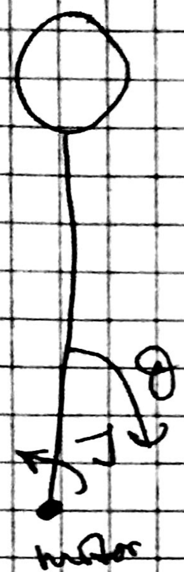
LQR



$m=1, \quad L=1$

We want to hold this upright!

θ = angle from vertical.

$$P \doteq \quad \ddot{\theta}_{t+1} = \tau - g\,\cancel{\sin\theta_t}\ \theta_t$$

$$\dot{\theta}_{t+1} = \dot{\theta}_t + \ddot{\theta}_t \Delta t$$

$$\theta_{t+1} = \theta_t + \dot{\theta}_t \Delta t + \frac{1}{2}\ddot{\theta}(\Delta t)^2$$

$s = [\theta, \dot{\theta}, \ddot{\theta}]$, $A = \tau$, $T = 100$.

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$$\begin{bmatrix} \ddot{\theta}_{t+1} \\ \dot{\theta}_{t+1} \\ \theta_{t} \end{bmatrix} = \underbrace{\begin{bmatrix} -g & 0 & 0 & J \\ 1 & 0 & \Delta t & 0 \\ 1 & \Delta t & \frac{1}{2}(\Delta t)^2 & 0 \end{bmatrix} \begin{bmatrix} \theta_t \\ \dot{\theta}_t \\ \ddot{\theta}_t \\ 1 \end{bmatrix}}$$

$$x_{t+1} = A x_t$$

$C(s) = \theta^2 + \lambda J^2 \rightarrow$ Don't want high torques.

Recall VI

for $t = T-1 \ldots 0$ (T time steps)

$$V(x, t) = \max_u r(x) + V(x', t+1)$$

where $x' = f(x, u)$

LQR Problem:

$$\text{cost} = \underline{\underline{x^T Q x}} + u^T R u$$

$\downarrow$

quadratic in x

+ve semi definite.

So cost ≥ 0 $(\forall x$, except $0)$

$Q = Q^T$

~~For the~~ pendulum problem we want $R > 0$.

$\forall u \quad u^T R u > 0, \quad R = R^T$.

$$\left.\begin{array}{l} R = \text{scalar} \\ Q = 3 \times 3 \end{array}\right\} \text{for pendulum problem.}$$

At last time step:

T-1

$u=0$

$$J(x_{T-1}, T-1) = x_{t-1}^T Q x_{T-1} + \cancel{u^T R u}^{\,0}$$

$$= x_{T-1}^T V_{T-1} x_{T-1}$$

So $V_{T-1} = Q$

T-2

$$J(x_{T-2}, T-2) = \min_{u_{T-2}} u_{T-2}^T R u_{T-2}$$

$$+ x_{T-2}^T Q x_{T-2} + (Ax_{T-2} + Bu_{T-2})^T V_{T-1}$$

$$(Ax_{T-2} + Bu_{T-2})$$

$$(\because Ax_{t-2} + Bu_{t-2} = x_{t-1})$$

Taking derivative

$$\frac{\partial J}{\partial u} = 2 R u_{t-2} + \frac{\partial J}{\partial u}\Big((A x_{t-2})^T V_{t-1} (A x_{t-2})$$

Same because of symmetry

$$+ u_{t-2}^T B^T V_{T-1} A x_{t-2}$$

$$+ x_{T-2}^T A^T V_{T-1} B u_{T-2}$$

$$+ u_{T-2}^T B^T U_{T-1} B U_{T-2} \Big)$$

$$= 2 R u_{T-2} + 2 x_{t-2}^T A^T V_{T-1} B + 2 B^T V_{T-1} B u_{T-2}$$

$$= 0$$

(Set it to 0)

$$(R + B^T V_{T-1} B) U_{T-2} = - X_{T-2}^T A^T V_{T-1} B$$

$$U_{T-2} = - (R + B^T U_{T-1} B)^{-1} (X_{T-2}^T A^T U_{T-1} B)$$

Is always invertible: R is +ve definite

$B^T V_{T-1} B$ is +ve semi definite.

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So $u_{T-2} = \boxed{-\left(R + B^T V_{T-1} B\right)^{-1} B^T V_{T-1} A} \, x_{T-2}$

$\|$

$K =$ Kalman Gain.

Sub U_{T-2} in $J(x_{T-2}, T-2)$

$$= x_{T-2}^T K^T R K x_{T-2}$$

$$+ x_{T-2}^T Q x_{T-2}$$

$$+ x^T (A + BK)^T V_{T-1} (A + BK) x$$

$$V_{T-2} = K^T R K + Q + (A + BK)^T V_{T-1} (A + BK)$$

VI for LQR

$V_T = 0 \rightarrow$ Initialization (fake time step)

for $t = T-1 \ldots 0$:

(Linear Controller) $K_t = -(R + B^T V_{t+1} B)^{-1} B^T V_{t+1} A$

(Quadratic value function) $V_t = (K_t^T R K_t + Q + (A+BK_t)^T V_{t+1} (A+BK_t))$

$O[(\dim(u)^3)(\dim(x)^3)T]$

Note

VI	VI for LQR
$O(T\|S\|\|A\|)$	$O((\dim u)^3 (\dim x)^3 T)$

DARE

Discrete Algebraic Ricatti Equation.

Once converged V_t dependence on time is removed so again can solve DARE equation as system at all states.

BIG WIN!