

# CSE-571 Robotics

## Probabilistic Robotics

Probabilities  
Bayes rule  
Bayes filters

1

## Probabilistic Robotics

Key idea: Explicit representation of uncertainty

(using the calculus of probability theory)

- Perception = state estimation
- Action = utility optimization

2

## Discrete Random Variables

- $X$  denotes a **random variable**.
- $X$  can take on a countable number of values in  $\{x_1, x_2, \dots, x_n\}$ .
- $P(X=x_i)$ , or  $P(x_i)$ , is the **probability** that the random variable  $X$  takes on value  $x_i$ .
- $P(\cdot)$  is called **probability mass function**.
- E.g.  $P(\text{Room}) = \langle 0.7, 0.2, 0.08, 0.02 \rangle$

3

## Joint and Conditional Probability

- $P(X=x \text{ and } Y=y) = P(x, y)$
- If  $X$  and  $Y$  are **independent** then
$$P(x, y) = P(x) P(y)$$
- $P(x | y)$  is the probability of  $x$  given  $y$ 
$$P(x | y) = P(x, y) / P(y)$$
$$P(x, y) = P(x | y) P(y)$$
- If  $X$  and  $Y$  are **independent** then
$$P(x | y) = P(x)$$

4

## Law of Total Probability, Marginals

### Discrete case

$$\sum_x P(x) = 1$$

$$P(x) = \sum_y P(x, y)$$

$$P(x) = \sum_y P(x | y)P(y)$$

### Continuous case

$$\int p(x) dx = 1$$

$$p(x) = \int p(x, y) dy$$

$$p(x) = \int p(x | y)p(y) dy$$

5

## Events

- $P(+x, +y)$  ?
- $P(+x)$  ?
- $P(-y \text{ OR } +x)$  ?
- Independent?

$P(X, Y)$

X	Y	P
+x	+y	0.2
+x	-y	0.3
-x	+y	0.4
-x	-y	0.1

6

## Marginal Distributions

$P(X, Y)$			$P(X)$	
X	Y	P	X	P
+x	+y	0.2	+x	
+x	-y	0.3	-x	
-x	+y	0.4		
-x	-y	0.1		

$P(x) = \sum_y P(x, y)$

$P(y) = \sum_x P(x, y)$

$P(Y)$	
Y	P
+y	
-y	

7

## Conditional Probabilities

$P(X, Y)$

X	Y	P
+x	+y	0.2
+x	-y	0.3
-x	+y	0.4
-x	-y	0.1

- $P(+x | +y)$  ?
- $P(-x | +y)$  ?
- $P(-y | +x)$  ?

8

## Bayes Formula

$$P(x, y) = P(x | y)P(y) = P(y | x)P(x)$$

$\Rightarrow$

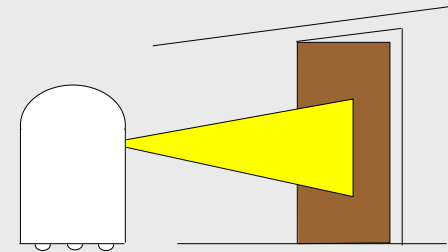
$$P(x | y) = \frac{P(y | x) P(x)}{P(y)} = \frac{\text{likelihood} \cdot \text{prior}}{\text{evidence}}$$

- Often **causal** knowledge is easier to obtain than **diagnostic** knowledge.
- Bayes rule allows us to use causal knowledge.

9

## Simple Example of State Estimation

- Suppose a robot obtains measurement  $z$
- What is  $P(\text{open} | z)$ ?



10

## Example

$$P(z | \text{open}) = 0.6 \quad P(z | \neg \text{open}) = 0.3$$

$$P(\text{open}) = P(\neg \text{open}) = 0.5$$

$$P(\text{open} | z) = \frac{P(z | \text{open})P(\text{open})}{P(z | \text{open})P(\text{open}) + P(z | \neg \text{open})P(\neg \text{open})}$$

$$P(\text{open} | z) = \frac{0.6 \cdot 0.5}{0.6 \cdot 0.5 + 0.3 \cdot 0.5} = \frac{2}{3} = 0.67$$

- $z$  raises the probability that the door is open.

11

## Normalization

$$P(x | y) = \frac{P(y | x) P(x)}{P(y)} = \eta P(y | x) P(x)$$

$$\eta = P(y)^{-1} = \frac{1}{\sum_{x'} P(y | x') P(x')}$$

Algorithm:

$$\forall x : \text{aux}_{x|y} = P(y | x) P(x)$$

$$\eta = \frac{1}{\sum_x \text{aux}_{x|y}}$$

$$\forall x : P(x | y) = \eta \text{aux}_{x|y}$$

12

## Conditioning

- Bayes rule and background knowledge:

$$P(x|y,z) = \frac{P(y|x,z) P(x|z)}{P(y|z)}$$

$$\begin{aligned} P(x|y) & \stackrel{?}{=} \int P(x|y,z) P(z) dz \\ & \stackrel{?}{=} \int P(x|y,z) P(z|y) dz \\ & \stackrel{?}{=} \int P(x|y,z) P(y|z) dz \end{aligned}$$

13

## Conditioning

- Bayes rule and background knowledge:

$$P(x|y,z) = \frac{P(y|x,z) P(x|z)}{P(y|z)}$$

$$P(x|y) = \int P(x|y,z) P(z|y) dz$$

14

## Conditional Independence

$$P(x,y|z) = P(x|z)P(y|z)$$

- Equivalent to

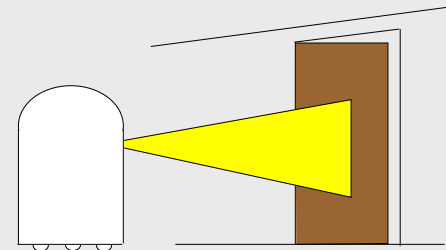
and  $P(x|z) = P(x|z,y)$

$$P(y|z) = P(y|z,x)$$

15

## Simple Example of State Estimation

- Suppose our robot obtains another observation  $z_2$ .
- What is  $P(\text{open}|z_1, z_2)$ ?



16

## Recursive Bayesian Updating

$$P(x | z_1, \dots, z_n) = \frac{P(z_n | x, z_1, \dots, z_{n-1}) P(x | z_1, \dots, z_{n-1})}{P(z_n | z_1, \dots, z_{n-1})}$$

**Markov assumption:**  $z_n$  is conditionally independent of  $z_1, \dots, z_{n-1}$  given  $x$ .

$$\begin{aligned} P(x | z_1, \dots, z_n) &= \frac{P(z_n | x) P(x | z_1, \dots, z_{n-1})}{P(z_n | z_1, \dots, z_{n-1})} \\ &= \eta P(z_n | x) P(x | z_1, \dots, z_{n-1}) \\ &= \eta_{1 \dots n} \prod_{i=1 \dots n} P(z_i | x) P(x) \end{aligned}$$

17

## Example: Second Measurement

$$\begin{aligned} P(z_2 | open) &= 0.5 & P(z_2 | \neg open) &= 0.6 \\ P(open | z_1) &= 2/3 & P(\neg open | z_1) &= 1/3 \end{aligned}$$

$$\begin{aligned} P(open | z_2, z_1) &= \frac{P(z_2 | open) P(open | z_1)}{P(z_2 | open) P(open | z_1) + P(z_2 | \neg open) P(\neg open | z_1)} \\ &= \frac{\frac{1}{2} \cdot \frac{2}{3}}{\frac{1}{2} \cdot \frac{2}{3} + \frac{3}{5} \cdot \frac{1}{3}} = \frac{5}{8} = 0.625 \end{aligned}$$

- $z_2$  lowers the probability that the door is open.

18

## Bayes Filters: Framework

- **Given:**

- Stream of observations  $z$  and action data  $u$ :  
 $d_t = \{u_1, z_2, \dots, u_{t-1}, z_t\}$
- Sensor model  $P(z|x)$ .
- Action model  $P(x|u, x')$ .
- Prior probability of the system state  $P(x)$ .

- **Wanted:**

- Estimate of the state  $X$  of a dynamical system.
- The posterior of the state is also called **Belief**:

$$Bel(x_t) = P(x_t | u_1, z_2, \dots, u_{t-1}, z_t)$$

19

## Bayes Filters

$z$  = observation  
 $u$  = action  
 $x$  = state

$$Bel(x_t) = P(x_t | u_1, z_1, \dots, u_t, z_t)$$

$$\text{Bayes} = \eta P(z_t | x_t, u_1, z_1, \dots, u_t) P(x_t | u_1, z_1, \dots, u_t)$$

$$\text{Markov} = \eta P(z_t | x_t) P(x_t | u_1, z_1, \dots, u_t)$$

$$\text{Total prob.} = \eta P(z_t | x_t) \int P(x_t | u_t, z_1, \dots, u_t, x_{t-1}) P(x_{t-1} | u_1, z_1, \dots, u_t) dx_{t-1}$$

$$\text{Markov} = \eta P(z_t | x_t) \int P(x_t | u_t, x_{t-1}) P(x_{t-1} | u_1, z_1, \dots, u_t) dx_{t-1}$$

$$= \eta P(z_t | x_t) \int P(x_t | u_t, x_{t-1}) Bel(x_{t-1}) dx_{t-1}$$

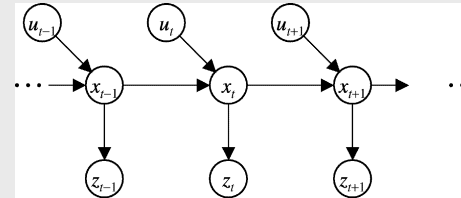
20

$$Bel(x_t) = \eta P(z_t | x_t) \int P(x_t | u_t, x_{t-1}) Bel(x_{t-1}) dx_{t-1}$$

1. Algorithm **Bayes\_filter**(  $Bel(x), d$  ):
2.  $n=0$
3. If  $d$  is a **perceptual** data item  $z$  then
4. For all  $x$  do
5.  $Bel'(x) = P(z | x) Bel(x)$
6.  $\eta = \eta + Bel'(x)$
7. For all  $x$  do
8.  $Bel(x) = \eta^{-1} Bel'(x)$
9. Else if  $d$  is an **action** data item  $u$  then
10. For all  $x$  do
11.  $Bel(x) = \int P(x | u, x') Bel(x') dx'$
12. Return  $Bel(x)$

21

## Markov Assumption



$$p(z_t | x_{0:t}, z_{1:t-1}, u_{1:t}) = p(z_t | x_t)$$

$$p(x_t | x_{1:t-1}, z_{1:t-1}, u_{1:t}) = p(x_t | x_{t-1}, u_t)$$

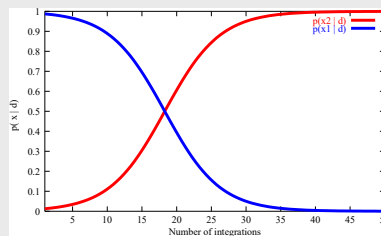
### Underlying Assumptions

- Static world
- Independent noise
- Perfect model, no approximation errors

22

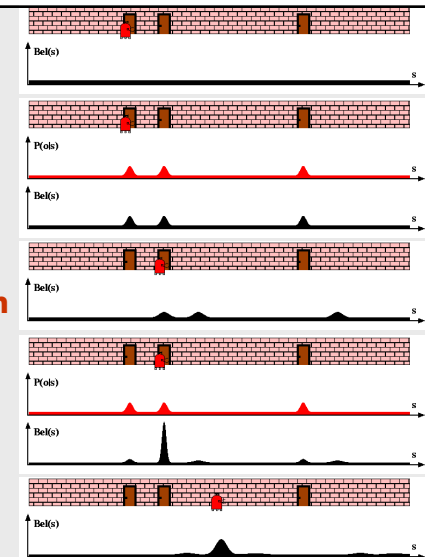
## Dynamic Environments

- Two possible locations  $x_1$  and  $x_2$
- $P(x_1) = 0.99$
- $P(z|x_2) = 0.09$   $P(z|x_1) = 0.07$

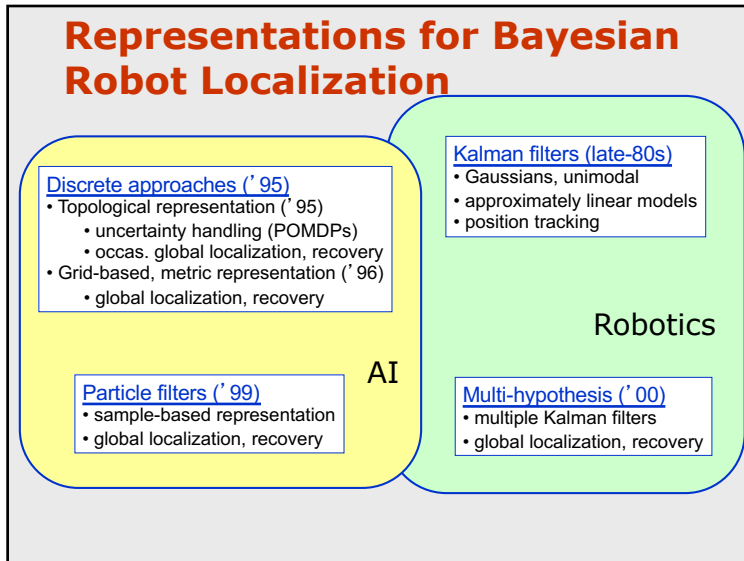


23

## Bayes Filters for Robot Localization



24



25

## Bayes Filters are Familiar!

$$Bel(x_t) = \eta P(z_t | x_t) \int P(x_t | u_t, x_{t-1}) Bel(x_{t-1}) dx_{t-1}$$

- Kalman filters
- Particle filters
- Hidden Markov models
- Dynamic Bayesian networks
- Partially Observable Markov Decision Processes (POMDPs)

26

## Summary

- Bayes rule allows us to compute probabilities that are hard to assess otherwise.
- Under the Markov assumption, recursive Bayesian updating can be used to efficiently combine evidence.
- Bayes filters are a probabilistic tool for estimating the state of dynamic systems.

27