ \eta = P(y)^{-1} = \frac{1}{\sum_{x'} P(y \mid x') P(x')} \]

Algorithm:

\[ \forall x : \text{aux}_{x \mid y} = P(y \mid x) \ P(x) \]

\[ \eta = \frac{1}{\sum_x \text{aux}_{x \mid y}} \]

\[ \forall x : P(x \mid y) = \eta \ \text{aux}_{x \mid y} \]
Conditioning

- Bayes rule and **background knowledge**:

\[
P(x \mid y, z) = \frac{P(y \mid x, z) \ P(x \mid z)}{P(y \mid z)}
\]

\[
P(x \mid y) = \int P(x \mid y, z) P(z) \ dz
\]

\[
= \int P(x \mid y, z) P(z \mid y) \ dz
\]

\[
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Conditioning

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P(x \mid y, z) = \frac{P(y \mid x, z) \ P(x \mid z)}{P(y \mid z)}
\]

\[
P(x \mid y) = \int P(x \mid y, z) \ P(z \mid y) \ dz
\]
Conditional Independence

\[ P(x, y \mid z) = P(x \mid z)P(y \mid z) \]

- Equivalent to

\[ P(x \mid z) = P(x \mid z, y) \]

and

\[ P(y \mid z) = P(y \mid z, x) \]
Simple Example of State Estimation

- Suppose our robot obtains another observation $z_2$.
- What is $P(\text{open}|z_1, z_2)$?
Recursive Bayesian Updating

\[ P(x \mid z_1, \ldots, z_n) = \frac{P(z_n \mid x, z_1, \ldots, z_{n-1}) \, P(x \mid z_1, \ldots, z_{n-1})}{P(z_n \mid z_1, \ldots, z_{n-1})} \]

**Markov assumption:** \( z_n \) is conditionally independent of \( z_1, \ldots, z_{n-1} \) given \( x \).

\[ P(x \mid z_1, \ldots, z_n) = \frac{P(z_n \mid x) \, P(x \mid z_1, \ldots, z_{n-1})}{P(z_n \mid z_1, \ldots, z_{n-1})} \]

\[ = \eta \, P(z_n \mid x) \, P(x \mid z_1, \ldots, z_{n-1}) \]

\[ = \eta_{1 \ldots n} \prod_{i=1 \ldots n} P(z_i \mid x) \, P(x) \]
Example: Second Measurement

\[ P(z_2 | open) = 0.5 \quad P(z_2 | \neg open) = 0.6 \]
\[ P(open | z_1) = \frac{2}{3} \quad P(\neg open | z_1) = \frac{1}{3} \]

\[
P(open | z_2, z_1) = \frac{P(z_2 | open) P(open | z_1)}{P(z_2 | open) P(open | z_1) + P(z_2 | \neg open) P(\neg open | z_1)}
\]

\[
= \frac{1 \cdot \frac{2}{3}}{1 \cdot \frac{2}{3} + \frac{3}{5} \cdot \frac{1}{3}} = \frac{5}{8} = 0.625
\]

\[ \bullet \text{ } z_2 \text{ lowers the probability that the door is open.} \]
Bayes Filters: Framework

- **Given:**
  - Stream of observations \( z \) and action data \( u: \)
    \[
    d_t = \{u_1, z_2, \ldots, u_{t-1}, z_t\}
    \]
  - Sensor model \( P(z|x) \).
  - Action model \( P(x|u,x') \).
  - Prior probability of the system state \( P(x) \).

- **Wanted:**
  - Estimate of the state \( X \) of a dynamical system.
  - The posterior of the state is also called **Belief**:
    \[
    Bel(x_t) = P(x_t | u_1, z_2, \ldots, u_{t-1}, z_t)
    \]
Bayes Filters

\[ Bel(x_t) = P(x_t | u_1, z_1 \ldots, u_t, z_t) \]

Bayes
\[ = \eta P(z_t | x_t, u_1, z_1, \ldots, u_t) P(x_t | u_1, z_1, \ldots, u_t) \]

Markov
\[ = \eta P(z_t | x_t) P(x_t | u_1, z_1, \ldots, u_t) \]

Total prob.
\[ = \eta P(z_t | x_t) \int P(x_t | u_1, z_1, \ldots, u_t, x_{t-1}) P(x_{t-1} | u_1, z_1, \ldots, u_t) \, dx_{t-1} \]

Markov
\[ = \eta P(z_t | x_t) \int P(x_t | u_t, x_{t-1}) P(x_{t-1} | u_1, z_1, \ldots, u_t) \, dx_{t-1} \]
\[ = \eta P(z_t | x_t) \int P(x_t | u_t, x_{t-1}) Bel(x_{t-1}) \, dx_{t-1} \]
\[ \text{Bel}(x_t) = \eta P(z_t \mid x_t) \int P(x_t \mid u_t, x_{t-1}) \text{Bel}(x_{t-1}) \, dx_{t-1} \]

1. Algorithm **Bayes_filter** (Bel(x), d):
2. \( n = 0 \)
3. If \( d \) is a perceptual data item \( z \) then
4. For all \( x \) do
5. \( \text{Bel}'(x) = P(z \mid x)\text{Bel}(x) \)
6. \( \eta = \eta + \text{Bel}'(x) \)
7. For all \( x \) do
8. \( \text{Bel}'(x) = \eta^{-1}\text{Bel}'(x) \)
9. Else if \( d \) is an action data item \( u \) then
10. For all \( x \) do
11. \( \text{Bel}'(x) = \int P(x \mid u, x') \text{Bel}(x') \, dx' \)
12. Return \( \text{Bel}'(x) \)
Markov Assumption

\[ p(z_t \mid x_{0:t}, z_{1:t-1}, u_{1:t}) = p(z_t \mid x_t) \]
\[ p(x_t \mid x_{1:t-1}, z_{1:t-1}, u_{1:t}) = p(x_t \mid x_{t-1}, u_t) \]

Underlying Assumptions

- Static world
- Independent noise
- Perfect model, no approximation errors
Dynamic Environments

- Two possible locations $x_1$ and $x_2$
- $P(x_1) = 0.99$
- $P(z|x_2) = 0.09$ $P(z|x_1) = 0.07$
Bayes Filters for Robot Localization
Representations for Bayesian Robot Localization

Discrete approaches (’95)
- Topological representation (’95)
  - uncertainty handling (POMDPs)
  - occas. global localization, recovery
- Grid-based, metric representation (’96)
  - global localization, recovery

Particle filters (’99)
- sample-based representation
- global localization, recovery

Kalman filters (late-80s)
- Gaussians, unimodal
- approximately linear models
- position tracking

Multi-hypothesis (’00)
- multiple Kalman filters
- global localization, recovery
Bayes Filters are Familiar!

\[ Bel(x_t) = \eta \ P(z_t \mid x_t) \int P(x_t \mid u_t, x_{t-1}) \ Bel(x_{t-1}) \ dx_{t-1} \]

- Kalman filters
- Particle filters
- Hidden Markov models
- Dynamic Bayesian networks
- Partially Observable Markov Decision Processes (POMDPs)
Summary

• Bayes rule allows us to compute probabilities that are hard to assess otherwise.

• Under the Markov assumption, recursive Bayesian updating can be used to efficiently combine evidence.

• Bayes filters are a probabilistic tool for estimating the state of dynamic systems.