CSE-571
Robotics

Probabilistic Robotics

Probabilities
Bayes rule
Bayes filters

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Probabilistic Robotics

Key idea: Explicit representation of uncertainty

(using the calculus of probability theory)

- Perception = state estimation
- Action = utility optimization

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Discrete Random Variables

- *X* denotes a random variable.
- *X* can take on a countable number of values in $\{x_1, x_2, \ldots, x_n\}$.
- $P(X=x_i)$, or $P(x_i)$, is the probability that the random variable *X* takes on value x_i.
- $P(\cdot)$ is called probability mass function.
- E.g. $P(Room) = \langle 0.7, 0.2, 0.08, 0.02 \rangle$

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Joint and Conditional Probability

- $P(X=x \text{ and } Y=y) = P(x,y)$
- If X and Y are independent then
  $P(x,y) = P(x)\,P(y)$
- $P(x \mid y)$ is the probability of *x given y*
  $P(x \mid y) = P(x,y) / P(y)$
  $P(x,y) = P(x \mid y)\,P(y)$
- If X and Y are independent then
  $P(x \mid y) = P(x)$

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Law of Total Probability, Marginals

Discrete case

$$\sum_x P(x) = 1$$

$$P(x) = \sum_y P(x, y)$$

$$P(x) = \sum_y P(x \mid y) P(y)$$

Continuous case

$$\int p(x)\, dx = 1$$

$$p(x) = \int p(x, y)\, dy$$

$$p(x) = \int p(x \mid y)\, p(y)\, dy$$

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Events

- P(+x, +y) ?
- P(+x) ?
- P(-y OR +x) ?
- Independent?

$P(X,Y)$

X	Y	P
+x	+y	0.2
+x	-y	0.3
-x	+y	0.4
-x	-y	0.1

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Marginal Distributions

$P(X,Y)$

X	Y	P
+x	+y	0.2
+x	-y	0.3
-x	+y	0.4
-x	-y	0.1

→ $P(x) = \sum_y P(x, y)$

$P(X)$

X	P
+x	
-x	

→ $P(y) = \sum_x P(x, y)$

$P(Y)$

Y	P
+y	
-y	

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Conditional Probabilities

$P(X,Y)$

X	Y	P
+x	+y	0.2
+x	-y	0.3
-x	+y	0.4
-x	-y	0.1

- P(+x | +y) ?
- P(-x | +y) ?
- P(-y | +x) ?

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Bayes Formula

$$P(x,y) = P(x \mid y)P(y) = P(y \mid x)P(x)$$

$$\Rightarrow$$

$$P(x \mid y) = \frac{P(y \mid x)\ P(x)}{P(y)} = \frac{\text{likelihood} \cdot \text{prior}}{\text{evidence}}$$

- Often causal knowledge is easier to obtain than diagnostic knowledge.
- Bayes rule allows us to use causal knowledge.

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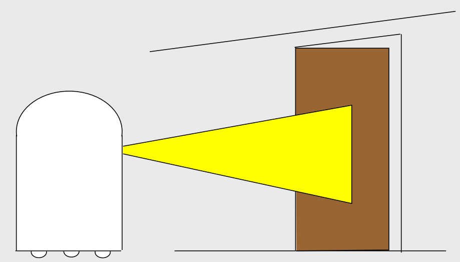
Simple Example of State Estimation

- Suppose a robot obtains measurement z
- What is *P(open|z)?*



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Example

$$P(z \mid open) = 0.6 \qquad P(z \mid \neg open) = 0.3$$

$$P(open) = P(\neg open) = 0.5$$

$$P(open \mid z) = \frac{P(z \mid open)P(open)}{P(z \mid open)p(open) + P(z \mid \neg open)p(\neg open)}$$

$$P(open \mid z) = \frac{0.6 \cdot 0.5}{0.6 \cdot 0.5 + 0.3 \cdot 0.5} = \frac{2}{3} = 0.67$$

- *z* raises the probability that the door is open.

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Normalization

$$P(x \mid y) = \frac{P(y \mid x)\ P(x)}{P(y)} = \eta\ P(y \mid x)\, P(x)$$

$$\eta = P(y)^{-1} = \frac{1}{\sum_{x'} P(y \mid x')P(x')}$$

Algorithm:

$$\forall x : \text{aux}_{x|y} = P(y \mid x)\ P(x)$$

$$\eta = \frac{1}{\sum_x \text{aux}_{x|y}}$$

$$\forall x : P(x \mid y) = \eta\ \text{aux}_{x|y}$$

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Conditioning

- Bayes rule and background knowledge:

$$P(x \mid y,z) = \frac{P(y \mid x,z)\ P(x \mid z)}{P(y \mid z)}$$

$$P(x \mid y) \stackrel{?}{=} \int P(x \mid y,z)\, P(z)\, dz$$

$$\stackrel{?}{=} \int P(x \mid y,z)\, P(z \mid y)\, dz$$

$$\stackrel{?}{=} \int P(x \mid y,z)\, P(y \mid z)\, dz$$

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Conditioning

- Bayes rule and background knowledge:

$$P(x \mid y,z) = \frac{P(y \mid x,z)\ P(x \mid z)}{P(y \mid z)}$$

$$P(x \mid y) = \int P(x \mid y,z)\, P(z \mid y)\, dz$$

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Conditional Independence

$$P(x,y \mid z) = P(x \mid z)P(y \mid z)$$

- Equivalent to

$$P(x \mid z) = P(x \mid z,y)$$

and

$$P(y \mid z) = P(y \mid z,x)$$

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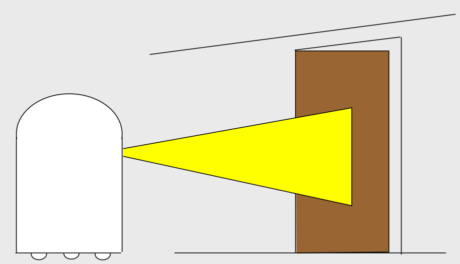
Simple Example of State Estimation

- Suppose our robot obtains another observation z_2.
- What is *P(open*|z_1, z_2*)?*



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Recursive Bayesian Updating

$$P(x \mid z_1,\ldots,z_n) = \frac{P(z_n \mid x, z_1,\ldots,z_{n-1})\ P(x \mid z_1,\ldots,z_{n-1})}{P(z_n \mid z_1,\ldots,z_{n-1})}$$

Markov assumption: z_n is conditionally independent of $z_1,\ldots,z_{n-1}$ given x.

$$P(x \mid z_1,\ldots,z_n) = \frac{P(z_n \mid x)\ P(x \mid z_1,\ldots,z_{n-1})}{P(z_n \mid z_1,\ldots,z_{n-1})}$$

$$= \eta\ P(z_n \mid x)\ P(x \mid z_1,\ldots,z_{n-1})$$

$$= \eta_{1\ldots n} \prod_{i=1\ldots n} P(z_i \mid x)\ P(x)$$

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Example: Second Measurement

$P(z_2 \mid open) = 0.5 \qquad P(z_2 \mid \neg open) = 0.6$

$P(open \mid z_1) = 2/3 \qquad P(\neg open \mid z_1) = 1/3$

$$P(open \mid z_2, z_1) = \frac{P(z_2 \mid open)\ P(open \mid z_1)}{P(z_2 \mid open)\ P(open \mid z_1) + P(z_2 \mid \neg open)\ P(\neg open \mid z_1)}$$

$$= \frac{\frac{1}{2}\cdot\frac{2}{3}}{\frac{1}{2}\cdot\frac{2}{3}+\frac{3}{5}\cdot\frac{1}{3}} = \frac{5}{8} = 0.625$$

- z_2 lowers the probability that the door is open.

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Bayes Filters: Framework

- **Given:**
  - Stream of observations z and action data u:
    $$d_t = \{u_1, z_2 \ldots, u_{t-1}, z_t\}$$
  - Sensor model $P(z|x)$.
  - Action model $P(x|u,x')$.
  - Prior probability of the system state $P(x)$.
- **Wanted:**
  - Estimate of the state X of a dynamical system.
  - The posterior of the state is also called **Belief**:
    $$Bel(x_t) = P(x_t \mid u_1, z_2 \ldots, u_{t-1}, z_t)$$

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Bayes Filters

z = observation
u = action
x = state

$$Bel(x_t) = P(x_t \mid u_1, z_1 \ldots, u_t, z_t)$$

Bayes
$$= \eta\ P(z_t \mid x_t, u_1, z_1, \ldots, u_t)\ P(x_t \mid u_1, z_1, \ldots, u_t)$$

Markov
$$= \eta\ P(z_t \mid x_t)\ P(x_t \mid u_1, z_1, \ldots, u_t)$$

Total prob.
$$= \eta\ P(z_t \mid x_t) \int P(x_t \mid u_1, z_1, \ldots, u_t, x_{t-1}) P(x_{t-1} \mid u_1, z_1, \ldots, u_t)\ dx_{t-1}$$

Markov
$$= \eta\ P(z_t \mid x_t) \int P(x_t \mid u_t, x_{t-1})\ P(x_{t-1} \mid u_1, z_1, \ldots, u_t)\ dx_{t-1}$$

$$= \eta\ P(z_t \mid x_t) \int P(x_t \mid u_t, x_{t-1})\ Bel(x_{t-1})\ dx_{t-1}$$

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$$Bel(x_t) = \eta \ P(z_t \mid x_t) \int P(x_t \mid u_t, x_{t-1}) \ Bel(x_{t-1}) \ dx_{t-1}$$

Algorithm **Bayes_filter**(*Bel(x),d*):
n=0
If *d* is a perceptual data item *z* then
  For all *x* do
    $Bel'(x) = P(z \mid x) Bel(x)$
    $\eta = \eta + Bel'(x)$
  For all *x* do
    $Bel'(x) = \eta^{-1} Bel'(x)$
Else if *d* is an action data item *u* then
  For all *x* do
    $Bel'(x) = \int P(x \mid u, x') \ Bel(x') \ dx'$
Return *Bel'(x)*

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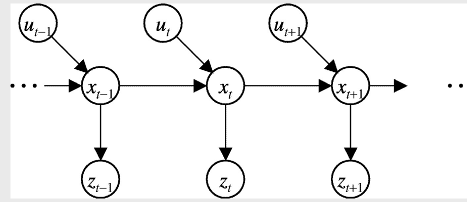
Markov Assumption



$$p(z_t \mid x_{0:t}, z_{1:t-1}, u_{1:t}) = p(z_t \mid x_t)$$
$$p(x_t \mid x_{1:t-1}, z_{1:t-1}, u_{1:t}) = p(x_t \mid x_{t-1}, u_t)$$

Underlying Assumptions
- Static world
- Independent noise
- Perfect model, no approximation errors

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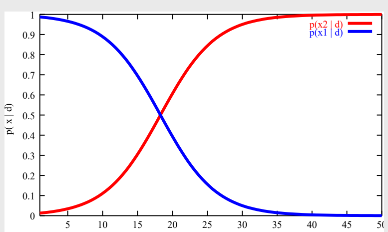
Dynamic Environments

- Two possible locations x_1 and x_2
- $P(x_1)=0.99$
- $P(z|x_2)=0.09$ $P(z|x_1)=0.07$



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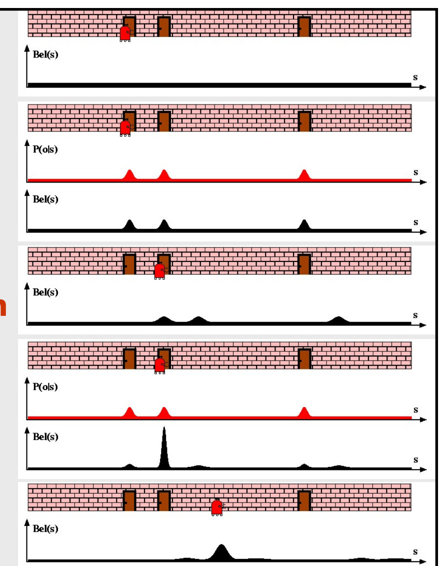
Bayes Filters for Robot Localization



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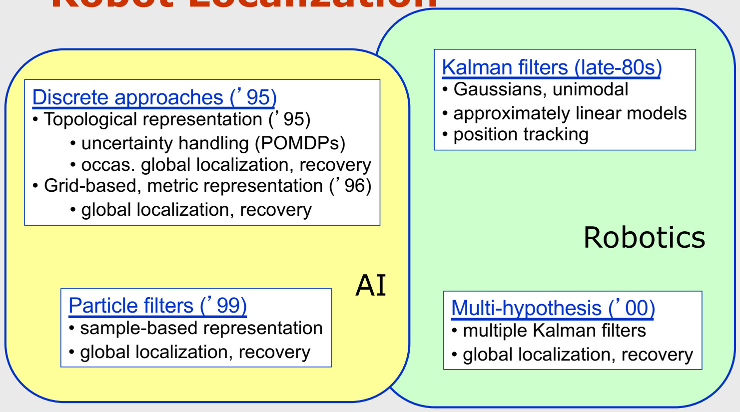
Representations for Bayesian Robot Localization



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Bayes Filters are Familiar!

$$Bel(x_t) = \eta \; P(z_t \mid x_t) \int P(x_t \mid u_t, x_{t-1}) \; Bel(x_{t-1}) \; dx_{t-1}$$

- Kalman filters
- Particle filters
- Hidden Markov models
- Dynamic Bayesian networks
- Partially Observable Markov Decision Processes (POMDPs)

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Summary

- Bayes rule allows us to compute probabilities that are hard to assess otherwise.
- Under the Markov assumption, recursive Bayesian updating can be used to efficiently combine evidence.
- Bayes filters are a probabilistic tool for estimating the state of dynamic systems.

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