CSE-571
Robotics

Probabilistic Robotics

Probabilities
Bayes rule
Bayes filters

Probabilistic Robotics

Key idea: Explicit representation of uncertainty
(using the calculus of probability theory)

- Perception = state estimation
- Action = utility optimization

Discrete Random Variables

- $X$ denotes a random variable.
- $X$ can take on a countable number of values in $\{x_1, x_2, \ldots, x_n\}$.
- $P(X=x_i)$, or $P(x_i)$, is the probability that the random variable $X$ takes on value $x_i$.
- $P(\cdot)$ is called probability mass function.

- E.g. $P(Room) = \{0.7, 0.2, 0.08, 0.02\}$

Joint and Conditional Probability

- $P(X=x \text{ and } Y=y) = P(x, y)$

- If $X$ and $Y$ are independent then
  $P(x, y) = P(x) P(y)$

- $P(x \mid y)$ is the probability of $x$ given $y$
  $P(x \mid y) = P(x, y) / P(y)$
  $P(x, y) = P(x \mid y) P(y)$

- If $X$ and $Y$ are independent then
  $P(x \mid y) = P(x)$
Law of Total Probability, Marginals

**Discrete case**

\[ \sum_x P(x) = 1 \]

\[ P(x) = \sum_y P(x, y) \]

\[ P(x) = \sum_y P(x \mid y) P(y) \]

**Continuous case**

\[ \int p(x) \, dx = 1 \]

\[ p(x) = \int p(x, y) \, dy \]

\[ p(x) = \int p(x \mid y) p(y) \, dy \]

Events

- \( P(\pm x, \pm y) \) ?
- \( P(\pm x) \) ?
- \( P(-y \text{ OR } +x) \) ?
- Independent?

Marginal Distributions

<table>
<thead>
<tr>
<th>( X \times Y )</th>
<th>P</th>
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<tbody>
<tr>
<td>(+x) (+y)</td>
<td>0.2</td>
</tr>
<tr>
<td>(+x) (-y)</td>
<td>0.3</td>
</tr>
<tr>
<td>(-x) (+y)</td>
<td>0.4</td>
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<tr>
<td>(-x) (-y)</td>
<td>0.1</td>
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Conditional Probabilities

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Bayes Formula

\[ P(x, y) = P(x \mid y)P(y) = P(y \mid x)P(x) \]

\[ P(x \mid y) = \frac{P(y \mid x)P(x)}{P(y)} = \frac{\text{likelihood} \cdot \text{prior}}{\text{evidence}} \]

- Often causal knowledge is easier to obtain than diagnostic knowledge.
- Bayes rule allows us to use causal knowledge.

Simple Example of State Estimation

- Suppose a robot obtains measurement \( z \).
- What is \( P(\text{open} \mid z) \)?

Example

- \( P(z \mid \text{open}) = 0.6 \)  \( P(z \mid \neg \text{open}) = 0.3 \)
- \( P(\text{open}) = P(\neg \text{open}) = 0.5 \)

\[
P(\text{open} \mid z) = \frac{P(z \mid \text{open})P(\text{open})}{P(z \mid \text{open})p(\text{open}) + P(z \mid \neg \text{open})p(\neg \text{open})}
\]

\[
P(\text{open} \mid z) = \frac{0.6 \cdot 0.5}{0.6 \cdot 0.5 + 0.3 \cdot 0.5} = \frac{2}{3} = 0.67
\]

- \( z \) raises the probability that the door is open.

Normalization

\[
P(x \mid y) = \frac{P(y \mid x) P(x)}{P(y)} = \eta \cdot P(y \mid x) P(x)
\]

\[
\eta = P(y) = \frac{1}{\sum_{x'} P(y \mid x')P(x')}
\]

Algorithm:

\[
\forall x : \text{aux}_{xy} = P(y \mid x) P(x)
\]

\[
\eta = \frac{1}{\sum_{x} \text{aux}_{xy}}
\]

\[
\forall x : P(x \mid y) = \eta \cdot \text{aux}_{xy}
\]
**Conditioning**

- Bayes rule and **background knowledge**:

\[
P(x | y, z) = \frac{P(y | x, z) P(x | z)}{P(y | z)}
\]

\[
P(x | y) = \int P(x | y, z) P(z) \, dz
\]

\[
= \int P(x | y, z) P(z | y) \, dz
\]

\[
= \int P(x | y, z) P(y | z) \, dz
\]

**Conditional Independence**

\[
P(x, y | z) = P(x | z) P(y | z)
\]

- Equivalent to

\[
P(x | z) = P(x | z, y)
\]

and

\[
P(y | z) = P(y | z, x)
\]

**Simple Example of State Estimation**

- Suppose our robot obtains another observation \( z_2 \).
- What is \( P(\text{open} | z_1, z_2) \)?
Recursive Bayesian Updating

\[ P(x \mid z_1, \ldots, z_n) = \frac{P(z_n \mid x, z_1, \ldots, z_{n-1}) P(x \mid z_1, \ldots, z_{n-1})}{P(z_n \mid z_1, \ldots, z_{n-1})} \]

**Markov assumption:** \( z_n \) is conditionally independent of \( z_1, \ldots, z_{n-1} \) given \( x \).

\[ P(x \mid z_1, \ldots, z_n) = \frac{P(z_n \mid x) P(x \mid z_1, \ldots, z_{n-1})}{P(z_n \mid z_1, \ldots, z_{n-1})} = \eta \frac{P(z_n \mid x) P(x \mid z_1, \ldots, z_{n-1})}{P(z_n \mid z_1, \ldots, z_{n-1})} = \eta \prod_{i=1}^{n} P(z_i \mid x) P(x) \]

**Example: Second Measurement**

\[ P(z_2 \mid \text{open}) = 0.5 \quad P(z_2 \mid \text{open}) = 0.6 \]
\[ P(\text{open} \mid z_1) = 2/3 \quad P(\text{open} \mid z_1) = 1/3 \]

\[ P(\text{open} \mid z_2, z_1) = \frac{P(z_2 \mid \text{open}) P(\text{open} \mid z_1)}{P(z_2 \mid \text{open}) P(\text{open} \mid z_1) + P(z_2 \mid \text{open}) P(\text{open} \mid z_1)} \]
\[ = \frac{1 \cdot 2}{1 \cdot 2 + 1 \cdot 3} = \frac{1}{2} \cdot \frac{2}{3} = \frac{5}{8} = 0.625 \]

• \( z_2 \) lowers the probability that the door is open.