# CSE-571
# Probabilistic Robotics

## Kalman Filters

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## Bayes Filter Reminder

- Prediction

$$\overline{bel}(x_t) = \int p(x_t \mid u_t, x_{t-1})\, bel(x_{t-1})\, dx_{t-1}$$

- Correction

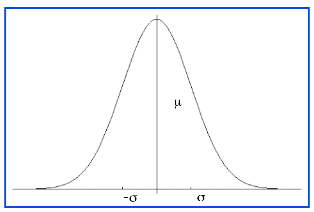
$$bel(x_t) = \eta\, p(z_t \mid x_t)\, \overline{bel}(x_t)$$

## Gaussians

$$p(x) \sim N(\mu, \sigma^2):$$

$$p(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}}$$

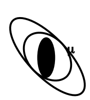
Univariate




$$p(\mathbf{x}) \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma}):$$

$$p(\mathbf{x}) = \frac{1}{(2\pi)^{d/2}|\boldsymbol{\Sigma}|^{1/2}} e^{-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^t \boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})}$$

Multivariate




## Properties of Gaussians

$$\left.\begin{array}{l} X \sim N(\mu, \sigma^2) \\ Y = aX + b \end{array}\right\} \Rightarrow Y \sim N(a\mu + b, a^2\sigma^2)$$

$$\left.\begin{array}{l} X_1 \sim N(\mu_1, \sigma_1^2) \\ X_2 \sim N(\mu_2, \sigma_2^2) \end{array}\right\} \Rightarrow p(X_1)\cdot p(X_2) \sim N\left(\frac{\sigma_2^2}{\sigma_1^2+\sigma_2^2}\mu_1 + \frac{\sigma_1^2}{\sigma_1^2+\sigma_2^2}\mu_2,\ \frac{1}{\sigma_1^{-2}+\sigma_2^{-2}}\right)$$

## Multivariate Gaussians

$$\left.\begin{array}{l} X \sim N(\mu,\Sigma) \\ Y = AX + B \end{array}\right\} \Rightarrow \quad Y \sim N(A\mu + B, A\Sigma A^T)$$

$$\left.\begin{array}{l} X_1 \sim N(\mu_1,\Sigma_1) \\ X_2 \sim N(\mu_2,\Sigma_2) \end{array}\right\} \Rightarrow p(X_1)\cdot p(X_2) \sim N\left(\frac{\Sigma_2}{\Sigma_1+\Sigma_2}\mu_1 + \frac{\Sigma_1}{\Sigma_1+\Sigma_2}\mu_2, \quad \frac{1}{\Sigma_1^{-1}+\Sigma_2^{-1}}\right)$$

- We stay in the "Gaussian world" as long as we start with Gaussians and perform only linear transformations.

## Discrete Kalman Filter

Estimates the state $x$ of a discrete-time controlled process that is governed by the linear stochastic difference equation

$$x_t = A_t x_{t-1} + B_t u_t + \varepsilon_t$$

with a measurement

$$z_t = C_t x_t + \delta_t$$



## Components of a Kalman Filter

$A_t$ — Matrix (nxn) that describes how the state evolves from $t$ to $t-1$ without controls or noise.

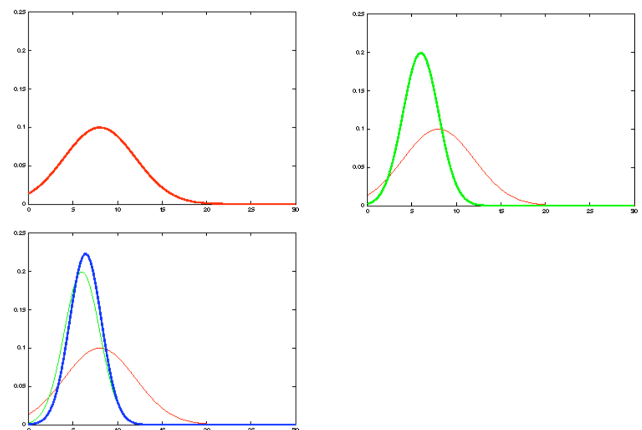
$B_t$ — Matrix (nxl) that describes how the control $u_t$ changes the state from $t$ to $t-1$.

$C_t$ — Matrix (kxn) that describes how to map the state $x_t$ to an observation $z_t$.

$\varepsilon_t$, $\delta_t$ — Random variables representing the process and measurement noise that are assumed to be independent and normally distributed with covariance $R_t$ and $Q_t$ respectively.



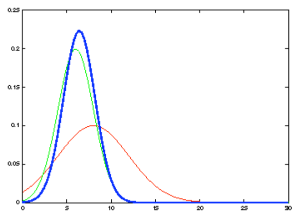
## Kalman Filter Updates in 1D



## Kalman Filter Updates in 1D

$$bel(x_t) = \begin{cases} \mu_t = \bar{\mu}_t + K_t(z_t - \bar{\mu}_t) \\ \sigma_t^2 = (1 - K_t)\bar{\sigma}_t^2 \end{cases} \quad \text{with} \quad K_t = \frac{\bar{\sigma}_t^2}{\bar{\sigma}_t^2 + \bar{\sigma}_{obs,t}^2}$$

$$bel(x_t) = \begin{cases} \mu_t = \bar{\mu}_t + K_t(z_t - C_t\bar{\mu}_t) \\ \Sigma_t = (I - K_t C_t)\bar{\Sigma}_t \end{cases} \quad \text{with} \quad K_t = \bar{\Sigma}_t C_t^T (C_t \bar{\Sigma}_t C_t^T + Q_t)^{-1}$$

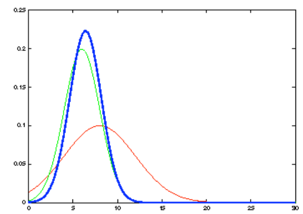


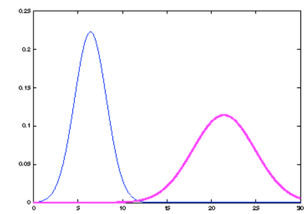


## Kalman Filter Updates in 1D

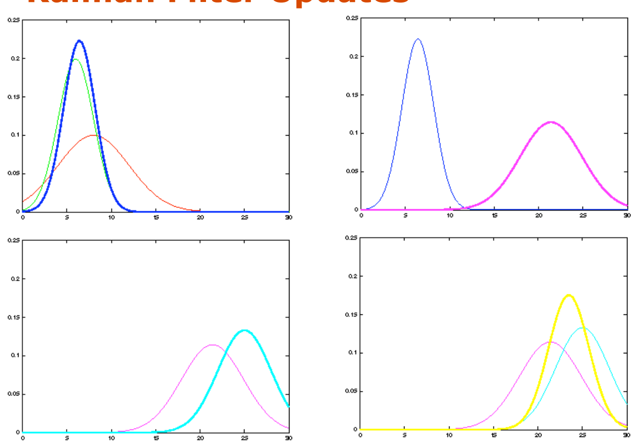
$$\overline{bel}(x_t) = \begin{cases} \bar{\mu}_t = a_t \mu_{t-1} + b_t u_t \\ \bar{\sigma}_t^2 = a_t^2 \sigma_t^2 + \sigma_{act,t}^2 \end{cases}$$

$$\overline{bel}(x_t) = \begin{cases} \bar{\mu}_t = A_t \mu_{t-1} + B_t u_t \\ \bar{\Sigma}_t = A_t \Sigma_{t-1} A_t^T + R_t \end{cases}$$





## Kalman Filter Updates



## Linear Gaussian Systems: Initialization

- Initial belief is normally distributed:

$$bel(x_0) = N(x_0; \mu_0, \Sigma_0)$$

## Linear Gaussian Systems: Dynamics

- Dynamics are linear function of state and control plus additive noise:

$$x_t = A_t x_{t-1} + B_t u_t + \varepsilon_t$$

$$p(x_t \mid u_t, x_{t-1}) = N\left(x_t; A_t x_{t-1} + B_t u_t, R_t\right)$$

$$\overline{bel}(x_t) = \int p(x_t \mid u_t, x_{t-1}) \qquad bel(x_{t-1})\, dx_{t-1}$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$\sim N\left(x_t; A_t x_{t-1} + B_t u_t, R_t\right) \quad \sim N\left(x_{t-1}; \mu_{t-1}, \Sigma_{t-1}\right)$$

## Linear Gaussian Systems: Dynamics

$$\overline{bel}(x_t) = \int p(x_t \mid u_t, x_{t-1}) \qquad bel(x_{t-1})\, dx_{t-1}$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$\sim N\left(x_t; A_t x_{t-1} + B_t u_t, R_t\right) \quad \sim N\left(x_{t-1}; \mu_{t-1}, \Sigma_{t-1}\right)$$

$$\Downarrow$$

$$\overline{bel}(x_t) = \eta \int \exp\left\{-\frac{1}{2}(x_t - A_t x_{t-1} - B_t u_t)^T R_t^{-1} (x_t - A_t x_{t-1} - B_t u_t)\right\}$$

$$\exp\left\{-\frac{1}{2}(x_{t-1} - \mu_{t-1})^T \Sigma_{t-1}^{-1} (x_{t-1} - \mu_{t-1})\right\} dx_{t-1}$$

$$\overline{bel}(x_t) = \begin{cases} \bar{\mu}_t = A_t \mu_{t-1} + B_t u_t \\ \bar{\Sigma}_t = A_t \Sigma_{t-1} A_t^T + R_t \end{cases}$$

## Linear Gaussian Systems: Observations

- Observations are linear function of state plus additive noise:

$$z_t = C_t x_t + \delta_t$$

$$p(z_t \mid x_t) = N\left(z_t; C_t x_t, Q_t\right)$$

$$bel(x_t) = \eta \quad p(z_t \mid x_t) \qquad \overline{bel}(x_t)$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$\sim N\left(z_t; C_t x_t, Q_t\right) \quad \sim N\left(x_t; \bar{\mu}_t, \bar{\Sigma}_t\right)$$

## Linear Gaussian Systems: Observations

$$bel(x_t) = \eta \quad p(z_t \mid x_t) \qquad \overline{bel}(x_t)$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$\sim N\left(z_t; C_t x_t, Q_t\right) \quad \sim N\left(x_t; \bar{\mu}_t, \bar{\Sigma}_t\right)$$

$$\Downarrow$$

$$bel(x_t) = \eta \exp\left\{-\frac{1}{2}(z_t - C_t x_t)^T Q_t^{-1} (z_t - C_t x_t)\right\} \exp\left\{-\frac{1}{2}(x_t - \bar{\mu}_t)^T \bar{\Sigma}_t^{-1} (x_t - \bar{\mu}_t)\right\}$$

$$bel(x_t) = \begin{cases} \mu_t = \bar{\mu}_t + K_t (z_t - C_t \bar{\mu}_t) \\ \Sigma_t = (I - K_t C_t) \bar{\Sigma}_t \end{cases} \quad \text{with} \quad K_t = \bar{\Sigma}_t C_t^T (C_t \bar{\Sigma}_t C_t^T + Q_t)^{-1}$$

## Kalman Filter Algorithm

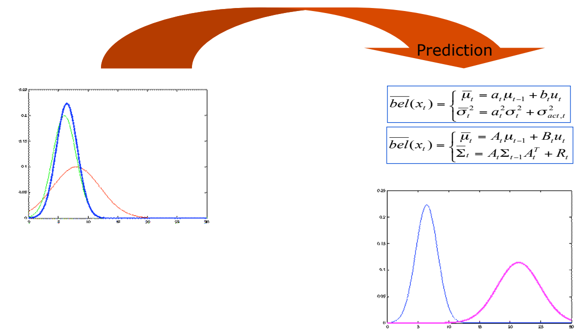
1. Algorithm **Kalman_filter**( $\mu_{t-1}, \Sigma_{t-1}, u_t, z_t$):

2. Prediction:
3. $\bar{\mu}_t = A_t \mu_{t-1} + B_t u_t$
4. $\bar{\Sigma}_t = A_t \Sigma_{t-1} A_t^T + R_t$

5. Correction:
6. $K_t = \bar{\Sigma}_t C_t^T (C_t \bar{\Sigma}_t C_t^T + Q_t)^{-1}$
7. $\mu_t = \bar{\mu}_t + K_t (z_t - C_t \bar{\mu}_t)$
8. $\Sigma_t = (I - K_t C_t) \bar{\Sigma}_t$

9. Return $\mu_t, \Sigma_t$

## The Prediction-Correction-Cycle

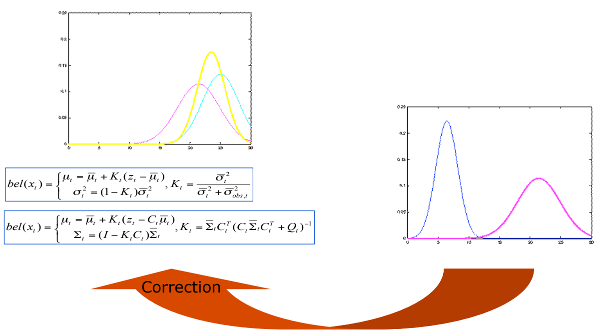



$$\overline{bel}(x_t) = \begin{cases} \bar{\mu}_t = a_t \mu_{t-1} + b_t u_t \\ \bar{\sigma}_t^2 = a_t^2 \sigma_t^2 + \sigma_{act,t}^2 \end{cases}$$

$$\overline{bel}(x_t) = \begin{cases} \bar{\mu}_t = A_t \mu_{t-1} + B_t u_t \\ \bar{\Sigma}_t = A_t \Sigma_{t-1} A_t^T + R_t \end{cases}$$



## The Prediction-Correction-Cycle

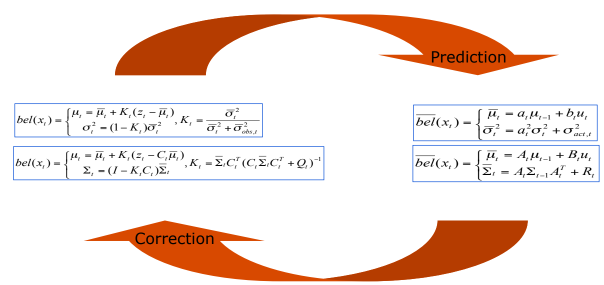



$$bel(x_t) = \begin{cases} \mu_t = \bar{\mu}_t + K_t (z_t - \bar{\mu}_t) \\ \sigma_t^2 = (1 - K_t) \bar{\sigma}_t^2 \end{cases}, K_t = \frac{\bar{\sigma}_t^2}{\bar{\sigma}_t^2 + \bar{\sigma}_{obs,t}^2}$$

$$bel(x_t) = \begin{cases} \mu_t = \bar{\mu}_t + K_t (z_t - C_t \bar{\mu}_t) \\ \Sigma_t = (I - K_t C_t) \bar{\Sigma}_t \end{cases}, K_t = \bar{\Sigma}_t C_t^T (C_t \bar{\Sigma}_t C_t^T + Q_t)^{-1}$$



## The Prediction-Correction-Cycle




$$bel(x_t) = \begin{cases} \mu_t = \bar{\mu}_t + K_t (z_t - \bar{\mu}_t) \\ \sigma_t^2 = (1 - K_t) \bar{\sigma}_t^2 \end{cases}, K_t = \frac{\bar{\sigma}_t^2}{\bar{\sigma}_t^2 + \bar{\sigma}_{obs,t}^2}$$

$$bel(x_t) = \begin{cases} \mu_t = \bar{\mu}_t + K_t (z_t - C_t \bar{\mu}_t) \\ \Sigma_t = (I - K_t C_t) \bar{\Sigma}_t \end{cases}, K_t = \bar{\Sigma}_t C_t^T (C_t \bar{\Sigma}_t C_t^T + Q_t)^{-1}$$

$$\overline{bel}(x_t) = \begin{cases} \bar{\mu}_t = a_t \mu_{t-1} + b_t u_t \\ \bar{\sigma}_t^2 = a_t^2 \sigma_t^2 + \sigma_{act,t}^2 \end{cases}$$

$$\overline{bel}(x_t) = \begin{cases} \bar{\mu}_t = A_t \mu_{t-1} + B_t u_t \\ \bar{\Sigma}_t = A_t \Sigma_{t-1} A_t^T + R_t \end{cases}$$

## Kalman Filter Summary

- Highly efficient: Polynomial in measurement dimensionality $k$ and state dimensionality $n$:
  $$O(k^{2.376} + n^2)$$
- Optimal for linear Gaussian systems!
- Most robotics systems are nonlinear!

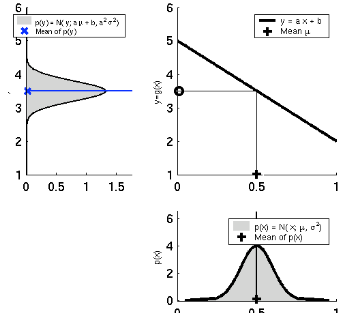
## Nonlinear Dynamic Systems

- Most realistic robotic problems involve nonlinear functions

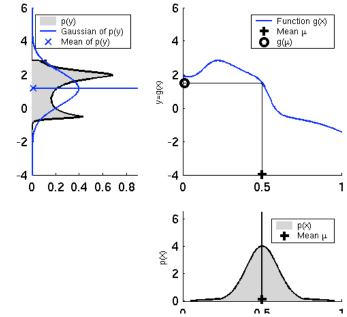
$$x_t = g(u_t, x_{t-1})$$

$$z_t = h(x_t)$$

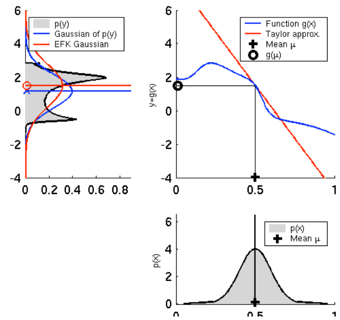
## Linearity Assumption Revisited



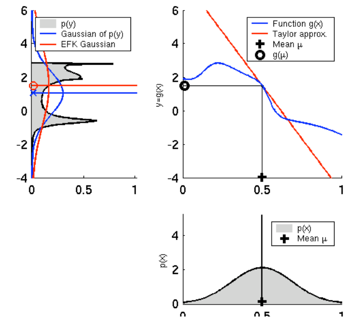

## Non-linear Function



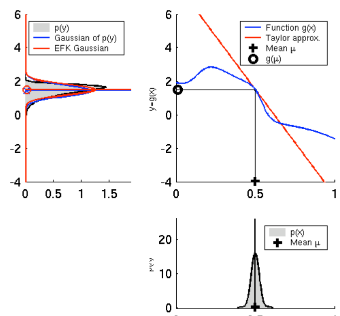
## EKF Linearization (1)




## EKF Linearization (2)




## EKF Linearization (3)




## EKF Linearization: First Order Taylor Series Expansion

- Prediction:

$$g(u_t, x_{t-1}) \approx g(u_t, \mu_{t-1}) + \frac{\partial g(u_t, \mu_{t-1})}{\partial x_{t-1}} (x_{t-1} - \mu_{t-1})$$

$$g(u_t, x_{t-1}) \approx g(u_t, \mu_{t-1}) + G_t \ (x_{t-1} - \mu_{t-1})$$

- Correction:

$$h(x_t) \approx h(\overline{\mu}_t) + \frac{\partial h(\overline{\mu}_t)}{\partial x_t} (x_t - \overline{\mu}_t)$$

$$h(x_t) \approx h(\overline{\mu}_t) + H_t \ (x_t - \overline{\mu}_t)$$

# EKF Algorithm

1. **Extended_Kalman_filter**( $\mu_{t-1}, \Sigma_{t-1}, u_t, z_t$):

2. Prediction:
3. $\bar{\mu}_t = g(u_t, \mu_{t-1})$ $\longleftarrow$ $\bar{\mu}_t = A_t \mu_{t-1} + B_t u_t$
4. $\bar{\Sigma}_t = G_t \Sigma_{t-1} G_t^T + R_t$ $\longleftarrow$ $\bar{\Sigma}_t = A_t \Sigma_{t-1} A_t^T + R_t$

5. Correction:
6. $K_t = \bar{\Sigma}_t H_t^T (H_t \bar{\Sigma}_t H_t^T + Q_t)^{-1}$ $\longleftarrow$ $K_t = \bar{\Sigma}_t C_t^T (C_t \bar{\Sigma}_t C_t^T + Q_t)^{-1}$
7. $\mu_t = \bar{\mu}_t + K_t (z_t - h(\bar{\mu}_t))$ $\longleftarrow$ $\mu_t = \bar{\mu}_t + K_t (z_t - C_t \bar{\mu}_t)$
8. $\Sigma_t = (I - K_t H_t) \bar{\Sigma}_t$ $\longleftarrow$ $\Sigma_t = (I - K_t C_t) \bar{\Sigma}_t$
9. Return $\mu_t, \Sigma_t$

$$H_t = \frac{\partial h(\bar{\mu}_t)}{\partial x_t} \qquad G_t = \frac{\partial g(u_t, \mu_{t-1})}{\partial x_{t-1}}$$

# Localization

> "Using sensory information to locate the robot in its environment is the most fundamental problem to providing a mobile robot with autonomous capabilities." [Cox '91]

- **Given**
  - Map of the environment.
  - Sequence of sensor measurements.
- **Wanted**
  - Estimate of the robot's position.
- **Problem classes**
  - Position tracking
  - Global localization
  - Kidnapped robot problem (recovery)

# Landmark-based Localization

1. **EKF_localization** ( $\mu_{t-1}, \Sigma_{t-1}, u_t, z_t, m$):

**Prediction:**

2. $$G_t = \frac{\partial g(u_t, \mu_{t-1})}{\partial x_{t-1}} = \begin{pmatrix} \frac{\partial x'}{\partial \mu_{t-1,x}} & \frac{\partial x'}{\partial \mu_{t-1,y}} & \frac{\partial x'}{\partial \mu_{t-1,\theta}} \\ \frac{\partial y'}{\partial \mu_{t-1,x}} & \frac{\partial y'}{\partial \mu_{t-1,y}} & \frac{\partial y'}{\partial \mu_{t-1,\theta}} \\ \frac{\partial \theta'}{\partial \mu_{t-1,x}} & \frac{\partial \theta'}{\partial \mu_{t-1,y}} & \frac{\partial \theta'}{\partial \mu_{t-1,\theta}} \end{pmatrix}$$ Jacobian of $g$ w.r.t location

3. $$V_t = \frac{\partial g(u_t, \mu_{t-1})}{\partial u_t} = \begin{pmatrix} \frac{\partial x'}{\partial v_t} & \frac{\partial x'}{\partial \omega_t} \\ \frac{\partial y'}{\partial v_t} & \frac{\partial y'}{\partial \omega_t} \\ \frac{\partial \theta'}{\partial v_t} & \frac{\partial \theta'}{\partial \omega_t} \end{pmatrix}$$ Jacobian of $g$ w.r.t control

4. $$M_t = \begin{pmatrix} (\alpha_1 |v_t| + \alpha_2 |\omega_t|)^2 & 0 \\ 0 & (\alpha_3 |v_t| + \alpha_4 |\omega_t|)^2 \end{pmatrix}$$ Motion noise

5. $\bar{\mu}_t = g(u_t, \mu_{t-1})$ Predicted mean
6. $\bar{\Sigma}_t = G_t \Sigma_{t-1} G_t^T + V_t M_t V_t^T$ Predicted covariance

1. **EKF_localization** ( $\mu_{t-1}, \Sigma_{t-1}, u_t, z_t, m$):

**Correction:**

2. $\hat{z}_t = \begin{pmatrix} \sqrt{(m_x - \bar{\mu}_{t,x})^2 + (m_y - \bar{\mu}_{t,y})^2} \\ \operatorname{atan2}(m_y - \bar{\mu}_{t,y}, m_x - \bar{\mu}_{t,x}) - \bar{\mu}_{t,\theta} \end{pmatrix}$ Predicted measurement mean

3. $H_t = \frac{\partial h(\bar{\mu}_t, m)}{\partial x_t} = \begin{pmatrix} \frac{\partial r_t}{\partial \bar{\mu}_{t,x}} & \frac{\partial r_t}{\partial \bar{\mu}_{t,y}} & \frac{\partial r_t}{\partial \bar{\mu}_{t,\theta}} \\ \frac{\partial \varphi_t}{\partial \bar{\mu}_{t,x}} & \frac{\partial \varphi_t}{\partial \bar{\mu}_{t,y}} & \frac{\partial \varphi_t}{\partial \bar{\mu}_{t,\theta}} \end{pmatrix}$ Jacobian of *h* w.r.t location

4. $Q_t = \begin{pmatrix} \sigma_r^2 & 0 \\ 0 & \sigma_r^2 \end{pmatrix}$

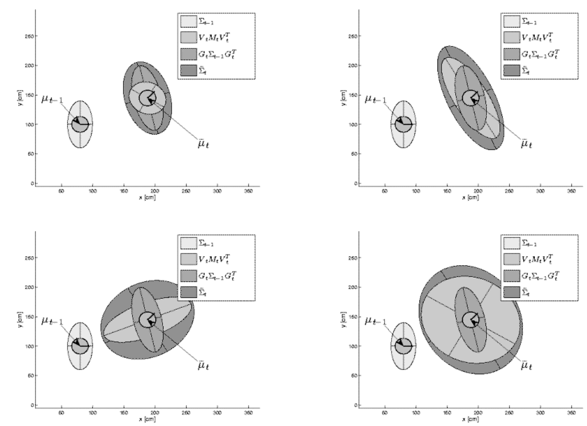
5. $S_t = H_t \bar{\Sigma}_t H_t^T + Q_t$ Pred. measurement covariance

6. $K_t = \bar{\Sigma}_t H_t^T S_t^{-1}$ Kalman gain

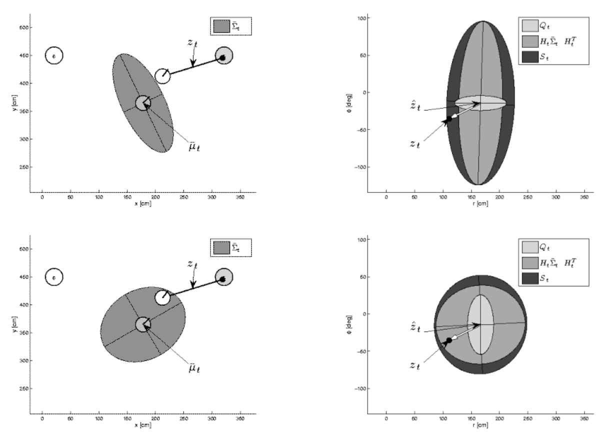
7. $\mu_t = \bar{\mu}_t + K_t (z_t - \hat{z}_t)$ Updated mean

8. $\Sigma_t = (I - K_t H_t) \bar{\Sigma}_t$ Updated covariance

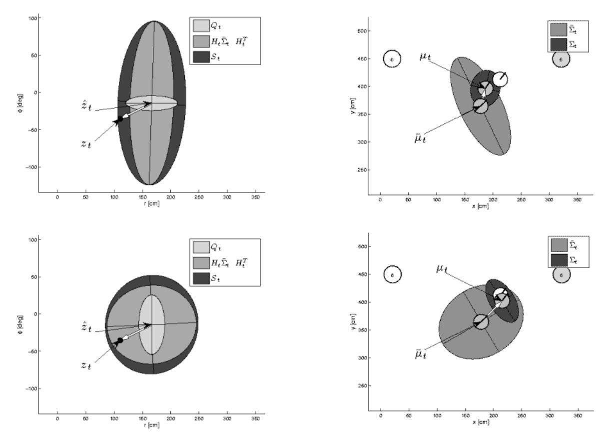
# EKF Prediction Step



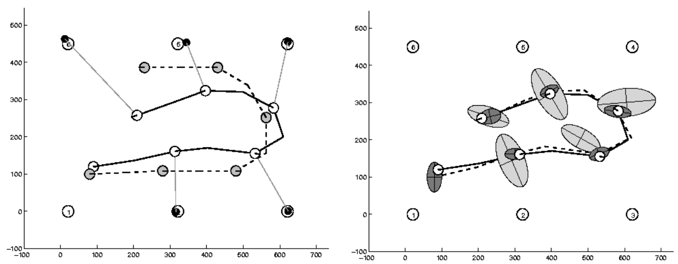
# EKF Observation Prediction Step



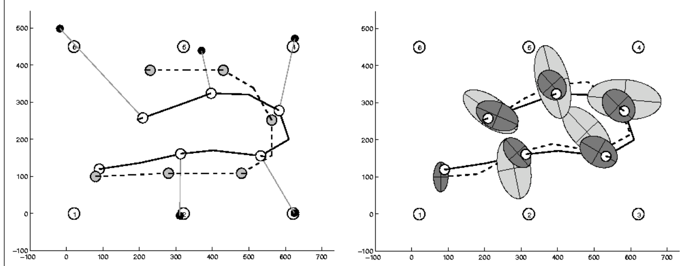
# EKF Correction Step



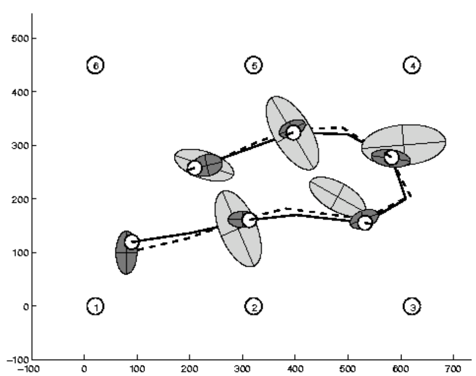
## Estimation Sequence (1)



## Estimation Sequence (2)



## Comparison to GroundTruth



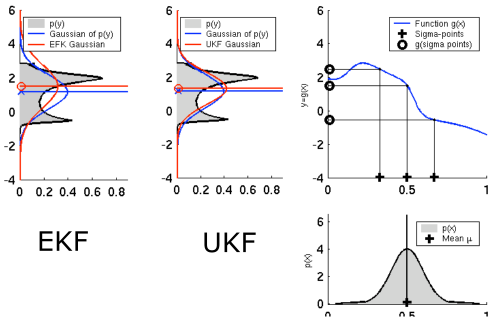

## EKF Summary

- Highly efficient: Polynomial in measurement dimensionality *k* and state dimensionality *n*:

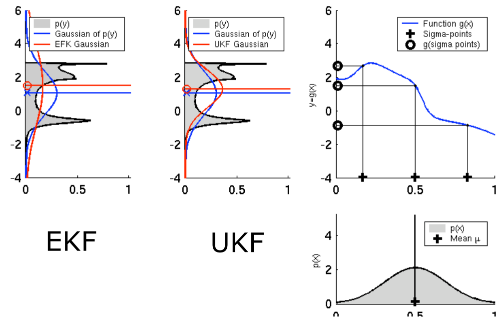
$$O(k^{2.376} + n^2)$$

- Not optimal!
- Can diverge if nonlinearities are large!
- Works surprisingly well even when all assumptions are violated!

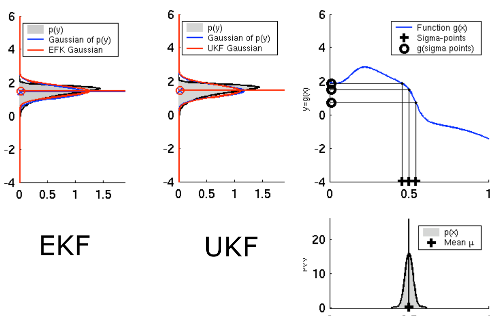
## Linearization via Unscented Transform




## UKF Sigma-Point Estimate (2)




## UKF Sigma-Point Estimate (3)




## Unscented Transform

Sigma points

$$\chi^0 = \mu$$

$$\chi^i = \mu \pm \left(\sqrt{(n+\lambda)\Sigma}\right)_i$$

Weights

$$w_m^0 = \frac{\lambda}{n+\lambda} \qquad w_c^0 = \frac{\lambda}{n+\lambda} + (1-\alpha^2+\beta)$$

$$w_m^i = w_c^i = \frac{1}{2(n+\lambda)} \qquad \text{for } i = 1,\dots,2n$$

Pass sigma points through nonlinear function

$$\psi^i = g(\chi^i)$$

Recover mean and covariance

$$\mu' = \sum_{i=0}^{2n} w_m^i \psi^i$$

$$\Sigma' = \sum_{i=0}^{2n} w_c^i (\psi^i - \mu)(\psi^i - \mu)^T$$

## UKF_localization ( $\mu_{t-1}, \Sigma_{t-1}, u_t, z_t, m$):

**Prediction:**

$$M_t = \begin{pmatrix} (\alpha_1 |v_t| + \alpha_2 |\omega_t|)^2 & 0 \\ 0 & (\alpha_3 |v_t| + \alpha_4 |\omega_t|)^2 \end{pmatrix}$$ Motion noise

$$Q_t = \begin{pmatrix} \sigma_r^2 & 0 \\ 0 & \sigma_r^2 \end{pmatrix}$$ Measurement noise

$$\mu_{t-1}^a = \begin{pmatrix} \mu_{t-1}^T & (0\ 0)^T & (0\ 0)^T \end{pmatrix}^T$$ Augmented state mean

$$\Sigma_{t-1}^a = \begin{pmatrix} \Sigma_{t-1} & 0 & 0 \\ 0 & M_t & 0 \\ 0 & 0 & Q_t \end{pmatrix}$$ Augmented covariance

$$\chi_{t-1}^a = \begin{pmatrix} \mu_{t-1}^a & \mu_{t-1}^a + \gamma \sqrt{\Sigma_{t-1}^a} & \mu_{t-1}^a - \gamma \sqrt{\Sigma_{t-1}^a} \end{pmatrix}$$ Sigma points

$$\bar{\chi}_t^x = g(u_t + \chi_t^u, \chi_{t-1}^x)$$ Prediction of sigma points

$$\bar{\mu}_t = \sum_{i=0}^{2L} w_m^i \chi_{i,t}^x$$ Predicted mean

$$\bar{\Sigma}_t = \sum_{i=0}^{2L} w_c^i (\chi_{i,t}^x - \bar{\mu}_t)(\chi_{i,t}^x - \bar{\mu}_t)^T$$ Predicted covariance

## UKF_localization ( $\mu_{t-1}, \Sigma_{t-1}, u_t, z_t, m$):

**Correction:**

$$\bar{Z}_t = h(\bar{\chi}_t^x) + \chi_t^z$$ Measurement sigma points

$$\hat{z}_t = \sum_{i=0}^{2L} w_m^i \bar{Z}_{i,t}$$ Predicted measurement mean

$$S_t = \sum_{i=0}^{2L} w_c^i (\bar{Z}_{i,t} - \hat{z}_t)(\bar{Z}_{i,t} - \hat{z}_t)^T$$ Pred. measurement covariance

$$\Sigma_t^{x,z} = \sum_{i=0}^{2L} w_c^i (\bar{\chi}_{i,t}^x - \bar{\mu}_t)(\bar{Z}_{i,t} - \hat{z}_t)^T$$ Cross-covariance

$$K_t = \Sigma_t^{x,z} S_t^{-1}$$ Kalman gain

$$\mu_t = \bar{\mu}_t + K_t (z_t - \hat{z}_t)$$ Updated mean

$$\Sigma_t = \bar{\Sigma}_t - K_t S_t K_t^T$$ Updated covariance

## 1. EKF_localization ( $\mu_{t-1}, \Sigma_{t-1}, u_t, z_t, m$):

**Correction:**

2. $$\hat{z}_t = \begin{pmatrix} \sqrt{(m_x - \bar{\mu}_{t,x})^2 + (m_y - \bar{\mu}_{t,y})^2} \\ \operatorname{atan2}(m_y - \bar{\mu}_{t,y}, m_x - \bar{\mu}_{t,x}) - \bar{\mu}_{t,\theta} \end{pmatrix}$$ Predicted measurement mean

3. $$H_t = \frac{\partial h(\bar{\mu}_t, m)}{\partial x_t} = \begin{pmatrix} \frac{\partial r_t}{\partial \bar{\mu}_{t,x}} & \frac{\partial r_t}{\partial \bar{\mu}_{t,y}} & \frac{\partial r_t}{\partial \bar{\mu}_{t,\theta}} \\ \frac{\partial \varphi_t}{\partial \bar{\mu}_{t,x}} & \frac{\partial \varphi_t}{\partial \bar{\mu}_{t,y}} & \frac{\partial \varphi_t}{\partial \bar{\mu}_{t,\theta}} \end{pmatrix}$$ Jacobian of *h* w.r.t location

4. $$Q_t = \begin{pmatrix} \sigma_r^2 & 0 \\ 0 & \sigma_r^2 \end{pmatrix}$$

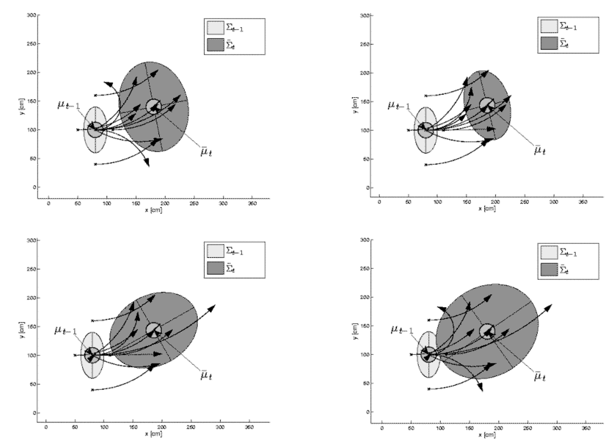
5. $S_t = H_t \bar{\Sigma}_t H_t^T + Q_t$ Pred. measurement covariance

6. $K_t = \bar{\Sigma}_t H_t^T S_t^{-1}$ Kalman gain

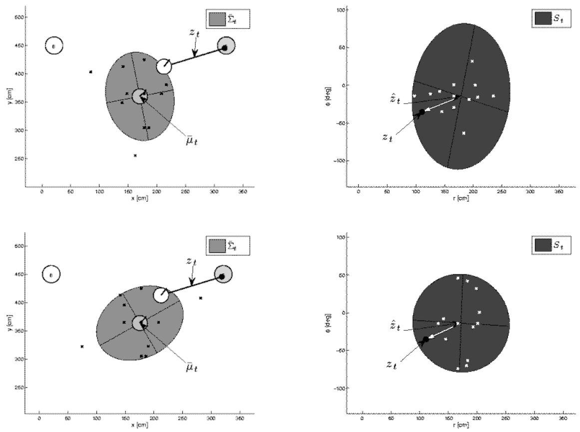
7. $\mu_t = \bar{\mu}_t + K_t (z_t - \hat{z}_t)$ Updated mean

8. $\Sigma_t = (I - K_t H_t) \bar{\Sigma}_t$ Updated covariance

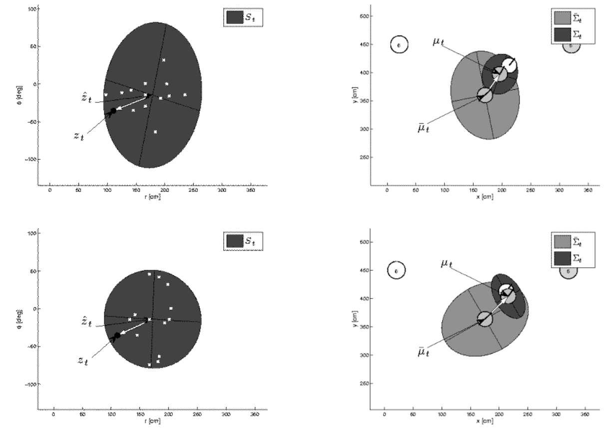
## UKF Prediction Step



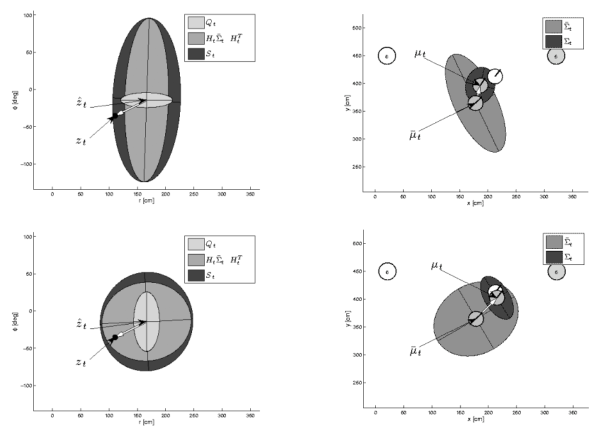
## UKF Observation Prediction Step



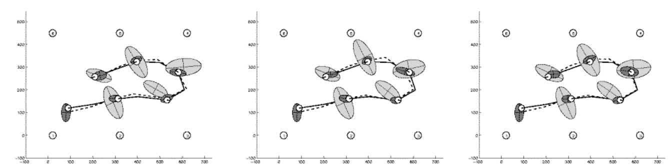
## UKF Correction Step



## EKF Correction Step

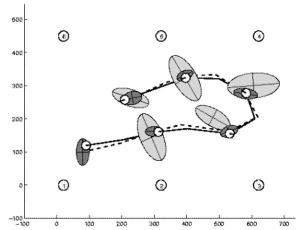


## Estimation Sequence

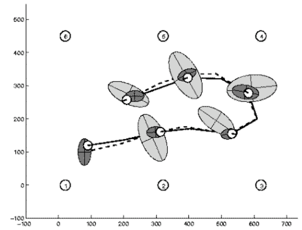


EKF PF UKF

## Estimation Sequence

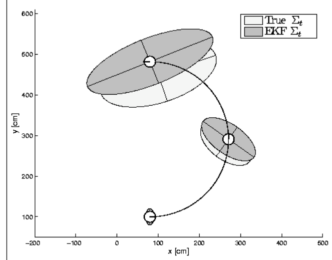


EKF

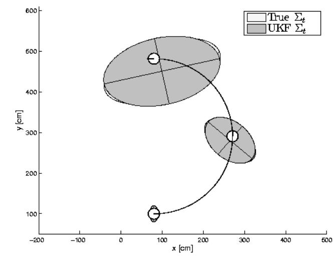


UKF

## Prediction Quality




EKF




UKF

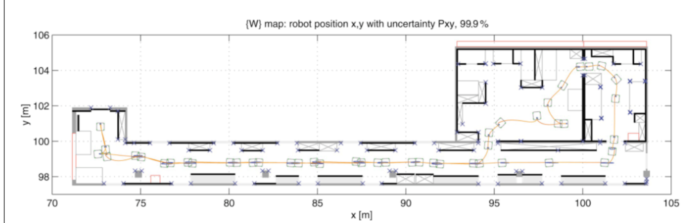
## UKF Summary

- Highly efficient: Same complexity as EKF, with a constant factor slower in typical practical applications
- Better linearization than EKF: Accurate in first two terms of Taylor expansion (EKF only first term)
- Derivative-free: No Jacobians needed
- Still not optimal!



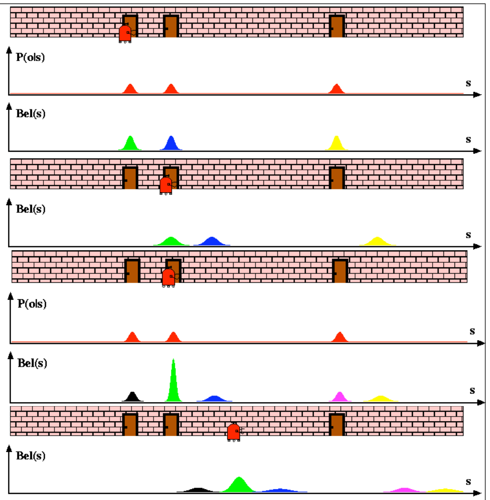
## Kalman Filter-based System

- [Arras et al. 98]:
  - Laser range-finder and vision
  - High precision (<1cm accuracy)




Courtesy of K. Arras

## Multi-hypothesis Tracking




## Localization With MHT

- Belief is represented by multiple hypotheses
- Each hypothesis is tracked by a Kalman filter

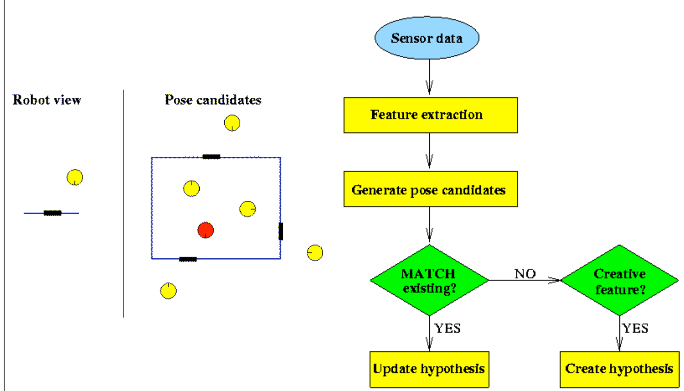
- **Additional problems**:
  - Data association: Which observation corresponds to which hypothesis?
  - Hypothesis management: When to add / delete hypotheses?
- Huge body of literature on target tracking, motion correspondence etc.

## MHT: Implemented System (1)

- Hypotheses are extracted from LRF scans
- Each hypothesis has probability of being the correct one: $$H_i = \{\hat{x}_i, \Sigma_i, P(H_i)\}$$
- Hypothesis probability is computed using Bayes' rule $$P(H_i \mid s) = \frac{P(s \mid H_i)P(H_i)}{P(s)}$$
- Hypotheses with low probability are deleted.
- New candidates are extracted from LRF scans. $$C_j = \{z_j, R_j\}$$

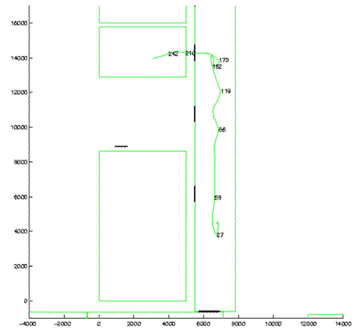
[Jensfelt et al. '00]

## MHT: Implemented System (2)

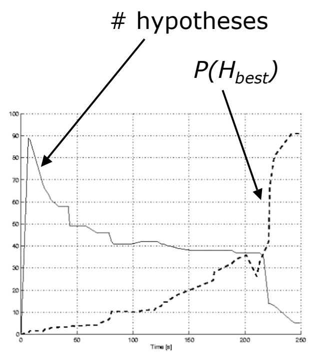



Courtesy of P. Jensfelt and S. Kristensen

# MHT: Implemented System (3)
# Example run



Map and trajectory




#hypotheses vs. time

Courtesy of P. Jensfelt and S. Kristensen